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Algorithmical Complexity of Some Statistical Decision Processes II

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In part II of presented paper some applications and examples of concepts introduced in part I are given.

24. Complexity of statistical decision processes will be studied as follows.

First, we separate every decision process into parts I, II,III (cf. section 8). This parts are not written as algorithmical processes, but as the very detailed procedures, from which the description of corresponding algorithmical processes will be obvious.

In every decision process E defined below only a few operations from S_1 occur, hence, we do not write the complexity as F-tuple of numbers (where $F = F_0 + 1$ or $F = F(E)$), but we shall consider only operations occurring in E and the number of theirs executions in E. (Compare with D10, D13.)

25. To be able to study the asymptotic properties of the complexity of statistical decision processes (and its change connected with the simplifications Z12, Z13, Z23 of dependence structure of random variables $Y_1, Y_2, \ldots, Y_n$) when the number n of observed values increases, we should proceed as follows: for every part (I, II, III) and every level Z1, Z2, Z3 (of dependence) we find the corresponding algorithmical processes $E_{ij}(n)$ (where i = I, II, III, $j = 1, 2, 3$) and the corresponding complexities $M_{ij}(n)$ and then consider the (asymptotical) operation savings connected with the transition from $E_{ij}(m)$ to $E_{ik}(m)$ ($m = 1, 2, 3, \ldots, \langle j, k\rangle = \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle$). We shall proceed as follows but in theorems we shall talk only about the (asymptotical) operation savings connected with the simplification Zjk.

26. Considerations of this section are connected with section 5. In what follows only decision processes with two possible decisions are considered. Hence,

$$D = \{0, 1\}, \tag{20}$$

where 0 corresponds to the hypothesis, 1 to the alternative (cf. section 4). A set X (of states of the environment) has the two members,

(21) $$X = \{a, b\}.$$

A priori probability of $a \in X$ equals $p \in (0, 1)$, so that

(22) $$P(\{a\}) = p \in (0, 1)$$

and

(24) $$P(\{b\}) = 1 - p \in (0, 1)$$

The loss function is defined as follows:

(26) $$W(a, 0) = W(b, 1) = 0$$

(27) $$W(a, 1) = W(b, 0) = 1$$

hence, the correct decision to $a \in X$ is 0, the correct decision to $b \in X$ is 1.

The decision process runs as follows: by a chance mechanism determined by P_a (if $a \in X$ is true) or P_b (if $b \in X$ is true) we sample values $y_1, y_2, \ldots, y_n$ and observe them. The decision $d \in D$ is determined by the Bayesian decision function mentioned below.

If $y_1, \ldots, y_n$ are the observed values and if

(a) the probability distributions P_a, P_b are discrete, then

$$\delta(y_1, y_2, \ldots, y_n) = \begin{cases} 0 & \text{if} \quad p \,.\, P_a(y_1, \ldots, y_n) > (1 - p) \,.\, P_b(y_1, \ldots, y_n) \\ 1 & \text{otherwise} \end{cases}$$

(b) the probability distributions P_a, P_b are absolutely continuous (with respect to the Lebesgue measure on E_1^n) and p_a, p_b are corresponding densities, then

$$\delta(y_1, y_2, \ldots, y_n) = \begin{cases} 0 & \text{if} \quad p \,.\, p_a(y_1, \ldots, y_n) > (1 - p) \,.\, p_b(y_1, \ldots, y_n) \\ 1 & \text{otherwise} \end{cases}$$

To be able to study the change of complexity of the decision process connected with the simplification of dependence structure of $Y_1, \ldots, Y_n$ we suppose, that

(28) $$n \geqq 2.$$

A transition from an elementary complexity to a complexity is done by this way: we assume, that every realization of each operation $+$, $-$, $<$, $[\,]$ is equally difficult as another realization of each of these operations. Also we assume, that the realization of operations ., :, $.^2$ is equally difficult, the same being valid for realizations inv and det. For a detailed discussion on these problems cf. [3] and also the remark in section 15.

27. a) Let us consider random variables $Y_1, Y_2, \ldots, Y_n$ with a simultaneous normal distribution and a statistical decision process with the two possible decisions:

hypothesis (0): density of the probability distribution of $Y_1, \ldots, Y_n$ is $p_a(y_1, \ldots, y_n)$,

alternative (1): density of the probability distribution of $Y_1, \ldots, Y_n$ is $p_b(y_1, \ldots, y_n)$.

A priori probabilities $P(\{a\}), P(\{b\})$ are given by (22), (24), the loss function by (26), (27).

b) In our case the Bayesian decision function is defined as follows

$$\delta(y_1, \ldots, y_n) = \begin{cases} 0 & \text{if } \; p \,.\, p_a(y_1, \ldots, y_n) > (1 - p) \,.\, p_b(y_1, \ldots, y_n) \\ 1 & \text{otherwise} \end{cases}$$

The result 0 corresponds to the hypothesis, 1 to the alternative (cf. 26b).

c) The density of simultaneous normal distribution (of n random variables) $X_1, \ldots, X_n$ is determined by

$$p(y_1, \ldots, y_n) = \frac{(\det C)^{1/2}}{(2\pi)^{n/2}} \,.\, \exp\left\{-\tfrac{1}{2} \sum_{k,j=1}^{n} c_{kj} \,.\, (y_k - a_k)(y_j - a_j)\right\},$$

where

$$\begin{aligned} a_k &= MX_k\,, \\ b_{kj} &= M(X_k - a_k)(X_j - a_j) \\ C &= B^{-1} \\ (j, k &= 1, 2, \ldots, n)\,. \end{aligned}$$

d) We assume, that values of $a_i, a_i', b_{kj}, b_{kj}'$ $(i, j, k = 1, \ldots, n)$ are known. Here $a_i = MY$ and $b_{kj} = M(Y_k - a_k)(Y_j - a_j)$ if the distribution of $Y_1, \ldots, Y_n$ is determined by P_a, $a_i' = MY$; and $b_{kj}' = M(Y_k - a_k')(Y_j - a_j')$ if the distribution of $Y_1, \ldots, Y_n$ is determined by P_b.

