Kybernetika

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A context free grammar describing synthesis of a phrase in free rhythm is presented. Free rhythm is that used in ancient proses and in music of early middle age in Europe and in byzantine influenced Asia. The grammar illustrates how modern means for communication analysis can exactly describe a matter which has got no corresponding expressing means in natural languages.

1. INTRODUCTION

Free rhythm as a certain system of time organization in artistical creations has been recognized in the modern time since about 100 years. After several years of initial development of its research certain categories concerning it have been fixed so that they could get a common spiritual property at least for specialists in corresponding scientific branches and that they have not been essentially modified by the following research of free rhythm. The categories have been described e.g. in [1] or [2] but in the greatest detail they have been presented in [3].

Free rhythm is not an absence of rhythm but a certain mode of ordering in time, which is different from the rhytmical rules known in classical poetry or music of the present millenium. It formed a basis of so called *plainsong* or *plainchant*, which was a unisone mode of singing general in both the Roman imperiums (West Roman and East one) since their separate existing and which was transferred into "barbarian" countries as a cultural heredity, where it lived almost a thousand years. In high middle age it began to be in decadence and with the end of middle age it was completely forgotten.

By modern scientists, the free rhythm system has been recognized first of all in the oldest musical compositions based on latine language texts, called milanese or ambrosian chant, and in the musical creation that followed: visigothic (or mozarabic) chant and namely gregorian one. The same rhythmical rules have been discovered also

in greek and latine prosaic creation well known from ancient time. Since about 40 years, when the original byzantine music has been redechiffred from medieval manuscripts after several centuries of ignorance (see [4]) the same system of rhythmical synthesis has been discovered also in music based on greek language, which rose and developed paralelly with the mentioned "latine" music. The explosion of its research and the richness of the results of this research implies a certain hope that the system of free rhythm holds also in other branches of musical eventually poetical creation of the early middle ages in Armenia, Syria, Egypt and Aethiopia, but one must wait until a sufficient number of compositions is authentically read.

The modern specialists have a certain terminology describing the rules of the free rhythm system, but the understanding is given by a certain practice in realizing the mentioned music: we can state that nowadays the musicologists, together with some other specialists in philology and aesthetics have understood something which was evident for the ancients but for which there are no suitable means in natural languages so that they could transfer an objective information about free rhythm outside the small group of specialists. In the present paper, the author tries to do a first step to prepare some essential properties of free rhythm so that they would be independent on musical sentiment and thus certain semiotical aspects of them could be communicated as notions of a formal theory. The step uses the phrase-structure-grammar mechanism, well known from mathematical linguistics (see e.g. [5]). The following part contains a description of the phrase-structure-grammar production rules of free rhythm: any exact description is always preceded by a mention of the corresponding property known from music.

2. DESCRIPTION OF THE SYSTEM

2.1. Elementary rhythm

In the free rhythm system we consider *elementary time values*. In natural languages they are considered as short syllabes while in music they are usually written as quavers. In the present paper we use a letter D to represent an elementary time value. The elementary time value cannot be divided but it can be repeated. Thus we can write DD, DDD etc. Let us mention that DD is an equivalent for a long syllabe in natural languages. In the mathematical theory of free rhythm we do not study real duration of an elementary time value, as it is an attribute which is outside the same theory: in latine chant, it is ordinarily permitted a slight *ritardando* or *accelerando* of any elementary time valne, but it is not specified of which size it has to be; the same principles have been transferred into the byzantine chant according to the school of Wellesz (see [4]) but after new considerations it seems that it will be possible to formulate more exact laws for any ritardando or accelerando. These laws seem to

be more rich in more oriental music, as e.g. in music of Armenia or Syria. But for
the present paper it is important that the synthesis of the elementary time values
does not depend on their duration in realization.

