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Michael Valášek

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DYNAMIC TIME PARAMETRIZATION OF MANIPULATOR TRAJECTORIES

MICHAEL VALÁŠEK

A fundamental time-scaling property of manipulator dynamics, a velocity profile scale modification procedure for bringing the movement along the given geometric trajectory within dynamic and actuating realizability, velocity profile approximating procedure on the given path, dynamically realizable velocity profile synthesis problem, nonlinear time scaling and its simple approximative solution have been developed. These procedures allow to use actuator torque limits and inverse dynamics for exact and efficient trajectory planning without repetitive dynamics recalculations.

1. INTRODUCTION

An intensive use of knowledge about actuator torque limits and inverse dynamics for efficient manipulator trajectory planning algorithms has been till now very rare. One past approach has used fixed velocity or acceleration limits of the joints [1], [2] what is at best a very coarse approximation of the true influence of actuator torque limits on movement speed. The other one has used the exact method of optimal control [3], [4], [5] what requires either iterations and in advance non-anticipated computational costs or again an approximation and substantial computational costs by dynamic programming in [3]. Only a promising algorithm of suboptimal approximations is developed in [6], [12].

There has been developed a fundamental time-scaling property of manipulator dynamics [7], [8], [12] that allows trajectory planning with exact and efficient use of actuator torque limits and inverse dynamics. The dynamic realizability of a proposed trajectory can be easy determined and a simple algorithm to modify the scale of the velocity profile can be applied in order to obtain the proposed trajectory to be realizable.

In the first part of this paper we describe the fundamental time-scaling property of manipulator dynamics for nonlinear time-parametrization of a geometric trajectory and the construction of a dynamically acceptable approximation of the proposed velocity profile on a geometric trajectory. A simple trajectory planning procedure by compounding dynamically acceptable velocity profiles on the given geometric trajectory is included.

In the second part of this paper we formulate the problem of dynamically acceptable velocity profile synthesis on the given geometric trajectory (i.e. also the problem of nonlinear time scaling formulated in [7]) for instance in the case when a proposed shape of velocity profile cannot be realized because of actuator torque limits violation by any scale change. A simple efficient solution of approximation of this problem is included.

2. TIME PARAMETRIZATION OF MANIPULATOR GEOMETRIC TRAJECTORIES

Suppose that we have a desired manipulator geometric trajectory

$$\mathbf{R} = \mathbf{R}(p) \tag{1}$$

where generally $\mathbf{R} = [R_i, O_i]^{\mathrm{T}}$, $[R_i]^{\mathrm{T}}$ is the radius vector of points on manipulator geometric trajectory in Cartesian coordinates, $[O_i]^{\mathrm{T}}$ is the manipulator gripper orientation in cartesian space and p is a geometric parameter

$$0 \leqq p \leqq p_{\max} . \tag{2}$$

We distinguish the geometric trajectory (or path) that is the pure shape of a geometric path in space, for example straight-line, circular arc, parabolic arc etc., and the trajectory that is the complete time behaviour of movement on the path in space containing information about velocities and accelerations for example straight-line movement with constant velocity, with constant acceleration, parabolic arc movement with deceleration by linear jerk etc.

Between geometric trajectory variables (1) and the manipulator joint variables $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_n]^{\mathrm{T}}$ the kinematic equations

$$\mathbf{R} = \mathbf{K}(\boldsymbol{\theta}) \tag{3}$$

are valid. We suppose that there exists an inverse kinematic solution for (3)

$$\boldsymbol{\theta} = \mathbf{K}^{-1}(\mathbf{R}) . \tag{4}$$

Therefore (1) includes together with radius vector the gripper orientation. The relations (1) and (4) can be however easily and without lost of generality replaced by the trajectory plan

$$\boldsymbol{\theta} = \boldsymbol{\theta}(p) \tag{5}$$

and further we can continue in the same way. In case that (5) cannot be obtained, e.g. we have redundant manipulator degrees of freedom and we cannot or we do not want to use the methods from Appendix 2, the solution is very difficult and probably can be achieved only by exact use of optimal control (cf. [6], [12]).

Further we have a requested velocity profile along the geometric trajectory

$$v = v(p) \tag{6}$$

where v is the magnitude of velocity vector along the path incuding the sign, but the velocity does not change its sign (the movement doesn't reverse).

Relation (6) is very useful for further consideration but it cannot be easily obtained; though we often need only its values in selected points. Either we choose velocity profile directy in the form (6) or we choose time dependence of trajectory length variable s

$$s = s(t)\,, \tag{7}$$

where according to the definition

$$s = \int_0^p \left| \frac{\mathrm{d}[R_i]^{\mathrm{T}}}{\mathrm{d}p} \right| \mathrm{d}p = s(p) \tag{8}$$

and

$$v = \frac{\mathrm{d}s(t)}{\mathrm{d}t} = v(t) \tag{9}$$

Now if we know the inverse function to (8) we can generate function (7) according to time, from it by the inverse function to (8) find p and to it from (9) v. If we know the inverse function to (7) we can start with p, from (8) find s, from it by the inverse function to (7) obtain the time t and from (9) finally v.

The dynamic equations of motion can be compactly written (cf. [9], [10])

$$\boldsymbol{n} = \boldsymbol{I}(\boldsymbol{\theta})\,\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{\theta}}^{\mathrm{T}}\,\boldsymbol{C}(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}} + \boldsymbol{V}(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{\theta}) \tag{10}$$

where

$\boldsymbol{n} = [n_1, \ldots, n_n]^{\mathrm{T}}$ is the n-dimensional vector of joint torques corresponding to the movement point,

$\boldsymbol{I}(\theta)$ is the (n, n) generalized inertia tensor,

$\boldsymbol{C}(\theta)$ is the (n, n, n) generalized tensor in the formulation of the Coriolis and centrifugal forces,

$\boldsymbol{V}(\theta)$ is the (n, n) generalized tensor in the formulation of the viscous friction forces,

$\boldsymbol{g}(\theta)$ is the n-dimensional position-dependent vector of gravity forces.