28. Let us consider a general case Z1 of the dependence of $Y_1, \ldots, Y_n$. We denote

$$B = \langle b_{kj} \rangle_{k,j=1}^{n}\,, \quad B' = \langle b_{kj}' \rangle_{k,j=1}^{n}\,, \quad C = B^{-1}\,, \quad C' = B'^{-1}\,.$$

(The existence of B^{-1}, B'^{-1} is always assumed.)

a) From 27d, c can be derived, after simple calculations that

$$p \,.\, p_a(y_1, \ldots, y_n) > (1 - p) \,.\, p_b(y_1, \ldots, y_n)$$

holds iff

$$\text{(30)} \qquad \log\left[\left(\frac{p}{1 - p}\right)^2 \,.\, \left(\frac{\det C}{\det C'}\right)\right] > \sum_{k,j=1}^{n} c_{kj} \,.\, (y_k - a_k)(y_j - a_j) -$$

$$- \sum_{k,j=1}^{n} c'_{kj} \cdot (y_k - a'_k)(y_j - a'_j) .$$

If (30) holds, our decision is 0 (hypothesis), otherwise it is 1 (alternative).

b) Decision process runs as follows.

Part I: is omitted, because a_i, a'_i, b_{kj}, b'_{kj}, $(i, j, k = 1, 2, \ldots, n)$ are known.

Part II: we compute

$$C = B^{-1} , \quad C' = B'^{-1} ,$$

$$\det C , \quad \det C' ,$$

$$\log \left[\left(\frac{p}{1 - p} \right)^2 \cdot \frac{\det C}{\det C'} \right] .$$

Part III: we calculate sums

$$\sum_{k,j=1}^{n} c_{kj} \cdot (y_k - a_k)(y_j - a_j) ; \quad \sum_{k,j=1}^{n} c'_{kj} \cdot (y_k - a'_k) \cdot (y_j - a'_j) ;$$

if the inequality (30) holds, our decision is 0 (hypothesis), otherwise the decision is 1 (alternative).

c) The detailed procedure corresponding to our decision process will be as follows:

II. – we calculate

$$C = B^{-1} , \quad C' = B'^{-1} , \quad \det C , \quad \det C' , \quad \frac{\det C}{\det C'} ,$$

$$1 - p , \quad \frac{p}{1 - p} , \quad \left(\frac{p}{1 - p} \right)^2 ,$$

$$\left(\frac{p}{1 - p} \right)^2 \cdot \frac{\det C}{\det C'} , \quad \log \left[\left(\frac{p}{1 - p} \right)^2 \cdot \frac{\det C}{\det C'} \right]$$

and put the last value to the cell $(2n + 1)$ of memory.

Algorithmical process corresponding to this part of decision process can be easily written. Its elementary r-complexity can be derived from the following table

operation	–	.	:	$.^2$	log	inv	det	$U.$
number of executions	1	1	2	1	1	2	2	1

and its r-complexity from Table 1.

III. Having observed the values $y_1, y_2, \ldots, y_n$:

– we compute the differences $(y_k - a_k)$, $(y_k - a'_k)$ and inscribe it to the cells $(2k - 1)$, $2k$ of memory $(k = 1, 2, \ldots, n)$.

– we compute, step by step, products $c_{kj}(y_k - a_k)(y_j - a_j)$ and currently sum them; the sum is inscribed into the cell 1.

– we compute, step by step, products $c'_{kj}(y_k - a'_k)(y_j - a'_j)$ and currently sum them; the sum is inscribed into the cell 2.

– we subtract the content of cell 2 from the content of cell 1 and the result compare with the content of cell $2n + 1$ of memory.

An algorithmical process corresponding to this part of decision process can easily be written. Its elementary r-complexity can be derived from the following table

operation	$+$	$-$	$<$	$.$	$U.$
number of exec.	$2(n^2 - 1)$	$2n + 1$	1	$4n^2$	$2n + 1$

and its r-complexity from Table 1.

29. Let us consider the case, when random variables $Y_1, Y_2, \ldots, Y_n$ are statistically independent (case Z3). In this case hold:

$$p_a(y_1, \ldots, y_n) = \prod_{i=1}^{n} p_i(y_i)\,,$$

$$p_b(y_1, \ldots, y_n) = \prod_{i=1}^{n} p'_i(y_i)\,,$$

where

$$p_i(y_i) = \frac{1}{\sqrt{(2\pi b_i)}} \cdot \exp\left\{-\frac{1}{2}\frac{(y_i - a_i)^2}{b_i}\right\},$$

$$p'_i(y_1) = \frac{1}{\sqrt{(2\pi b_i)}} \cdot \exp\left\{-\frac{1}{2}\frac{(y_i - a'_i)^2}{b'_i}\right\},$$

$$b_i = b_{ii}\,, \quad b'_i = b'_{ii}\,;$$

and values a_i, b_{ii}, a'_i, b'_{ii} are defined in 27d.

a) It can be easily found that

$$p \cdot p_a(y_1, \ldots, y_n) > (1 - p)\, p_b(y_1, \ldots, y_n)$$

holds if

$$\text{(31)} \qquad \log\left[\left(\prod_{i=1}^{n} b_i \Big/ \prod_{i=1}^{n} b'_i\right) \cdot \left(\frac{1 - p}{p}\right)^2\right] < \sum_{i=1}^{n} \frac{(y_i - a_i)^2}{b_i} - \sum_{i=1}^{n} \frac{(y_i - a'_i)^2}{b'_i}\,.$$

 If (31) holds, our decision is 0, otherwise it is 1.

b) The statistical decision process will proceed as follows:

Part I is omitted, because a_i, a'_i, b_i, b'_i, p are known.