Elementary rhythm reflects the ordering of elementary time values at the lowest
level: they can form either elementary arses or elementary theses. An *elementary
arsis* is one elementary time value and it introduces one instance of the elementary
rhythm by a certain elevation of the flow of the elementary time values. As that
elevation has no exact relation to any physical property of the elementary time value
forming the arsis, we shall write it so that we put a letter H before the corresponding
D. Thus we obtain the first production rule of the grammar of the free rhythm:

(1) $$A_E \rightarrow HD .$$

An *elementary thesis* can be either one elementary time value or a pair of them; we
consider it as a certain relaxation of the flow of the elementary time values which
can take one or two elementary time values. Also that relaxation has no exact relation
to any physical property of the corresponding elementary time values so that we must
introduce a new symbol I for it. It corresponds to the symbol of *vertical episeme*
which is used in books prepared for practical realization in modern time (naturally,
there are a lot of identification where one should put a thesis according to the context
or to the graphical form of neumes, so that the episemes are not to be necessarily
in all cases; in our system, where the neumes are not used and where only context-free
rules are applied, we must write symbol I before any thesis). Therefore we get the
production rules for the elementary thesis:

(2) $$T_E \rightarrow ID ,$$

(3) $$T_E \rightarrow IDD .$$

The *elementary rhythm* is a pair where after an elementary arsis an elementary
thesis follows. It can be reflected in the grammar by the following rule:

(4) $$R_E \rightarrow A_E T_E .$$

From the view point of *elementary level* any instance of a musical or linguistical
form we consider is a sequence of elementary rhythms.

2.2. Simple rhythm

Elementary rhythms form no efficient components for a higher rhythmical synthe-
sis. Instead of them we must introduce *compound time values* which begin by an
elementary thesis which is followed by an elementary arsis; we shall represent them

by an auxiliary symbol D_C for which it is possible formulate immediately the following rule:

(5) $$D_C \rightarrow T_E A_E .$$

Therefore any compound time value has a form of either $IDHD$ or $IDDHD$. Using the musical terminology, we can state that from the view point of *basic level* any composition is a sequence of compound time values. They can be ordered in simple rhythms so that some of them form *simple arses* while the other form *simple theses*:

(6) $$A_S \rightarrow F D_C ,$$

(7) $$T_S \rightarrow G D_C .$$

In these production rules the symbols F and G have a certain analogy with the symbols H and I introduced for the elementary rhythm: F initiates a certain elevation which is present during the whole following compound time value, while G initiates a certain relaxation with a similar duration. Let us mention that these relaxations or elevations can function against those of the elementary rhythm: any compound time value begins with an elementary thesis.

The *simple rhythm* is formed by a pair of a simple arsis and a simple thesis:

(8) $$R_S \rightarrow A_S T_S .$$

2.3. Composed rhythm

We could be in a temptation to state that any instance of musical or linguistical creation we consider is a sequence of simple rythms (such a statement would be an analogy to that we have already formulated for the elementary and basic levels). In fact it does not hold: the simple arses and theses can form chains which form arses and theses on a higher level; moreover in these chains there can be present also whole rhythms and thus one rhythm can be recursively composed of rhythms on a lower level which need not be simple. This is an affair which cannot be expressed by means of natural languages but which can be formalized by recursive means of phrase structure grammars.

We introduce a notion of *opened arsis*, which has not been known by specialists in music but which can serve us as a useful auxiliary mean:

(9) $$A_O \rightarrow A P ,$$

(10) $$A_O \rightarrow A A_O ,$$

(11) $$A_O \rightarrow R A_O .$$

These rules tell that an opened arsis is a sequence of arses (rule (10)) and rhythms
(rule (11)) which must be concluded by an arsis (not by a rhythm — rule (9)), after
which a symbol P takes place. That symbol corresponds to the sign of *pétasté* well
known in byzantine notation, where it indicates — similarly as P in our grammar —
the place of the greatest elevation effect in a rhythm. Let us mentioned that rule (9)
tells that a chain forming an arsis cannot be concluded by a thesis.

