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Connectedness in Fuzzy Topology

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The concepts of subspace topology and connectedness in the context of a fuzzy topological space are introduced.

I. INTRODUCTION

With the emergence of the fundamental paper [5] by Zadeh in 1965 number of papers have appeared in literature featuring the application of fuzzy sets to pattern recognition, decision problems, function approximation, system theory, fuzzy logic, fuzzy algorithms, fuzzy automata, fuzzy grammars, fuzzy languages, fuzzy algebras, fuzzy topology, etc. [2], [7]. In this note, our interests are in the study of certain concepts in fuzzy topology.

The concepts of continuity, compactness in the context of a fuzzy topological space are well known [1], [7]. In this note the concept of subspace topology is introduced and connectedness of a fuzzy topological space are studied. Our notation and terminology follow that of [1].

II. SUBSPACE TOPOLOGY

Definition 1. [1] A *fuzzy topology* is a family $\boldsymbol{T}$ of fuzzy sets in X which satisfies the following conditions:

a) $\emptyset, X \in \boldsymbol{T}$.

b) If $A, B \in \boldsymbol{T}$, then $A \cap B \in \boldsymbol{T}$.

c) If $A_i \in \boldsymbol{T}$, for each $i \in I$, then $\bigcup_i A_i \in \boldsymbol{T}$.

$\boldsymbol{T}$ is called a fuzzy topology for X, and the pair $(X, \boldsymbol{T})$ is called a fuzzy topological space (FTS in short).

Definition 2. [1] Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function $\mu_B(Y)$. Then the inverse image of B written as $f^{-1}(B)$ is a fuzzy set in X whose membership function is defined by

$$\mu_{f^{-1}(B)}(x) = \mu_B\{f(x)\} \quad \text{for all } x \text{ in } X .$$

Conversely, let A be a fuzzy set in X with membership function $\mu_A(x)$. The image of A written as $f(A)$ is a fuzzy set in Y whose membership function is given by

$$\begin{aligned} \mu_{f(A)}(y) &= \operatorname*{Sup}_{x \in f^{-1}(y)} \{\mu_A(x)\} \quad \text{if} \quad f^{-1}(y) \quad \text{is not empty} , \\ &= 0 \quad \text{otherwise} \end{aligned}$$

for all $y \in Y$ where $f^{-1}(y) = \{x/f(x) = y\}$.

Definition 3. [1] A function 'f' from a fuzzy topological space $(X, \boldsymbol{T})$ to a fuzzy topological space $(Y, \boldsymbol{U})$ is *F-continuous* if and only if the inverse of each $\boldsymbol{U}$-open fuzzy set is $\boldsymbol{T}$-open.

Let $(X, \boldsymbol{T})$ be a fuzzy topological space. Let A be an ordinary subset of X. Then the relative fuzzy topology of A can be defined in the following way.

The subset A of X (in the ordinary sense) has a characteristic function say μ_A such that

$$\begin{aligned} \mu_A(X) &= 1 \quad \text{if} \quad x \in A , \\ &= 0 \quad \text{if} \quad x \notin A . \end{aligned}$$

Let
$$\boldsymbol{T}_A = \{B \cap A \mid B \in \boldsymbol{T}\} .$$

Then $\boldsymbol{T}_A$ is called a fuzzy subspace topology on A.

For, choose $\emptyset \in \boldsymbol{T}$, with membership function $\mu_\emptyset$. Then, as $\min(\mu_\emptyset(x), \mu_A(x)) = \mu_\emptyset(x)$, we have $\emptyset \in \boldsymbol{T}_A$.

To show that $A \in \boldsymbol{T}_A$, choose μ_X characteristic function for X, then

$$\min(\mu_X(x), \mu_A(x)) = \mu_A(x) .$$

Again, if B_A and $C_A \in \boldsymbol{T}_A$ then $B_A \cap C_A \in \boldsymbol{T}_A$ for $B_A \cap C_A = (B \cap C) \cap A$, and for $B_A^i \in \boldsymbol{T}_A$, then

$$\bigcup_i B_A^i = \bigcup_i (B_i \cap A) = (\bigcup B_i) \cap A .$$

Hence, $\boldsymbol{T}_A$ defines the fuzzy subspace topology for A.

III. CONNECTEDNESS

Definition 4. A fuzzy topological space X is said to be *disconnected* if $X = A \cup B$, where A and B are non-empty open fuzzy sets in X such that $A \cap B = \emptyset$.

Hence, a connected space is defined as follows:

Definition 5. A fuzzy topological space X is said to be *connected* if X cannot be represented as the union of two non-empty, disjoint open fuzzy sets on X.

Remark 1. If $X = A \cup B$ where $A \cap B = \emptyset$, A and B are non-empty fuzzy open sets of X, then they are complements to each other; and hence, both are open and closed.

For

$$\min \{\mu_A(x), \mu_B(x)\} = \mu_\emptyset(x), \quad x \in X,$$
$$= \mu_A(x) \quad \text{or} \quad \mu_B(x).$$

Suppose for a given x,

$$\min \{\mu_A(x), \mu_B(x)\} = \mu_A(x), \quad \text{then} \quad \mu_B(x) = 1.$$

Theorem. The F-continuous image of a connected fuzzy space X to a fuzzy topological space is connected.

Proof. Let $f : X \to Y$ be a F-continuous map of X to Y where $(X, \mathbf{T})$ and $(Y, \mathbf{T}_{f(X)})$ are fuzzy topological spaces, X is connected.

Suppose $f(X)$ is disconnected. Then there exist non-empty open fuzzy sets G and $H \in \mathbf{T}_{f(X)}$ such that $f(X) = G \cup H$. This implies that G and H are obtained from non-empty open fuzzy sets say G_y and H_y in Y such that $G = G_y \cap f(X)$; $H = H_y \cap f(X)$.

We shall show that $f^{-1}(G)$ and $f^{-1}(H)$ give a disconnection for X. That is, $X = f^{-1}(G) \cup f^{-1}(H)$.

For by Definition 4,

$$\max (\mu_{f^{-1}(G)}(x), \mu_{f^{-1}(H)}(x)) = \max (\mu_G(f(x)), \mu_H(f(x))),$$
$$= 1 \quad (\text{since } G \cup H = f(X)),$$
$$= \mu_X(x).$$

This is a contradiction to X being connected.

Hence, $f(X)$ is connected. □

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