Part II. We compute:

$$\log \frac{\prod_{i=1}^{n} b_i}{\prod_{i=1}^{n} b'_i} \cdot \left(\frac{1-p}{p}\right)^2$$

and the result inscribe into the cell 1 of memory.

Part III. We compute sums

$$\sum_{i=1}^{n} \frac{(y_i - a_i)^2}{b_i}, \quad \sum_{i=1}^{n} \frac{(y_i - a'_i)^2}{b'_i}.$$

If the inequality (31) holds the decision is 0, otherwise 1.

c) Detailed procedure corresponding to statistical decision process in question will be as follows:

Part II. – we compute $b_1 . b_2, b_1 . b_2 . b_3, \ldots, \prod_{i=1}^{n} b_i$ and inscribe the last value to the cell 1 of memory.

– we compute $b'_1 . b'_2, b'_1 . b'_2 . b'_3, \ldots, \prod_{i=1}^{n} b'_i$,

– we compute $\prod_{i=1}^{n} b_i / \prod_{i=1}^{n} b'_i$ and inscribe the result to the cell 2.

– we compute $(1 - p)$, $(1 - p)/p$, $((1 - p)/p)^2$, multiply the content of the cell 1 of memory by $((1 - p)/p)^2$ and find the logarithm of this product; the value of this logarithm is inscribed to the cell 1.

The algorithmical process corresponding to the Part II of decision process can easily be found. Its elementary r-complexity can be derived from the following table.

operations	—	.	:	$.^2$	log	$U.$
number of exec.	1	$2n - 1$	2	1	1	1

and its r-complexity from the Table 1.

Part III. Having observed the values $y_1, y_2, \ldots, y_n$:

– we compute $y_i - a_i$, $(y_i - a_i)^2$, $(y_i - a_i)^2/b_i$, $(i = 1, 2, \ldots, n)$ and currently sum the values of $(y_i - a_i)^2/b_i$; the result of summation will be inscribed to the cell 1 of memory.

– we compute $y_i - a_i'$, $(y_i - a_i')^2$, $(y_i - a_i')^2/b_i'$, $(i = 1, 2, \ldots, n)$ and currently sum the values of $(y_i - a_i')^2/b_i'$; we subtract the result of summation from the contain of the cell 2 and inscribe the last result to the cell 2.

– we compare the content of the cells 1 and 2; if (31) is satisfied, the decision is 0, otherwise 1.

An algorithmical process corresponding to this part of decision process can easily be written. Its elementary r-complexity can be found from the table

operation	+	−	<	:	$.^2$	$U.$
number of exec.	$2(n-1)$	$2n-1$	1	$2n$	$2n$	1

and its r-complexity from the Table 1.

30. Now we consider the case Z2, when random variables $Y_1, Y_2, \ldots, Y_n$ are separated into $q \geqq 2$ independent blocks (cf. section 9). In such a case hold

$$p_a(y_1, \ldots, y_n) = \prod_{l=1}^{q} p_l(y_{i_0+\ldots+i_{l-1}+1}, \ldots, y_{i_0+\ldots+i_l}),$$

$$p_b(y_1, \ldots, y_n) = \prod_{l=1}^{q} p_l'(y_{i_0+\ldots+i_{l-1}+1}, \ldots, y_{i_0+\ldots+i_l}),$$

where

$$p_l(y_{i_0+\ldots+i_{l-1}}, \ldots, y_{i_0+\ldots+i_l}) =$$

$$= \frac{(\det C_l)^{1/2}}{(2\pi)^{i_l/2}} \exp\{-\tfrac{1}{2} \sum_{k,j=i_0+\ldots+i_{l-1}+1}^{i_0+i_1+\ldots+i_l} c_{kj}(y_k - a_k) \cdot (y_j - a_j)\},$$

$$p_l'(y_{i_0+\ldots+i_{l-1}+1}, \ldots, y_{i_0+\ldots+i_l}) =$$

$$= \frac{(\det C_l')^{1/2}}{(2\pi)^{i_l/2}} \exp\{-\tfrac{1}{2} \sum_{k,j=i_0+\ldots+i_{l-1}+1}^{i_0+\ldots+i_l} c_{kj}'(y_k - a_k') \cdot (y_j - a_j')\}$$

and

$$C_l = \langle c_{kj} \rangle_{k,j=i_0+\ldots+i_{l-1}+1}^{i_0+\ldots+i_l}$$

is the inverse matrix to the matrix

$$B_l = (b_{kj})_{k,j=i_0+\ldots+i_{l-1}+1}^{i_0+\ldots+i_l}$$

and

$$C_l' = \langle c_{kj}' \rangle_{k,j=i_0+\ldots+i_{l-1}+1}^{i_0+\ldots+i_l}$$

is the inverse matrix to

$$B'_l = \langle b'_{kj} \rangle_{k,j=i_0+\ldots+i_{l-1}+1}^{i_0+\ldots+i_l}$$

for $l = 1, 2, \ldots, q$.