Similarly we introduce a notion of *opened thesis*:

$$(12) \qquad\qquad T_O \rightarrow GT,$$

$$(13) \qquad\qquad T_O \rightarrow T_O T,$$

$$(14) \qquad\qquad T_O \rightarrow T_O R.$$

These rules have symetrical properties if compared with those for the opened arsis:
any opened thesis is a sequence of theses (rule (13)) and rhythms (rule (14)) which
begins with symbol G followed by a thesis (rule (12)). The symbol G has been intro-
duced in rule (7) for simple rhythm; here it has a similar semantics, i.e. it introduces
a certain relaxation which durates during the whole chain forming the thesis. Let us
mention that rule (12) exclude any case that after G an arsis would be present.

The last 6 rules have used symbols A, T and R which correspond to the notions
of arsis, thesis and rhythm which are understood in their complete extent. We must
present rules for their forming. Those rules, presented further, expresses that any
rhythm is an ordered pair of *arsis* and *thesis* and arses and theses rise from opened
arses and theses respectively so that we "close" them. The opened arses must be closed
at their left hand sides by a symbol F: it has been introduced in rule (6) and its
semantics is generalized for the whole chain forming the arsis, similarly as it has been
alreday mentioned for the symbol G in case of thesis. Rule (15) expresses that closing:

$$(15) \qquad\qquad A \rightarrow FA_O.$$

Similarly any thesis is obtained from an opened thesis by closing it at its right hand
side by a symbol M. It has not been introduced; it means the point of the greatest
size of relaxation and we have used a letter M for it because its syntactical function
ressembles that of *membre* (or similarly of *incisus* or *phrase*) *pause* (see e.g. [2],
p. 90 and p. 108 – 116).

$$(16) \qquad\qquad T \rightarrow T_O M.$$

Rules (15) and (16) must be completed by rules which would allow to form the most
simple arses and theses: they are simple ones closed similarly as the other arses and
theses.

$$(17) \qquad\qquad A \rightarrow A_S P,$$

$$(18) \qquad\qquad T \rightarrow T_S M.$$

The last rule defines the forming of *rhythm*:

$$(19) \qquad\qquad R \to AT.$$

It tells that any rhythm is an ordered pair of arsis and thesis. Let us mention that the notion of rhythm which has been just introduced is not a direct generalization of elementary rhythm or simple one. The most simple rhythm defined by rule (19) is the following one:

$$FIDHDPGIDHDM$$

Its arsis $FIDHDP$ and thesis $GIDHDM$ have risen from simple arsis $FIDHD$ and a simple thesis $GIDHD$ by closing (adding P and M respectively). The corresponding simple arsis and thesis can form a corresponding simple rhythm

$$FIDHDGIDHD$$

which is not a rhythm according to the rule (19).

2.4. Phrases

Forming of more composed rhythms according to the rules of the last section can be repeated until forming a *phrase*. It is a rhythm which is sufficiently rich from the point of view of the author intention but which cannot form a component in forming more complicated rhythm (according to rules (11) and (14)) because such rhythms would be too long than they could be taken by any human as rhythmical entiers. Thus we can formulate a rule defining the notion of phrase which is identified by K:

$$(20) \qquad\qquad K \to R.$$

The rule must be completed by another one so that it would be possible to allow also elementary rhythm to form a phrase, because we meet with such instances of phrases in plainsong. Thus we formulate the last rule

$$(21) \qquad\qquad K \to R_E.$$

One can state that any composition of the array which we recognize is a sequence of phrases. Any phrase is a result of the highest rhythmical synthesis of the elementary time values while the ordering of phrases is no more an affair of rhythm but of that which is commonly called *musical* or *stylistical form*. It does not concern the interests and the method of the present paper.