The manipulator movement realizability is restricted by actuator torque limits that can be generally position – and velocity-dependent. For example, for the most often DC electric motors with permanent magnets the restrictions have the form

$$\boldsymbol{n}^- - \boldsymbol{F}(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}} \leqq \boldsymbol{n} \leqq \boldsymbol{n}^+ - \boldsymbol{F}(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}} \tag{11}$$

where

$\boldsymbol{n}^+ = [n_1^+, n_2^+, \ldots, n_n^+]^{\mathrm{T}}$ and $\boldsymbol{n}^- = [n_1^-, n_2^-, \ldots, n_n^-]^{\mathrm{T}}$ are maximum and minimum constant torque limits,

$\boldsymbol{F}(\boldsymbol{\theta})$ is the (n, n) generalized tensor in the formulation of velocity actuator torque dependency such as for the back EMF of electric motors.

The generalization of (11) is described in Appendix 1.

We will synthetise dynamically realizable movement along the requested geometric trajectory as the time-parametrization of a geometric parameter p

$$p = p(t)\,. \tag{12}$$

Differentiating from (1)

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}t}\,. \tag{13}$$

Restricting $\boldsymbol{R}$ on $[R_i]^{\mathrm{T}}$ from (13)

$$\frac{\mathrm{d}[R_i]^{\mathrm{T}}}{\mathrm{d}t} = \frac{\mathrm{d}[R_i]^{\mathrm{T}}}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}t}\,. \tag{14}$$

But

$$\left|\frac{\mathrm{d}[R_i]^{\mathrm{T}}}{\mathrm{d}t}\right| = |v(p)| \tag{15}$$

and

$$\left|\frac{\mathrm{d}p}{\mathrm{d}t}\right| = \frac{\left|\frac{\mathrm{d}[R_i]^{\mathrm{T}}}{\mathrm{d}t}\right|}{\left|\frac{\mathrm{d}[R_i]^{\mathrm{T}}}{\mathrm{d}p}\right|} = \frac{|v(p)|}{\left|\frac{\mathrm{d}[R_i]^{\mathrm{T}}}{\mathrm{d}p}\right|} = |f(p)|\,. \tag{16}$$

By considering the movement sign in (1) and (6) we can determine (16) in such way that it is valid (including the sign)

$$\frac{\mathrm{d}p}{\mathrm{d}t} = f(p)\,. \tag{17}$$

Because of easy computation of dynamically synthetised trajectory we will look for an approximation of (6) by supposing (12) in the form of a polynomial

$$p = P_0 + P_1 t + P_2 t^2 + \ldots = \sum_{i=0}^{N} P_i t^i\,. \tag{18}$$

An approximation is necessary even in case that (6) is directly a polynomial because $|\mathrm{d}[R_i]^{\mathrm{T}}/\mathrm{d}p|$ can be very complex function. Because of validity of (16) we must suppose that $|\mathrm{d}[R_i]^{\mathrm{T}}/\mathrm{d}p|$ is nonzero, otherwise (1) is not a satisfactory geometric parametrization of the manipulator trajectory that cannot be used for manipulator movement synthesis. The approximation can however be as accurate as necessary by increasing the degree N of polynomial (18) or by dividing the interval (2) in to more subintervals and using (18) in each of them (spline approximation).

For determining the approximation of (6) we use information about (17) such as points (p_i, f_i) on (17), i.e.

$$p_i = \sum_{k=0}^{N} P_k t_i^k = p(t)|_{t=t_i} \tag{19}$$

$$f_i = \sum_{k=0}^{N} k P_k t_i^{k-1} = \left.\frac{\mathrm{d}p(t)}{\mathrm{d}t}\right|_{t=t_i}$$

or the derivatives in points of (17) because from (17)

$$\frac{d^2p}{dt^2} = \frac{d}{dt}(f(p)) = \frac{df}{dp}\frac{dp}{dt} = \frac{df}{dp}f \tag{20}$$

$$e = \frac{df}{dp}$$

and

$$q = \frac{d^2p}{dt^2}$$

$$p_i = \sum_{k=0}^{N} P_k t_i^k = p(t)|_{t=t_i} \tag{21}$$

$$q_i = \sum_{k=0}^{N} k(k-1)\, P_k t_i^{k-2} = f_i e_i = \frac{df}{dp} f|_{p=p_i} = \left.\frac{d^2p(t)}{dt^2}\right|_{t=t_i}$$

Both (19) and (21) are valid for some unknown time point t_i which is to be determined.

Other procedure to obtain either separate values dp/dt and d^2p/dt^2 for (19) and (21) or by using the inverse function of (7) all dependences (17) and (20) is the following. From (8)

$$\frac{ds}{dt} = \frac{ds}{dp}\frac{dp}{dt} \tag{22}$$

and

$$\frac{dp}{dt} = \frac{\frac{ds}{dt}}{\frac{ds}{dp}} = \frac{\frac{ds}{dt}}{\left|\frac{d[R_i]^T}{dp}\right|} \tag{23}$$

and ds/dt is from (7). Further from (22)

$$\frac{d^2s}{dt^2} = \frac{d^2s}{dp^2}\left(\frac{dp}{dt}\right)^2 + \frac{ds}{dp}\frac{d^2p}{dt^2} \tag{24}$$

and

$$\frac{d^2p}{dt^2} = \frac{\frac{d^2s}{dt^2} - \frac{d^2s}{dp^2}\left(\frac{dp}{dt}\right)^2}{\frac{ds}{dp}} \tag{25}$$

where dp/dt is from (23), d^2s/dt^2 from (7), $ds/dp = |d[R_i]^T/dp|$ and $d^2s/dp^2 = d/dp|d[R_i]^T/dp|$.

Now we transform the time-parametrization (12), resp. (18), by the following transformation with c = constant $\neq 0$ that scales the velocity profile

$$t' = t/c\,, \quad f_i' = cf_i\,, \quad e_i' = ce_i\,, \quad q_i' = c^2 q_i\,. \tag{26}$$