a) It can be shown that

$$p \cdot p_a(y_1, \ldots, y_n) > (1 - p) \cdot p_b(y_1, \ldots, y_n)$$

holds iff

(32)

$$\log \left[\frac{\prod_{l=1}^{q} \det C_l}{\prod_{l=1}^{q} \det C'_l} \cdot \left(\frac{p}{1-p} \right)^2 \right] > \sum_{l=1}^{q} \sum_{k,j=i_0+\ldots+i_{l-1}+1}^{i_0+\ldots+i_l} c_{kj}(y_k - a_k) \cdot (y_k - a_k) -$$

$$- \sum_{l=1}^{q} \sum_{k,j=i_0+\ldots+i_{l-1}+1}^{i_0+\ldots+i_l} c'_{kj}(y_k - a'_k) \cdot (y_j - a'_j) .$$

b) The statistical decision process in question will proceed as follows:

Part I is omitted, because all parameters are known.

Part II: we compute

$$C_l = B_l^{-1}, \quad C'_l = B_l'^{-1}, \quad \det C_l, \quad \det C'_l, \quad l = (1, 2, \ldots, q)$$

and

$$\log \frac{\prod_{l=1}^{q} \det C_l}{\prod_{l=1}^{q} \det C'_l} \cdot \left(\frac{p}{1-p} \right)^2 .$$

Part III: we compute sums

$$\sum_{l=1}^{q} \sum_{k,j=i_0+\ldots+i_{l-1}+1}^{i_0+\ldots+i_l} c_{kj}(y_k - a_k) \cdot (y_j - a_j) ,$$

$$\sum_{l=1}^{q} \sum_{k,j=i_0+\ldots+i_{l-1}+1}^{i_0+\ldots+i_l} c'_{kj}(y_k - a'_k) \cdot (y_j - a'_j) ,$$

if inequality (32) holds the decision is 0, otherwise 1.

c) A detailed procedure of computation corresponding to the statistical decision process in question will be as follows.

– we compute $C_1 = B_1^{-1}$, det C_1; $C_2 = B_2^{-1}$, det C_2, det C_1 . det C_2; $C_3 = B_3^{-1}$, det C_3, det C_1 . det C_2 . det C_3; ...; $C_q = B_q^{-1}$, det C_q; $\prod_{l=1}^{q}$ det C_l; and the last value inscribe to the cell 1 of memory.

– we compute $C'_1 = B_1'^{-1}$, det C'_1; $C'_2 = B_2'^{-1}$; det C'_2; det C'_1 . det C'_2; ...; $C'_q = B_q'^{-1}$; det C'_q; $\prod_{l=1}^{q}$ det $C'_l = \prod$; the content of the cell 1 will be divided by $\prod$ and the result inscribed to the cell 1.

Algorithmical process corresponding to the Part III of the decision process can easily be found. Its elementary r-complexity can be derived from the following table

operation	—	.	:	$.^2$	log	inv	det	U.
number of exec.	1	$2q - 1$	2	1	1	$2q$	$2q$	1

and its r-complexity from the Table 1.

Part III. Having observed the values $y_1, \ldots, y_n$.

– we compute $y_i - a_i$, $y_i - a'_i$ and inscribe them to the cells $2i$, $2i + 1$ of memory ($i = 1, 2, \ldots, n$).

– we compute $c_{kj}(y_k - a_k) . (y_j - a_j)$ and currently sum the values of these products, $k, j = i_0 + \ldots + i_{l-1} + 1, \ldots, i_0 + \ldots + i_l$, $l = 1, \ldots, q$. We inscribe the result to the cell 2 of memory.

– we compute $c'_{kj}(y_k - a'_k) . (y_j - a'_j)$ and currently sum the values of these products; $k, j = i_0 + \ldots + i_{l-1} + 1, \ldots, i_0 + \ldots + i_l$; $l = 1, \ldots, q$. We subtract the result from the content of the cell 2 and inscribe the last computed value to the cell 2.

– we compare the content of the cells 1 and 2; if inequality (32) holds, the result of decision process in question is 0, otherwise it is 1.

An algorithmical process corresponding to this Part III of the decision process can easily be written. Its elementary r-complexity can be found from the table

operation	+	—	<	.	U.
number of exec.	$2 . \sum_{l=1}^{q} i_l^2 - 2$	$2n + 1$	1	$4 . \sum_{i=1}^{q} i_l^2$	$2n$

and its r-complexity from the following Table 1.

31. The results of sections 27–30 on r-complexity are summarized in Table 1.

Table 1.

operations	type of depend.	Z1	Z2	Z3	Z1	Z2	Z3
	Part	II	II	II	III	III	III
$+, -, <$		1	1	1	$2n^2 + 2n$	$2m + 2n$	$4n - 2$
$., :, .^2$		4	$2q + 2$	$2n + 1$	$4n^2$	$4m$	$4n$
log		1	1	1	0	0	0
inv, det		4	$4q$	0	0	0	0
$U.$		1	1	1	$2n + 1$	$2n$	1

m is given by

$$m = \sum_{l=1}^{q} i_l^2 .$$

Remark (cf. with section 9). From Table 1 can be easily seen that in Part II of decision process in question the simplification (of the type) Z12 results in an increasing number of operations (inv, det) and ($., :, .^2$). Hence, the complexity of Patr II of our decision process increases, too. It is caused by the assumption, that realization of operations inv and det to matrix of each rank has the same complexity (cf. section 18). It can be easily shown that if operations inv and det are resolved to operations $+, ., :$, then the corresponding complexity of Part II is not increasing when the simplification Z12 is used.