Therefore we can conclude our derivation of the rules of the free rhythm system
and close their system into a phrase structure grammar:

Grammar of free rhythm is an ordered 4-tuplet $\Gamma = (V, V_1, K, S)$ where V is a set of terminal symbols which are namely D, I, F, G, H, P and M, where V_1 is a set of auxiliary symbols which are namely A_E, T_E, R_E, D_C, A_S, T_S, R_S, A_O, T_O, A, T, R and K, where $K \in V_1$ is the initial symbol and S is a set of production rules (1) to (21).

3. SIMPLIFICATION OF THE GRAMMAR

We have tried to transform all the rules known from the non-mathematical theory of free rhythm into production ones and for the legibility we have let them enter into the set S of rules. We can see that for any ordinary composition of a phrase the rule (8) is not necessary (it does not take place in the right hand side of any production rule) and thus we can omit it from S. Then we are able to omit symbol R_S from V_1.

Another simplification can be made by omitting rule (21). It can be admitted because any elementary rhythm forming a phrase can be completed to a simple rhythm (and then according to rules (17) and (18) to a rhythm) so that we put an elementary time value before that which forms its arsis, and we eventually put another elementary time value after that which forms its thesis (in case that the elementary thesis consists of one elementary time value — such a case takes place only in byzantine music while in the latine one it has not been met). So e.g. an elementary rhythm $HDIDD$ which forms a phrase (and which is thus not used as a component of a rhythm in a higher level) can be transforfed into $FIDHDPGIDHDM$ which satisfies rule (19) and which really differs from the original one only by the first occurence of D which can be considered as a quaver pause, i.e. which is not realized but silently felt (such a practice has been introduced also in many modern transcriptions of plainchant).

After omitting rule (21) from S we can omit also rules (1) to (4) from the same set and the corresponding symbols R_E, A_E and T_E from set V_1. Moreover we can omit also terminal symbols H and I from set V because their occurence (and thus the corresponding semantical function) can be determined from the context: if we omit all the occurences of both symbols from any chain of terminal symbols forming a phrase we can return them simply according to the following procedure: we read the chain from left to right; if we meet a configuration DD we replace it immediately by a configuration $IDHD$; if the following symbol is D we let it be present in the chain; in any case we repeat the test for occurence of DD at the right of the put configuration $IDHD$ or at the right of an eventual repeated occurence of D. The algorithm is exactly described at the end of the present paper.

If we omit H and I from V we must correspondingly modify rule (5) replacing it by 2 new rules:

(5a) $$D_C \rightarrow DD \,,$$

(5b) $$D_C \rightarrow DDD \,.$$

They tell that any compound time value is either a pair or a triplet of elementary time values.

Another simplification is caused by the fact that after introducing the notions of the simple rhythm in 2.2 we had to formulate special production rules for further synthesis of them (rules (17) and (18)); it is possible to express another set of production rules so that they would be common for simple arses and theses (closed naturally by all necessary symbols F, P, G and M) and for those composed at a higher level according to the rules of 2.3. Therefore we omit rules (6), (7), (17) and (18) and we add two new rules; they should logically complete the triplets of rules (9) to (11) and (12) to (14) respectively:

(22) $$A_O \rightarrow D_C P \,,$$

(23) $$T_O \rightarrow GD_C \,.$$

They introduce partial cases of opened arses and theses, which could be called opened simple arsis and thesis and which can be closed by the same way as the other types of opened arses and theses using rules (15) and (16).

The last simplification consists in the fact that after omitting rule (21) there is only one rule with K, namely rule (20). It can be omitted, if R becames the initial symbol of the grammar. It reflects the fact that every rhythm can be a phrase.