From (19) and (21) it is obvious that this transformation is satisfied by

(27) $$P'_k = c^k P_k .$$

But now from (27) follows

(28) $$p' = \sum_{k=0}^{N} P'_k t'^k = \sum_{k=0}^{N} c^k P_k \left(\frac{t}{c}\right)^k = \sum_{k=0}^{N} P_k t^k = p$$

$$\frac{\mathrm{d}p'}{\mathrm{d}t'} = \sum_{k=0}^{N} k P'_k t'^{k-1} = \sum_{k=0}^{N} k c^k P_k \left(\frac{t}{c}\right)^{k-1} = \sum_{k=0}^{N} c k P_k t^{k-1} = c \frac{\mathrm{d}p}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2 p'}{\mathrm{d}t'^2} = \sum_{k=0}^{N} k(k-1) P'_k t'^{k-2} = \sum_{k=0}^{N} k(k-1) c^k P_k \left(\frac{t}{c}\right)^{k-2} = \sum_{k=0}^{N} c^2 k(k-1) P_k t^{k-2} = c^2 \frac{\mathrm{d}^2 p}{\mathrm{d}t^2}$$

and from this and (1) further

(29) $$\boldsymbol{R}' = \boldsymbol{R}$$

$$\frac{\mathrm{d}\boldsymbol{R}'}{\mathrm{d}t'} = \frac{\mathrm{d}\boldsymbol{R}'}{\mathrm{d}p'}\frac{\mathrm{d}p'}{\mathrm{d}t'} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}p} c \frac{\mathrm{d}p}{\mathrm{d}t} = c \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2\boldsymbol{R}'}{\mathrm{d}t'^2} = \frac{\mathrm{d}^2\boldsymbol{R}'}{\mathrm{d}p'^2}\left(\frac{\mathrm{d}p'}{\mathrm{d}t'}\right)^2 + \frac{\mathrm{d}\boldsymbol{R}'}{\mathrm{d}p'}\frac{\mathrm{d}^2 p'}{\mathrm{d}t'^2} = \frac{\mathrm{d}^2\boldsymbol{R}}{\mathrm{d}p^2}\left(c\frac{\mathrm{d}p}{\mathrm{d}t}\right)^2 + \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}p} c^2 \frac{\mathrm{d}^2 p}{\mathrm{d}t^2} = c^2 \frac{\mathrm{d}^2\boldsymbol{R}}{\mathrm{d}t^2}$$

and finally from (4)

(30) $$\boldsymbol{\theta}' = \boldsymbol{\theta}$$

$$\frac{\mathrm{d}\boldsymbol{\theta}'}{\mathrm{d}t'} = \frac{\mathrm{d}\boldsymbol{K}^{-1}(\boldsymbol{R}')}{\mathrm{d}p'}\frac{\mathrm{d}p'}{\mathrm{d}t'} = c\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2\boldsymbol{\theta}'}{\mathrm{d}t'^2} = \frac{\mathrm{d}^2\boldsymbol{K}^{-1}(\boldsymbol{R}')}{\mathrm{d}p'^2}\left(\frac{\mathrm{d}p'}{\mathrm{d}t'}\right)^2 + \frac{\mathrm{d}\boldsymbol{K}^{-1}(\boldsymbol{R}')}{\mathrm{d}p'}\frac{\mathrm{d}^2 p'}{\mathrm{d}t'^2} = c^2\frac{\mathrm{d}^2\boldsymbol{\theta}}{\mathrm{d}t^2} .$$

Using (30) in (10) and (11) we obtain

(31) $$\boldsymbol{n}' = \boldsymbol{I}(\boldsymbol{\theta}')\ddot{\boldsymbol{\theta}}' + \dot{\boldsymbol{\theta}}'^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{\theta}')\dot{\boldsymbol{\theta}}' + \boldsymbol{V}(\boldsymbol{\theta}')\dot{\boldsymbol{\theta}}' + \boldsymbol{g}(\boldsymbol{\theta}')$$

$$\boldsymbol{n}' = c^2(\boldsymbol{I}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{\theta}}_p\boldsymbol{C}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}) + c(\boldsymbol{V}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}) + \boldsymbol{g}(\dot{\boldsymbol{\theta}})$$

and

(32) $$\boldsymbol{n}^- - \boldsymbol{F}(\boldsymbol{\theta}')\dot{\boldsymbol{\theta}}' \leqq \boldsymbol{n}' \leqq \boldsymbol{n}^+ - \boldsymbol{F}(\boldsymbol{\theta}')\dot{\boldsymbol{\theta}}'$$

$$\boldsymbol{n}^- - c(\boldsymbol{F}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}) \leqq \boldsymbol{n}' \leqq \boldsymbol{n}^+ - c(\boldsymbol{F}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}) .$$

Analogically as in [7] we designate by

(33) $$\boldsymbol{n}'_a = c^2(\boldsymbol{I}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}) = c^2\boldsymbol{n}_a$$

$$\boldsymbol{n}'_b = c(\boldsymbol{V}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \boldsymbol{F}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}) = c\boldsymbol{n}_b$$

(34) $$\boldsymbol{n}_c^{+\prime} = \boldsymbol{n}^+ - \boldsymbol{g}(\boldsymbol{\theta}) = \boldsymbol{n}_c^+$$

$$\boldsymbol{n}_c^{-\prime} = \boldsymbol{n}^- - \boldsymbol{g}(\boldsymbol{\theta}) = \boldsymbol{n}_c^-$$

the forces $\boldsymbol{n}_a$ dependent on accelerations and multiplications of velocities, the forces $\boldsymbol{n}_b$ dependent on velocity and the efficient torque limits $\boldsymbol{n}_c^+$, $\boldsymbol{n}_c^-$ dependent only on position. Restriction by actuator torque limits can be expressed

$$n_c^- \leqq c^2 n_a + c n_b \leqq n_c^+ \tag{35}$$

where particular members n_a, n_b, n_c^+, n_c^- are only position-dependent for the given trajectory plan (5) and determined time-parametrization (12).

3. SCALING OF TIME PARAMETRIZATION OF GEOMETRIC TRAJECTORIES TO SATISFY TORQUE LIMITATIONS

How fast or slow a manipulator can move along a trajectory is restricted by actuator torque limits. In order to determine the violation of the torque limits, the inverse dynamics must be solved and the computed needed torques compared to their limits. From these comparisons we are looking for the constant c in the transformation (26) that brings the motion within the actuator torque limits.

First we must determine the approximation (18) for the velocity profile (6) and trajectory plan (5). We describe how to determine the polynomial (18) for $N \leqq 4$ from the conditions (19) and (21).

Case $N = 1$. We can choose only one velocity value in every interval

$$p = 0 \quad \frac{\mathrm{d}p}{\mathrm{d}t} = f = f_0 \quad \text{for} \quad t = 0 \, .$$

We obtain (cf. Fig. 1a)

$$p = f_0 t \tag{36}$$

The velocity profile is completely determined by the path parametrization (1).