Lemma L1. If $x_1, \ldots, x_q$ are non-negative numbers and $x_1 + x_2 + \ldots + x_q = n$ for $n > 0$, then the sum $\sum_{l=1}^{q} x_i^2$ reaches its maximum iff $x_1 = x_2 = \ldots = x_q = n/q$. In this case

$$\sum_{l=1}^{q} x_i^2 = \frac{n^2}{q}$$

holds.

Theorem T1. Let us consider the Part III of decision process described in section 27 and simplification Z12 of dependence structure of random variables $Y_1, \ldots, Y_n$. Then

(a1) The greatest operation savings are obtained when random variables $Y_1, \ldots, Y_n$ are divided into blocks of the same length, i.e. when $i_1 = i_2 = \ldots = i_q$;

(a2) then the saving of operations ($+, -, <$) and operations ($., :, .^2$) is asymptotically characterized by the constant function $h(n) = q$.

(b) the operation saving caused by the separation of random variables $Y_1, \ldots, Y_n$ into independent blocks depend only on number of variables contained in every block and no on the way of separation of $Y_1, \ldots, Y_n$ into these blocks.

Proof. Validity of (a1) and (b) can be easily seen from L1 and Table 1. We denote by $m_1(n)$, $m_2(n)$ the number of operations $+, -, <$ and by $m_3(n)$, $m_4(n)$ the number of operations $., :, .^2$ realized in Part III when the types Z1, Z2 of dependence structure of $Y_1, \ldots, Y_n$ are assumed. It is sufficient to prove that

(33) $$\lim_{n\to\infty} \frac{m_1 n}{m_2 n} = \lim_{n\to\infty} \frac{m_3(n)}{m_4(n)} = q\,.$$

Random variables $Y_1, \ldots, Y_n$ are divided in q blocks and every block contains n/q random variables. Hence, $i_1 = i_2 = \ldots = i_q = n/q$ and $\sum_{l=1}^{q} l_l^2 = n^2/q$, so that $m_2(n) = 2\sum_{l=1}^{q} i_l^2 + 2n = 2n^2/q + 2n$, $m_4(n) = 4\sum_{l=1}^{q} i_l^2 = 4n^2/q$ (and $m_1(n) = 2n^2 + 2n$, $m_3(n) = 4n^2$) from which can be easily seen, that (33) holds.

Lemma L2. Let $q, n \in N^+$, $q \geqq 2$, $x_1, \ldots, x_q \in N^+$. If $\sum_{l=1}^{q} x_l = n$ and $\max\{x_1, \ldots \ldots, x_q\} = d$, then

(34) $$\sum_{l=1}^{q} x_l^2 \leqq {}_{[}n/d_{]} \,.\, d^2 + (q - {}_{[}n/d_{]})$$

and the equality holds iff ${}_{[}n/d_{]}$ naturals from $\langle x_1, \ldots, x_q \rangle$ equal d and the others equal 1.

Theorem T1(c). Let us consider the Part III of decision process described in 27 and simplification Z12 of dependence structure of $Y_1, \ldots, Y_n$. We denote

$$d = \max(i_1, \ldots, i_q)\,, \quad q' = {}_{[}n/d_{]}\,.$$

Then the operation savings is minimal if random variables $Y_1, \ldots, Y_n$ are divided into q' blocks by d variables and $n - q'$ blocks by 1 variable.

Proof. Validity of T1(c) can be easily seen from L2 and Table 1.

DISCRETE MARKOV PROCESS

32. Considerations of this section links to considerations of section 26.

a) In sections 32–36 we shall consider random variables $Y_1, \ldots, Y_n$ taking values $1, 2, \ldots, w$ (for $w \in N$, $w \geqq 2$).

(Statistical) dependence of $Y_1, \ldots, Y_n$ will be supposed to be of Markov type, i.e.,

$$P[Y_j = y_j \mid Y_{j-1} = Y_{i-1}, \ldots, Y_1 = y_1] = P[Y_j = y_j \mid Y_{j-1} = y_{j-1}]$$

for $y_1, \ldots, y_n \in \{1, 2, \ldots, w\}$, $j = 2, 3, \ldots, n$.

There are the two possible decisions:

hypothesis (0): random variables $Y_1, \ldots, Y_n$ are distributed with respect to the probability P_a;

alternative (1): random variables $Y_1, \ldots, Y_n$ are distributed with respect to the probability P_b.

A priori probabilities $P(\{a\})$, $P(\{b\})$ are given by (22), (24), the loss function by (26), (27).

We suppose that for $y, y' \in \{1, 2, \ldots, w\}$, $j = 2, 3, \ldots, n$ the probabilities

$$P_a[Y_j = y \mid Y_{i-1} = y'] ,$$
$$P_b[Y_i = y \mid Y_{i-1} = y'] ,$$
$$P_a[Y_1 = y] ,$$
$$P_b[Y_1 = y']$$

are known.

b) In the case mentioned above the Bayesian decision function δ is defined as follows:

$$\delta(y_1, \ldots, y_n) = \begin{cases} 0 & \text{if} \quad p \,.\, P_a[Y_n = y_n, \ldots, Y_1 = y_1] > (1 - p) \,. \\ & \quad .\, P_b[Y_n = y_n, \ldots, Y_1 = y_1] , \\ 1 & \text{otherwise} . \end{cases}$$

The value 0 of the decision function δ corresponds to the hypothesis, the value 1 of δ corresponds to the alternative.