Now we can present the modified grammar $\Gamma^* = (V^*, V_1^*, R, S^*)$ which is more simple than Γ but which offer an equivalent rhythmical synthesis: $V^* = (D, F, G, P, M)$, $V^* = (D_C, A_O, T_O, A, T, R)$ and S^* is the set of the following production rules:

(5a) $$D_C \rightarrow DD \,,$$

(5b) $$D_C \rightarrow DDD \,,$$

(22) $$A_O \rightarrow D_C P \,,$$

(9) $$A_O \rightarrow AP \,,$$

(10) $$A_O \rightarrow AA_O \,,$$

(11) $$A_O \rightarrow RA_O \,,$$

(23) $$T_O \rightarrow GD_C \,,$$

$$(12) \qquad\qquad T_0 \rightarrow GT\,,$$

$$(13) \qquad\qquad T_0 \rightarrow T_0 T\,,$$

$$(14) \qquad\qquad T_0 \rightarrow T_0 R\,,$$

$$(15) \qquad\qquad A \rightarrow F A_0\,,$$

$$(16) \qquad\qquad T \rightarrow T_0 M\,,$$

$$(19) \qquad\qquad R \rightarrow AT\,.$$

4. EXTENSION OF THE GRAMMAR

The presented grammars have been constructed according to the theoretical informations about the free rhythm system. Sometimes we meet examples detailly prepared by the same theoreticians, which follow more general rules.. It would be useful to extend the grammar so that it could reflect also those rules, which have not been explicitly formulated but which have been implicitly felt when realizing the considered type of music.

As an illustration we present an extension which rises from the example presented in [6], page XIV. The example contains a short phrase with prepared arses and theses of all levels. They have been mapped by usual signs for crescendo (arses) and decrescendo (theses), and the points where any thesis is joined with the arsis of the corresponding rhythm are marked by a sign of accent (it corresponds to our symbol P or G). The example is presented in Fig. 1 (the text of the example, the signs concerning

Fig. 1.

elementary rhythm and some historical names of certain rhythms have been omitted from it as they have no relation to the matter of the present paper). We can see that there are two places in the example which are not consistent with the theory: they have been identified by letters A and B under the notation; in both places a rhythm begins which has "empty" arsis. Such an affair has not been admitted by the theory but it can be simply felt by musicians as a certain discontinuity of the rhythmic

flow. It can be simply introduced into the presented mathematical theory; the set S^* of the production rules of the grammar Γ^* is to be enlarged by one rule, namely

(15a) $$A \to FP.$$

Then the presented example can be simply transformed to the following string:

$$FDDPGGGGDDMGDDMMFPGGDDMFDDPGDDMGDDDMMMFFPGDD -$$
$$MFDDDPPGGDDMGDDMMM$$

The symbols D correspond to the quavers (elementary time values) of the original notation; let us mention that any crotchet must be replaced by 2 instances of D.

5. CONCLUSIONS

We have presented certain tools which can exactly reflect rhythmical laws of ancient music and of related linguistical creations. We do not intend them to replace usual notation in order to enable more exact definition of any relalization, because the texts generated by the corresponding production rules are very complicated, as we could see. The mathematical theory can serve more for research than for practice. Nevertheless one can try to simplify the presented rules in order to get more legible sequences of terminal symbols. The other way to develope the presented theory is to include other rules of free rhythm which have been implicitly felt but which have not been built into non-mathematical theories of free rhythm: such considerations would serve for direct enriching of musical science.

APPENDIX

We present an algorithm which completes texts generated by grammar Γ^* by symbols H and I and so it transforms them to texts generated by grammar Γ. In other words, the algorithm determines the elementary rhythm according to the rhythm of a higher level. The algorithm is written as a program in ALGOL 68 (see [7]); it reads a sequence of letters (a text generated by grammar Γ^*) which is concluded by a "sentinel", which is a symbol which is not included in V and which informs that the input sequence of symbols is finished. We use letter Z for it. The algorithm generates another sequence of symbols (a text generated by grammar Γ corresponding to the read text).

```
begin char x; bool p := false;
L  : read(x);
       if x ≠ "D" then go to M fi;
       if p then p := false; print ("IDHD") else p := true fi;
       go to L;
M : if p then p := false; print ("D") fi;
     print (x); if x ≠ "Z" then go to L f i
end;
```

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