Case $N = 2$. A very important case because this approximation enables to obtain

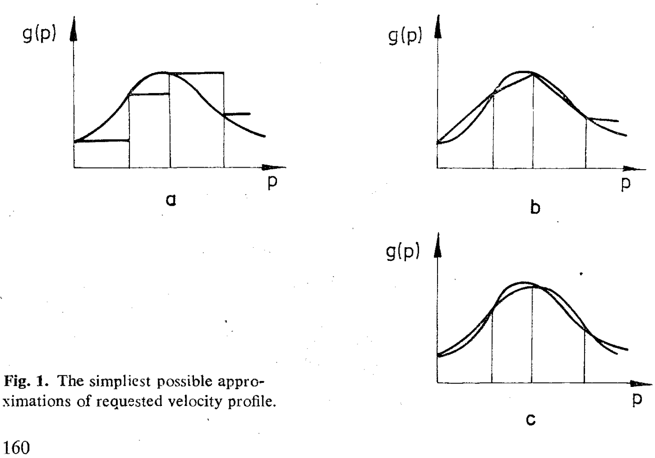

Fig. 1. The simpliest possible approximations of requested velocity profile.

acceleration/deceleration (generally the change of velocity) that is different from the velocity profile implicitely involved in (1) and on the top of it by this simple approximation we do not need to know the function (6) (at most only two values). We can request the velocity at the start and at the end of the interval

$$p = 0\,, \quad \frac{dp}{dt} = f = f_1 \quad \text{for} \quad t_1 = 0$$

$$p = p_{\max}\,, \quad \frac{dp}{dt} = f = f_2 \quad \text{for} \quad t_2 = T = ?$$

and we receive (cf. Fig. 1b)

$$p = f_1 t + \frac{f_2^2 - f_1^2}{4p_{\max}}\, t^2 \tag{37}$$

$$0 \leqq t \leqq T = \frac{2p_{\max}}{f_1 + f_2}\,.$$

Case $N = 3$. We can choose three points in each of the intervals (for example at the start, at the end and between these) and we receive a system of nonlinear equations that can be solved but not so easy, or we can choose two points and one derivative what is however asymmetric and so we use the case $N = 4$, or we can choose two points and the time length T.

Case $N = 4$. We can choose two points and two derivatives at the start and at the end of the interval

$$\left.\begin{aligned} p_1 &= 0\,, \quad \frac{dp}{dt} = f = f_1 \\ p_1 &= 0\,, \quad \frac{d^2p}{dt^2} = q = q_1 \end{aligned}\right\} \quad \text{for} \quad t_1 = 0$$

$$\left.\begin{aligned} p_2 &= p_{\max}\,, \quad \frac{dp}{dt} = f = f_2 \\ p_2 &= p_{\max}\,, \quad \frac{d^2p}{dt^2} = q = q_2 \end{aligned}\right\} \quad \text{for} \quad t_2 = T = ?$$

and we receive (cf. Fig. 1c)

$$p = f_1 t + \frac{q_1}{2}\, t^2 + \left(\frac{f_2 - f_1 - q_1 T}{T^2} - \frac{q_2 - q_1}{3T}\right) t^3 + \left(\frac{q_2 - q_1}{4T^2} - \frac{f_2 - f_1 - q_1 T}{2T^3}\right) t^4 \tag{38}$$

$$0 \leqq t \leqq T = \frac{-\dfrac{f_1 + f_2}{2} \pm \sqrt{\left[\left(\dfrac{f_1 + f_2}{2}\right)^2 + 4\,\dfrac{q_1 - q_2}{12}\, p_{\max}\right]}}{2\,\dfrac{q_1 - q_2}{12}}\,.$$

This case is equivalent to the approximation of (17) by Bezier's cubic curves. Similarly other procedures of curve fitting can be used.

Now from the computed approximation of time-parametrization for each time $t \in [0; T]$ and each joint i we find the minimum and maximum scaling constant values of c that satisfy the torque limits by solving quadratic inequality (35) together with the condition

$$c \geqq 0\,, \tag{39}$$

that expresses motion irreversibility. The result is an interval $[c_i^-(t), c_i^+(t)]$, where any value inside this interval is a possible scale of the time-parametrization (12) for this joint and this point in the trajectory. This scaling interval may be violated by torque limits at other joints and times (or trajectory points). The resulting interval of scaling constant c for all movement is found by intersecting all such intervals

$$[c^-, c^+] = \bigcap_{i,t} [c_i^-(t), c_i^+(t)] \tag{40}$$

The intervals can be parametrized according to p instead of t.

Thus we have the requested geometric trajectory (1) and the requested velocity profile (6). From that by the above described procedure we have derived suitable approximation (18) of this shape of velocity profile and we have calculated the interval (40) of transformation (26) constant c. The intersection of all intervals (40) may be empty or $c^+ = c^- = 0$. If this happens, the movement along the given geometric trajectory with the given shape of velocity profile is unrealizable at any scale of this profile. If this interval is not null, the movement along the given geometric trajectory with the given shape of velocity profile is realizable with any profile scale within this interval. If $c^+ < 1$, the velocities should be slowed down by at least the factor c^+. If $c^- > 1$, the velocities should be speeded up by at least the factor c^-. If $c^- \leqq 1 \leqq c^+$, the chosen scale of velocity profile is realizable or the movement can be speeded up by the factor c^+ or slowed down by the factor c^-.

If we have the given geometric trajectory and the requested movement along this path by a rough shape of velocity profile even described in parts (for example acceleration from zero velocity, then constant velocity movement and deceleration to zero velocity), we do not need to order the velocity profile (6) all at once, but we can compound the velocity profile from parts (in our example from three parts). We compute the admissible interval of profile scale for every velocity profile described separately. Now we can link up the scaled velocity profiles by satisfying at least the continuity conditions of position on the geometric trajectory and velocities, prospectively further continuity conditions.

4. EXAMPLES

The above procedure will be illustrated for the same two-link planar manipulator as in [7] (Fig. 2). The procedure can be easily applied to manipulators with more degrees of freedom and more complex paths. This manipulator has two rotary joints with joint angles θ_1 and θ_2 and parallel axes, so that the manipulator can generate movement only in $x - y$ plane. Gravity is acting in the negative direction with magnitude g. The length, mass and moment of inertia about the proximal joint for each link are designated by l_i, m_i and I_i.