The dependence among $Y_1, \ldots, Y_n$ is of Markov type, hence

$$P_a[Y_n = y_n, \ldots, Y_1 = y_1] = \prod_{j=2}^{n} P_a[Y_j = y_j \mid Y_{i-1} = y_{i-1}] \,.\, P_a[Y_1 = y_1] ,$$

$$P_b[Y_n = y_n, \ldots, Y_1 = y_1] = \prod_{j=2}^{n} P_b[Y_j = y_j \mid Y_{j-1} = y_{j-1}] \,.\, P_b[Y_1 = y_1] .$$

33. First, the case Z1 of statistical dependence of $Y_1, \ldots, Y_n$ (cf. section 9) is considered. In this case

$$p \,.\, P_a[Y_n = y_n, \ldots, Y_1 = y_1] > (1 - p) \,.\, P_b[Y_n = y_n, \ldots, Y_1 = y_1]$$

holds iff

(35) $$p \cdot \prod_{j=n,(n-1),\dots,2} P_a[Y_j = y_j \mid Y_{j-1} = y_{j-1}] \cdot P_a[Y_1 = y_1] >$$

$$> (1 - p) \prod_{j=n,(n-1),\dots,2} P_b[Y_j = y_j \mid Y_{j-1} = y_{j-1}] \cdot P_b[Y_1 = y_1] \,.$$

It is obvious from (35), that statistical decision process turns as follows:

Part II. Values of probabilities $P_a[Y_j = y \mid Y_{j-1} = y']$, $P_b[Y_j = y \mid Y_{j-1} = y']$, $(j = 2, 3, \dots, n)$, $P_a[Y_1 = y]$, $P_b[Y_1 = y']$ for $y, y' \in \{1, 2, \dots, w\}$ are inscribed to memory.

Part III. Having observed the values $y_1, \dots, y_n$, we compute

$$p \prod_{j=2}^{n} P_a[Y_j = y_j \mid Y_{j-1} = y_{j-1}] \cdot P_a[Y_1 = y_1]$$

$$(1 - p) \prod_{j=2}^{u} P_b[Y_j = y_j \mid Y_{i-1}] \cdot P_b[Y_1 = y_1]$$

and compare this two values. If (35) holds, the decision is 0, otherwise it is 1.

The preceding considerations obviously imply that $2(n - 1)$ matrices of rank $w \times w$ and two w-dimensional vectors must be inscribed to memory, hence in Part II $2(n - 1) w^2 + 2w$ cells of memory are occupied.

In Part III it is necessary to execute the multiplication operation $2n$ times, $<$ once, $-$ once.

The algorithmical processes corresponding to Parts II and III of the statistical decision process in question can be easily found. Its elementary r-complexity can be derived from the following table:

	Part II	Part III		
Operations	$U.$	$-$	$<$	$\cdot$
Number of exec.	$2(n - 1) w^2 + 2w$	1	1	$2n$

and r-complexity from the Table 2.

34. Now it is considered the case Z2, when random variables $Y_1, \dots, y_n$ are divided into $q \geqq 2$ independent blocks (cf. section 9).

In that case random variables $Y_{i_0 + i_1 + \dots + i_l}$ and $Y_{i_0 + i_1 + \dots i_l + 1}$ are independent for $l = 1, 2, \dots, q - 1$, hence,

$$P_a[Y_{i_0 + i_1 + \dots + i_l + 1} = y \mid Y_{i_0 + i_1 + \dots + i_l} = y'] = P_a[Y_{i_0 + i_1 + \dots + i_l + 1} = y] \,,$$

$$P_b[Y_{i_0 + i_1 + \dots + i_l + 1} = y \mid Y_{i_0 + i_1 + \dots + i_l} = y'] = P_b[Y_{i_0 + i_1 + \dots + i_l + 1} = y]$$

(for $l = 1, 2, \ldots, q - 1$). Hence, in the Part II of the decision process mentioned above $2(n - 1) - 2(q - 1)$ matrices of rank $w \times w$ and $2(q - 1) + 2$ w-dimensional vectors must be inscribed to memory, so in Part II $2(n - q)\, w^2 + 2qw$ cells of memory are occupied.

In Part III is necessary to execute the multiplication operation $2n$-times, $<$ once, $-$ once. The algorithmical process corresponding to Parts II and III of the statistical decision process can be easily described. Its elementary r-complexity can be found from the table

	Part II	Part III		
Operations	$U.$	$-$	$<$	$\cdot$
Number of exec.	$2(n - q)\, w^2 + 2qw$	1	1	$2n$

and its r-complexity from the Table 2.

35. Now is considered the case when random variables $Y_1, \ldots, Y_n$ are independent (Z3). It is obvious that Z3 is a special case of Z2, when $Y_1, \ldots, Y_n$ are separated in $q = n$ independent blocks. Hence, the r-complexity of Parts II, III of statistical decision process in question is characterized by the Table 2.

36. The results of sections 32–35 on r-complexity are summarized in Table 2.

Table 2.

Operations	Type of depend. Z1	Z2	Z3	Z1	Z2	Z3
	Part II	II	II	III	III	III
$\cdot, <$	0	0	0	2	2	2
$\cdot$	0	0	0	$2n$	$2n$	$2n$
$U.$	$2(n - 1)\, w^2 + 2w$	$2(n - q)\, w^2 + 2qw$	$2nw$	0	0	0

Complexity of Part III of decision process described in section 32 does not change when the dependence structure of $Y_1, \ldots, Y_n$ is simplified.

Theorem T2. Let us consider the Part II of the decision process described in section 32 and its r-complexity.

(a) In the type Z13 of simplification of dependence structure of random variables $Y_1, \ldots, Y_n$ the saving of operations $U.$ is asymptotically characterized by constant function $h(n) = w$ $(n = 1, 2, 3, \ldots)$.