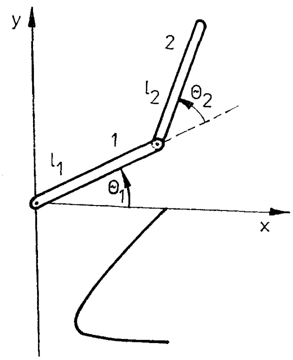

Fig. 2. A planar two-link manipulator and a parabolic arc trajectory.

The equations of motion are (cf. [7])

$$(41) \qquad n_2 = \ddot{\theta}_1\left(I_2 + \frac{m_2 l_1 l_2}{2}\cos\theta_2 + \frac{m_2 l_2^2}{4}\right) + \ddot{\theta}_2\left(I_2 + \frac{m_2 l_2^2}{4}\right) + \\ + \frac{m_2 l_1 l_2}{2}\dot{\theta}_1^2 \sin\theta_2 + \frac{m_2 l_2 g}{2}\cos(\theta_1 + \theta_2)$$

$$n_1 = \ddot{\theta}_1\left(I_1 + I_2 + m_2 l_1 l_2 \cos\theta_2 + \frac{m_1 l_1^2 + m_2 l_2^2}{4} + m_2 l_1^2\right) + \\ + \ddot{\theta}_2\left(I_2 + \frac{m_2 l_2^2}{4} + \frac{m_2 l_1 l_2}{2}\cos\theta_2\right) - \frac{m_2 l_1 l_2}{2}\dot{\theta}_2^2 \sin\theta_2 + \\ - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2 + \left(\frac{m_2 l_2}{2}\cos(\theta_1 + \theta_2) + l_1\left(\frac{m_1}{2} + m_2\right)\cos\theta_1\right) g\,.$$

The inverse kinematic equation according to (4) are (cf. [7], [11])

$$(42) \qquad h = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

$$\theta_2 = \operatorname{atan} 2(-\sqrt{(1 - h^2)}, h)$$

$$\theta_1 = \operatorname{atan} 2((l_1 + l_2 \cos \theta_2)\, y - l_2 \sin \theta_2 x,\ (l_1 + l_2 \cos \theta_2)\, x + l_2 \sin \theta_2 y)$$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} = \frac{1}{l_1 l_2 \sin \theta_2} \begin{bmatrix} l_2 \cos(\theta_1 + \theta_2) & l_2 \sin(\theta_1 + \theta_2) \\ -l_1 \cos \theta_1 & -l_1 \sin \theta_1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix} = \frac{1}{l_1 l_2 \sin \theta_2} \begin{bmatrix} l_2 \cos(\theta_1 + \theta_2) & l_2 \sin(\theta_1 + \theta_2) \\ -l_1 \cos \theta_1 & -l_1 \sin \theta_1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} +$$

$$+ \frac{1}{l_1 l_2 \sin \theta_2} \begin{bmatrix} l_1 l_2 \cos \theta_2 & -l_2^2 \\ -l_1^2 & -l_1 l_2 \cos \theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ (\dot{\theta}_1 + \dot{\theta}_2)^2 \end{bmatrix}.$$

The parameters are $l_1 = l_2 = 0{\cdot}5$ m, $m_1 = m_2 = 1$ kg, $I_1 = I_1 = m_1 l_1^2/12 + m_1 R^2/4$, $R = 0{\cdot}1 l_1$, $g = 9{\cdot}8\ \mathrm{m\,s^{-2}}$.

4.1. Straight Line Movement

A straight line motion from $[x_0, y_0] = [0{\cdot}5, -0{\cdot}5]$ to $[x_1, y_1] = [0{\cdot}5, 0]$ is to be generated from zero velocity into zero velocity. The straight line is parametrized such as $p = s$ and the relations (14), (22), (24) are very simple. The torque limits for the actuators are $n_1^+ = -n_1^- = 8$ Nm, and $n_2^+ = -n_2^- = 2$ Nm. A comparison between n_c^+, n_c^- and n_a for the acceleration of $2\ \mathrm{ms^{-2}}$ from zero velocity according (37) is presented in Figure 3. Carrying out the computations in (35) and (40), it is found that $[c^-, c^+] = [0, 0{\cdot}6976]$. The limitation $c^+ = 0{\cdot}6976$ arises from joint 1

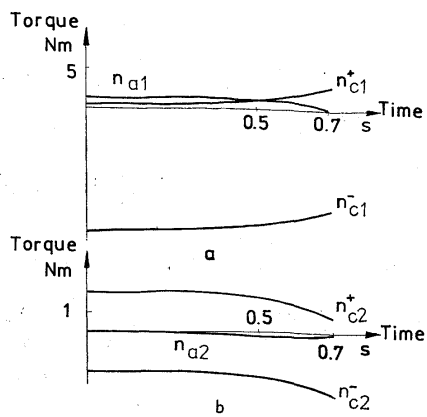

Fig. 3. Torque profiles for acceleration on straight-line trajectory of $2\ \mathrm{ms^{-2}}$ from $[x, y] = [0{\cdot}5, -0{\cdot}5]$ to $[0{\cdot}5, 0]$ when $n_1^+ = -n_1^- = 8$ Nm and $n_2^+ = -n_2^- = 2$ Nm. a — joint 1 — n_c^+, n_c^- and n_a versus time. b — corresponding ones for joint 2.

at the time $t = 0$ s. Thus the fastest realizable acceleration is $2 \, . \, 0{\cdot}6976^2 = 0{\cdot}9733 \ \mathrm{ms}^{-2}$ The relation between n_c^+, n_c^- and n_a for the motion at constant velocity $1 \ \mathrm{ms}^{-1}$ is depicted in Figure 4. The interval of factor c is $[c^-, c^+] = [0, 3{\cdot}4531]$. The limita-

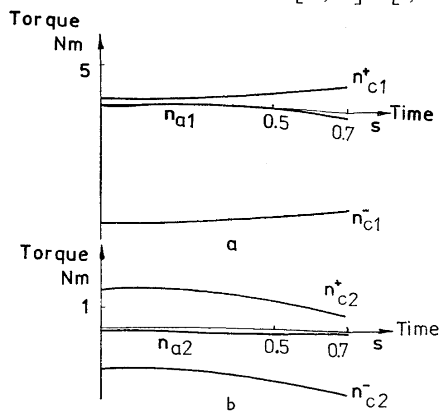

Fig. 4. Torque profiles for a constant velocity movement of $1 \ \mathrm{ms}^{-1}$ with other conditions the same as in Figure 3.

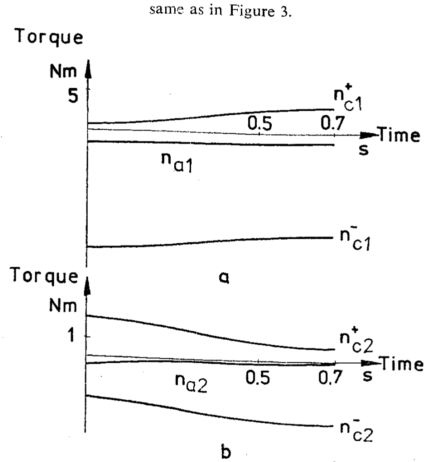

Fig. 5. Torque profiles for a deceleration movement of $-2 \ \mathrm{ms}^{-2}$ with other conditions the same as in Figure 3.

tion $c^+ = 3{\cdot}4531$ arises from joint 2 at the time $t = 0{\cdot}1125$ s. Thus the fastest realizable constant velocity is $1 \,.\, 3{\cdot}4531 = 3{\cdot}4531\ \mathrm{ms}^{-1}$. Comparison between n_c^+, n_c^- and n_a for the deceleration of $-2\ \mathrm{ms}^{-2}$ into zero velocity according (37) is shown in Figure 5. The interval of factor c is $[c^-, c^+] = [0, 3{\cdot}0371]$. The limitation $c^+ = 3{\cdot}0371$ arises from joint 1 at the time $t = 0$ s. Thus the fastest realizable deceleration into zero velocity is $2 \,.\, 3{\cdot}0371^2 = 18{\cdot}4476\ \mathrm{ms}^{-1}$. By compounding investigated velocity profiles we can synthetise the movement for example as follows – first the acceleration of $0{\cdot}9733\ \mathrm{ms}^{-2}$ from zero velocity into the velocity of $0{\cdot}5\ \mathrm{ms}^{-1}$ for the time of $0{\cdot}5137$ s, then the motion at constant velocity of $0{\cdot}5\ \mathrm{ms}^{-1}$ and at the end the deceleration of $-18{\cdot}4476\ \mathrm{ms}^{-2}$ into zero velocity for the time $0{\cdot}02710$ s. During acceleration and deceleration the manipulator covers the path of $0{\cdot}1284 + 0{\cdot}0068 = 0{\cdot}1352$ m. The remaining path of $0{\cdot}5 - 0{\cdot}1352 = 0{\cdot}3648$ m the manipulator covers at the constant velocity of $0{\cdot}5\ \mathrm{ms}^{-1}$ during the time interval of $0{\cdot}7296$ s.

4.2. Parabolic Arc Movement

A parabolic arc movement in Figure 2 with the velocity profile determined by a linear jerk, by a zero jerk at the end of motion and by zero velocities at both ends of the trajectory. The parametric equations of the path are

(43)
$$\begin{aligned} x &= 0{\cdot}5 - p + p^2 \\ y &= -0{\cdot}5 + 0{\cdot}5p^2 \\ &0 \leqq p \leqq 1 \,. \end{aligned}$$

The requested velocity profile v and arc length time dependence s are determined according (7) and (8) (the length of arc is $0{\cdot}77015$ m)

(44)
$$\begin{aligned} v &= 0{\cdot}77015(4\tau^3 - 12\tau^2 + 8\tau) \\ s &= 0{\cdot}77015(\tau^4 - 4\tau^3 + 4\tau^2) \\ &0 \leqq \tau \leqq 1 \,. \end{aligned}$$

The time parametrization of p can be approximated according to (38). For computation of the needed values we use (23) and (25), where $\mathrm{d}s/\mathrm{d}p = \sqrt{(5p^2 - 4p + 1)}$. It can be easily found that

$$\left.\frac{\mathrm{d}p}{\mathrm{d}t}\right|_{p=0} = f_1 = 0 \,,$$

$$\left.\frac{\mathrm{d}^2p}{\mathrm{d}t^2}\right|_{p=0} = q_1 = 8 \,.\, 0{\cdot}77015 = 6{\cdot}16120 \,, \quad \left.\frac{\mathrm{d}p}{\mathrm{d}t}\right|_{p=1} = f_2 = 0 \quad \text{and}$$

$$\left.\frac{\mathrm{d}^2p}{\mathrm{d}t^2}\right|_{p=1} = q_2 = \frac{-4}{\sqrt{2}}\, 0{\cdot}77015 = -2{\cdot}1783 \,.$$

Thus

(45)
$$\begin{aligned} p &= 3{\cdot}0806t^2 - 2{\cdot}8188t^3 + 0{\cdot}6920t^4 \\ &0 \leqq t \leqq 1{\cdot}2\ \mathrm{s} \,. \end{aligned}$$

Now by (45) we can generate the motion and by (35) and (40) we determine the interval of factor c as $[c^-, c^+] = [0, 0.916]$ where the limitation $c^+ = 0.916$ arises from the joint 1 at the time of 0·480 s. We must scale down the time parametrization (45). According (27) the realizable time parametrization is

$$p = 2.5848t^2 - 2.1665t^3 + 0.4872t^4 \tag{46}$$
$$0 \leq t \leq 1.31 \text{ s.}$$

The comparison among the requested profile of velocity (44), the approximated profile of velocity (45) and the scaled profile of velocity (46) is shown in Figure 6.

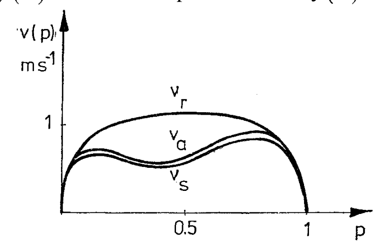

Fig. 6. Velocity profiles for a parabolic arc movement in Figure 2. The comparison among the requested velocity profile v_r(44), the approximated one v_a (45) and the scaled one v_s (46) versus parameter p is shown.

Not very nice approximation is due to a very bad implicit profile of velocity involved in (43). For better approximation of the velocity profile shape (44) would be necessary to subdivide the interval of p.