(b) If random variables $Y_1, \ldots, Y_n$ are divided into $q = q(n)$ independent blocks and

(b1)

(36) $$\lim_{n\to\infty} \frac{q(n)}{n} = \beta,$$

then the saving of operations $U.$ is asymptotically characterized by constant function

$$h(n) = \frac{w}{w + (1 - w)\beta}.$$

(b2) function $q(n)$ is constant, then the saving of operations $U.$ is asymptotically characterized by constant function $h(n) = 1$.

Proof. (a) It is obvious that

$$\lim_{n\to\infty} \frac{2(n-1)w^2 + 2w}{2nw} = w,$$

which proves T2(a).

(b1) We have to show, that if (36) holds, then

$$\lim_{n\to\infty} \frac{2(n-1)w^2 + 2w}{2(n - q(n))w^2 + 2q(n)w} = \frac{w}{w + (1 - w)\beta};$$

this can be easily proved.

(b2) If $q(n)$ is constant, then $\lim_{n\to\infty} q(n)/n = 0$, hence, the proof of (b2) immediately follows from (b1).

DISCRETE PROCESSES WITH COMPLETE DEPENDENCE

37. Considerations of this paragraph links to considerations of section 26.

a) We consider the random variables $Y_1, \ldots, Y_n$ $(n \geqq 2)$ taking values $1, 2, \ldots, w$ $(w \in N^+, w \geqq 2)$ and the decision process with the two possible decisions:

hypothesis (0): random variables $Y_1, \ldots, Y_n$ are distributed with respect to the probability measure P_a;

alternative (1): random variables $Y_1, \ldots, Y_n$ are distributed with respect to the probability measure P_b; a priori probabilities $P[\{a\}]$, $P[\{b\}]$ are given by (22), (24), the loss function by (26), (27).

b) In that case the Bayesian decision function is defined as follows:

$$\delta(y_1, \ldots, y_n) = \begin{cases} 0 & \text{if } \; p \,.\, P_a(Y_n = y_n, \ldots, Y_1 = y_1) > (1 - p) \,. \\ & \quad .\, P_b(Y_n = y_n, \ldots, Y_1 = y_1) \\ 1 & \text{otherwise} \,. \end{cases}$$

We suppose, that for $y_1, y_2, \ldots, y_n \in \{1, 2, \ldots, w\}$; $j = 2, 3, \ldots, n$ the probabilities

$$P_a(Y_j = y_j \mid Y_{j-1} = y_{j-1}, \ldots, Y_1 = y_1),$$
$$P_b(Y_j = y_j \mid Y_{j-1} = y_{j-1}, \ldots, Y_1 = y_1),$$
$$P_a(Y_1 = y_1),$$
$$P_b(Y_1 = y_1)$$

are known. It is obvious, that

$$P_a(Y_n = y_n, \ldots, Y_1 = y_1) =$$
$$= \prod_{j=2}^{n} P_a[Y_i = y_i \mid Y_{i-1} = y_{i-1}, \ldots, Y_1 = y_1] \,.\, P_a[Y_1 = y_1],$$
$$P_b(Y_n = y_n, \ldots, Y_1 = y_1) =$$
$$= \prod_{j=2}^{n} P_b[Y_j = y_j \mid Y_{j-1} = y_{i-1}, \ldots, Y_1 = y_1] \,.\, P_b[Y_1 = y_1] \,.$$

38. First the case Z1 of statistical dependence of $Y_1, \ldots, Y_n$ (cf. section 9) is considered. In this case

$$p \,.\, P_a(Y_n = y_n, \ldots, Y_1 = y_1) > (1 - p) \,.\, P_b(Y_n = y_n, \ldots, Y_1 = y_1)$$

holds iff

$$(37) \qquad p \prod_{j=2}^{n} P_a[Y_j = y_j \mid Y_{j-1} = y_{i-1}, \ldots, Y_1 = y_1] \,.\, P_a[Y_1 = y_1] >$$
$$> (1 - p) \prod_{j=2}^{n} p_b[Y_j = y_j \mid Y_{j-1} = y_{j-1}, \ldots, y_1 = y_1] \,.\, P_b[Y_1 = y_1] \,.$$

From (37) can be easily seen that statistical decision process mentioned above turns as follows:

Part II. Values of $P_a[Y_j = y_j \mid Y_{j-1} = y_{i-1}, \ldots, Y_1 = y_1]$, $P_b[Y_j = y_j \mid Y_{j-1} = y_{j-1}, \ldots, Y_1 = y_1]$. $P_a[Y_1 = y_1]$, $P_b[Y_1 = y_1]$ for $y_1, \ldots, y_n \in \{1, 2, \ldots, w\}$, $j = 2, 3, \ldots, n$ are inscribed into memory.

Part III. Having observed the values of $y_1, \ldots, y_n$ we compute values of the expressions on the left and the right sides of the inequality (37). If (37) holds, then the decision is 0, otherwise it is 1.

The above considerations immediately imply, that it is necessary to inscribe

$$2(w^n + w^{n-1} + \ldots + w) = 2 \cdot \frac{w^{n+1} - 1}{w - 1} - 2$$

numbers into memory. In Part III it is necessary to apply the operation $-$ once, $<$ once and multiplication $2n$ times.

The algorithmical processes corresponding to Parts II, III of the decision process in question can be easily described. Its elementary r-complexity can be found from the table.

	Part II	Part III		
Operation	$U.$	$-$	$<$	$.$
Number of exec.	$2 \cdot \frac{w^{n+1} - 1}{w - 1} - 2$	1	1	$2n$

and r-complexity from the Table 3.