5. SYNTHESIS OF DYNAMICALLY REALIZABLE TIME PARAMETRIZATION OF GEOMETRIC TRAJECTORIES

According to the above described procedure we can determine whether the movement along the given geometric trajectory with the given shape of velocity profile is possible. If the movement is impossible for the given shape of velocity profile (i.e. for the given velocity profile with any constant scaling factor), does there exist any other velocity profile for which the movement along the given geometric trajectory is possible? We shall try to answer this in what follows.

The relation between $\mathrm{d}p/\mathrm{d}t$ and $\mathrm{d}^2p/\mathrm{d}t^2$ in (35) is fixed for the given shape of velocity profile. By its varying we can get any shape of velocity profile. Let us designate $\mathrm{d}p/\mathrm{d}t = d_1$ and $\mathrm{d}^2p/\mathrm{d}t^2 = d_2$. Employing (35) togethei with (29)–(34)

wa can write

$$
\boldsymbol{n}^- - \boldsymbol{g}(\boldsymbol{\theta}) \leqq \left(\boldsymbol{I}(\boldsymbol{\theta})\frac{\mathrm{d}^2\boldsymbol{K}^{-1}(\boldsymbol{R})}{\mathrm{d}p^2} + \left(\frac{\mathrm{d}\boldsymbol{K}^{-1}(\boldsymbol{R})}{\mathrm{d}p}\right)^{\mathrm{T}} \boldsymbol{C}(\boldsymbol{\theta})\frac{\mathrm{d}\boldsymbol{K}^{-1}(\boldsymbol{R})}{\mathrm{d}p}\right) d_1^2 + \tag{46}
$$

$$
+ \left((\boldsymbol{V}(\boldsymbol{\theta}) + \boldsymbol{F}(\boldsymbol{\theta}))\frac{\mathrm{d}\boldsymbol{K}^{-1}(\boldsymbol{R})}{\mathrm{d}p}\right) d_1 + \left(\boldsymbol{I}(\boldsymbol{\theta})\frac{\mathrm{d}\boldsymbol{K}^{-1}(\boldsymbol{R})}{\mathrm{d}p}\right) d_2 \leqq \boldsymbol{n}^+ - \boldsymbol{g}(\boldsymbol{\theta})
$$

$$
\boldsymbol{n}_c^- \leqq \boldsymbol{A} d_1^2 + \boldsymbol{B} d_1 + \boldsymbol{D} d_2 \leqq \boldsymbol{n}_c^+ \tag{47}
$$

$$
n_{ci}^- + \frac{B_i^2}{4A_i} \leqq A_i\left(d_1 + \frac{B_i}{2A_i}\right)^2 + D_i d_2 \leqq n_{ci}^+ + \frac{B_i^2}{4A_i}
$$

where

$$
[A_i] = \boldsymbol{A} = \boldsymbol{I}(\boldsymbol{\theta})\frac{\mathrm{d}^2\boldsymbol{K}^{-1}(\boldsymbol{R})}{\mathrm{d}p^2} + \left(\frac{\mathrm{d}\boldsymbol{K}^{-1}(\boldsymbol{R})}{\mathrm{d}p}\right)^{\mathrm{T}} \boldsymbol{C}(\boldsymbol{\theta})\frac{\mathrm{d}\boldsymbol{K}^{-1}(\boldsymbol{R})}{\mathrm{d}p} \tag{48}
$$

$$
[B_i] = \boldsymbol{B} = (\boldsymbol{V}(\boldsymbol{\theta}) + \boldsymbol{F}(\boldsymbol{\theta}))\frac{\mathrm{d}\boldsymbol{K}^{-1}(\boldsymbol{R})}{\mathrm{d}p}
$$

$$
[D_i] = \boldsymbol{D} = \boldsymbol{I}(\boldsymbol{\theta})\frac{\mathrm{d}\boldsymbol{K}^{-1}(\boldsymbol{R})}{\mathrm{d}p}.
$$

The relation (47) is with the condition $d_1 \geqq 0$ in the plane $d_1 - d_2$ for joint i the domain in Figure 7. But for the other joint there can be acceptable another domain and the admissible domain for the given position is found by intersecting the domains

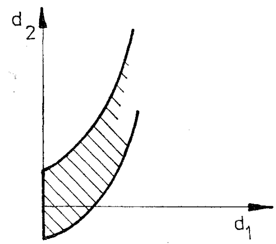

Fig. 7. A possible mutual relation between $d_1 = \mathrm{d}p/\mathrm{d}t$ and $d_2 = \mathrm{d}^2p/\mathrm{d}t^2$ for one joint and one position.

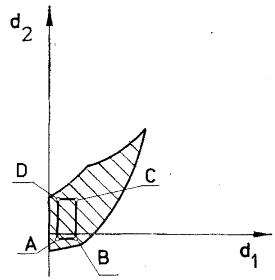

Fig. 8. A possible mutual relation between d_1 and d_2 for one position and more joints.

of all joints (see Fig. 8). i.e. the domain in the plane with parabolic arc boundaries. Such domain is for every position on the geometric trajectory. In one such domain for one position d_1 and d_2 can fill all points. The mutual position of d_1 and d_2

for one time behaviour of p for the domains of different positions is however dependent, because between d_1 and d_2 the differential relation is valid.

Finding of the domain boundary of possible functions in the functional space, that satisfy the conditions (47) for all positions on the path, is a very difficult problem. We can however find different approximations of the solution of this problem. A very simple one is the following.

We intersect the domains (47) for all positions in some segment of the geometric trajectory. We receive a general domain in the $d_1 - d_2$ plane. In this domain we place the rectangle ABCD with sides parallel to the coordinate axis (cf. Fig. 8). This is possible in different ways and this can be used for continuous attaching of approximative rectangles according to the reached velocities in intervals on the all trajectory. Now the admissible velocities and accelerations on the all trajectory segment can change independently in the limits

(49)
$$p_0 \leqq p \leqq p_f$$
$$d_{1A} = d_{1D} = d_1^- \leqq d_1 \leqq d_1^+ = d_{1B} = d_{1C}$$
$$d_{2A} = d_{2B} = d_2^- \leqq d_2 \leqq d_2^+ = d_{2D} = d_{2C} .$$

For the synthesis of the time parametrization of p we can use relations (37), where $p_{max} = p_f - p_0$ and where the start velocity f_1 and the end velocity f_2 in the interval satisfy

(50)
$$2(p_f - p_0)\, d_2^- \leqq f_2^2 - f_1^2 \leqq 2(p_f - p_0)\, d_2^+ .$$

It is possible to construct further approximations in the similar way.