39. Now we consider the case Z2, when random variables $Y_1, \ldots, Y_n$ are divided into $q \geqq 2$ statistically independent blocks.

Let $l \in \{2, 3, \ldots, q\}$; $j \in \{i_0 + i_1 + \ldots + i_{l-1} + 1, \ldots, i_0 + i_1 + \ldots + i_l\}$. Then for $k = 1, 2, \ldots, i_0 + i_1 + \ldots + i_{l-1}$ random variables y_j, y_k are independent, so that

$$(38) \qquad P_a[Y_j = y_j \mid Y_{j-1} = y_{j-1}, \ldots, Y_1 = y_1] =$$

$$= P_a[Y_j = y_j \mid Y_{j-1} = y_{j-1}, \ldots, Y_{i_0+\ldots+i_{l-1}+1} = y_{i_0+\ldots+i_{l-1}+1}],$$

$$(39) \qquad P_b[Y_j = y_j \mid Y_{j-1} = y_{j-1}, \ldots, Y_1 = y_1] =$$

$$= P_b[Y_j = y_j \mid Y_{j-1} = y_{j-1}, \ldots, Y_{i_0+\ldots+i_{l-1}+1} = Y_{i_0+\ldots+i_{l-1}+1}].$$

((38), (39) holds also for $l = 1$.) Hence, in Part II it is necessary to inscribe into memory

$$2(w^{i_l} + w^{i_l-1} + \ldots + w)$$

numbers for $l = 1, 2, \ldots, q$, hence, in Part II

$$\frac{2}{w-1} \cdot \left(\sum_{l=1}^{q} w^{i_l+1} - wq\right)$$

cells of memory are occupied.

In Part III it is necessary to apply the operation $-$ once, multiplication $2n$ times, $<$ once.

The algorithmical processes corresponding to the Parts II, III can be easily written. Its elementary r-complexity can be derived from the table

	Part II	Part III		
Operations	$U.$	$-$	$<$	$.$
Number of exec.	$\frac{2}{w-1} \cdot (\sum_{l=1}^{q} w^{i_l+1} - wq)$	1	1	$2n$

and r-complexity from the Table 3.

40. Let us consider the case Z3, when random variables $Y_1, \ldots, Y_n$ are statistically independent. Z3 is a special case for Z2 when random variables $Y_1, \ldots, Y_n$ are divided into $q = n$ independent blocks. Hence, $i_1 = i_2 = \ldots = i_q = 1$ and

$$\frac{2}{w-1} \cdot (\sum_{l=1}^{q} w^{i_l+1} - wq) = 2wn\,,$$

so the r-complexity of Parts II, III of statistical decision process in question if characterized by the Table 3.

41. The results of sections 37–40 on r-complexity are summarized in Table 3.

Table 3.

Operations	Tape of dependence	Z1	Z2	Z3	Z1	Z2	Z3
	Part	II	II	II	III	III	III
$-, <$		0	0	0	2	2	2
$.$		0	0	0	$2n$	$2n$	$2n$
$U.$		m_1	m_2	$2wn$	0	0	0

m_1, m_2 are given by

$$m_1 = m_1(n) = 2 \cdot \frac{w^{n+1} - 1}{w - 1} - 2\,, \tag{40}$$

$$m_2 = \frac{2}{w-1} \cdot (\sum_{l=1}^{q} w^{i_l+1} - wq)\,. \tag{41}$$

The complexity of Part III of decision process described in section 37 does not changes when the dependence structure of $Y_1, \ldots, Y_n$ simplifies.

Lemma L3. Let $x_1, \ldots, x_q$ be non-negative real numbers, $\sum_{i=1}^{q} x_i = n$ (for $q, n \in N^+$), let $w > 0$.

a) Then $\sum_{i=1}^{q} w^{x_i+1}$ reaches its minimum iff

$$x_1 = x_2 = \ldots = x_q . \tag{42}$$

b) If (42) holds, then $x_1 = n/q$ and

$$\sum_{i=1}^{q} w^{x_i+1} = wq \, . \, w^{n/q} . \tag{43}$$

Theorem T3. Let us consider the Part II of the decision process described in section 37 and its r-complexity. Then

(a) the saving of operations U. corresponding to the simplification Z13 of dependence structure of $Y_1, \ldots, Y_n$ is asymptotically characterized by the function $h(n) = w^n/n$.

(b1) If random variables $Y_1, \ldots, Y_n$ are divided into q statistically independent blocks, then the saving of operations U. is maximal when $i_1 = i_2 = \ldots = i_q$.

(b2) In that case the saving of operations U. is asymptotically characterized by the function $h : N \to E_1$

$$h(n) = \frac{1}{q} \, . \, w^{n(1-1/q)} .$$

Proof. (a) We must prove that

$$\lim_{n\to\infty} \frac{m_1(n)}{2wn} \, . \left(\frac{w^n}{n}\right)^{-1} = 1 \tag{44}$$

where $m_1(n)$ is given by (41). To prove (44) is a matter of an easy calculation.

(b1) Validity of (b1) is implied by L3 and Table 3.

(b2) It is sufficient for proving (b2) to prove that if $i_1 = i_2 = \ldots = i_q = n/q$ then

$$\lim_{n\to\infty} \frac{m_1(n)}{m_2} \, . \, \frac{1}{\frac{1}{q} \, w^{n(1-1/q)}} = 1$$

where $m_1(n)$, m_2 are given by (41), (42). If $i_1 = i_2 = \ldots = i_q = n/q$ then $m_2 = (2/(w-1)) . \, wq \, . \, (w^{n/q} - 1)$, from which the validity of (44) can be easily proved.

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