Another problem is the construction of a suitable rectangle or rather the general domain valid for the all trajectory segment. Again we can use the following approximation. If for one joint the parameters of two domains (47) 1 and 2 (in two positions) satisfy

(51)
$$D_1 > 0\,, \quad D_2 > 0$$
$$\frac{A_2}{D_2} \geqq \frac{A_1}{D_1} > 0$$
$$B_2 \geqq B_1$$
$$\frac{n_{c2}^+}{D_2} \leqq \frac{n_{c1}^+}{D_1} \quad \text{or} \quad \frac{n_{c2}^-}{D_2} \leqq \frac{n_{c1}^-}{D_1}$$

then the upper parabolic boundary of the domain in Figure 7 for the parameters 2 is all within the domain for the parameters 1 or the lower parabolic boundary of the domain in Figure 7 for the parameters 1 is all within the domain for the parameters 2. For other combinations of signs in (51) analogical considerations are valid. By using of this property we can find the parabolic boundaries (if (51) is valid then the boundary parabolic arcs have the parameters max A/D, max B, min n_c^+/D and min A/D, min B, max n_c^-/D) as in Figure 8 for one position, but now valid for the all trajectory segment. The intersection for all joints we find easily.

We have also derived the solution of the problem of nonlinear time scaling formulated in [7].

6. EXAMPLE

In [7] there is an example of an unrealizable movement at any (constant) velocity for the same manipulator and the same straight-line path as in Section 4.1. The actuator limits were $n_1^+ = -n_1^- = 6{\cdot}9$ Nm and $n_2^+ = -n_2^- = 1$ Nm. We shall show that there does still exist such a velocity profile which enables the movement along this straight-line.

Using the considerations (51) the admissible (approximated) domain of d_1 and d_2 for the whole of straight-line is constructed in Figure 9. From Figure 9 it is apparent that only the movement with deceleration is realizable and thus the following time parametrization of the straight-line with the geometric parametrization $p = s$ according to (37) is dynamically admissible

$$p = 2t - 0{\cdot}8t^2 \tag{52}$$

$$0 \leqq t \leqq 0{\cdot}282 \text{ s}$$

The time parametrization (52) is illustrated in Figure 9 as the line BA and its re-

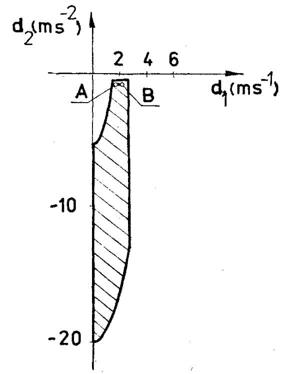

Fig. 9. An approximated domain of possible d_1 and d_2 for both joints and for the whole straight-line movement from $[x, y] = [0{\cdot}5, -0{\cdot}5]$ to $[0{\cdot}5, 0]$ when $n_1^+ = -n_1^- = 6{\cdot}9$ Nm and $n_2^+ = -n_2^- = 1$ Nm.

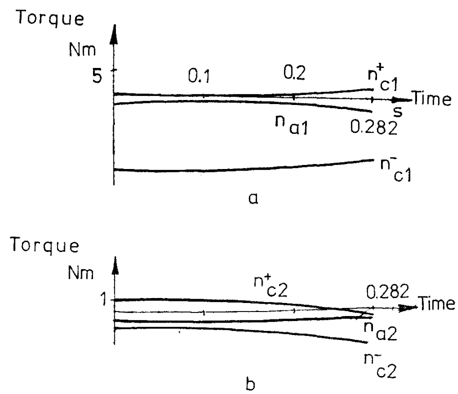

Fig. 10. Torque profiles for deceleration on straight-line movement as in Figure 3–5 of $-1{\cdot}6\ \mathrm{ms}^{-2}$ from the velocity of $2\ \mathrm{ms}^{-1}$ when $n_1^+ = -n_1^- = 6{\cdot}9$ Nm and $n_2^+ = -n_2^- = 1$ Nm. *a* — *joint* 1 and *b* — joint 2 as in Figure 3.

alizability is demonstrated in Figure 10 where it is shown the comparison between n_c^+, n_c^- and n_a for both joints.

7. CONCLUSIONS

The time-scaling property of manipulator dynamics allows to use the actuator torque limits and inverse dynamics for an exact and efficient trajectory planning without repetitive dynamics recalculations. In the first part of the paper there are described a simple algorithm for velocity profile scale modification so that the movement to be dynamically realizable along the given geometric trajectory and several procedures for an approximation of the given velocity profile on the given path as well. In the second part of the paper the problem of dynamically realizable velocity profile synthesis along the given geometric trajectory and its simple approximative solution are presented. This is also the solution of the problem of nonlinear time scaling formulated in [7]. It is also possible to use the described procedures for a manipulator geometric trajectory planning in the space or modifications of unsuitable manipulator paths [8].

APPENDIX 1

If the actuator torque limits cannot be written in the simple form (11) as in the case of permanent magnet dc electrical motors we have instead of (11) a more complex inequality. The actuator torque limits are given by the limits of the actuator control variable and the actuator dynamics is determined by a differential equation. Generally

$$\boldsymbol{u}^- \leqq \boldsymbol{H}\left(\boldsymbol{n}, \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t}, \ldots, \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}, \ldots\right) \leqq \boldsymbol{u}^+ \tag{53}$$

where

$$\boldsymbol{u}^+ = [u_1^+, u_2^+, \ldots, u_n^+]^\mathrm{T} \quad \text{and} \quad \boldsymbol{u}^- = [u_1^-, u_2^-, \ldots, u_n^-]^\mathrm{T}\,.$$

are maximum and minimum constant torque control variable limits, and $\boldsymbol{H}$ is the differential equation of the actuator torque dynamics written as a vector. Nevertheless we can use the above developed theory.

According to the considerations in (28)–(30)

$$\frac{\mathrm{d}^j\boldsymbol{\theta}'}{\mathrm{d}t'^j} = c^j \frac{\mathrm{d}^j\boldsymbol{\theta}}{\mathrm{d}t^j} \qquad j = 0, 1, 2, 3, \ldots \tag{54}$$

and from (31)–(34)

$$\frac{\mathrm{d}^j\boldsymbol{n}'}{\mathrm{d}t'^j} = c^{j+2} \frac{\mathrm{d}^j\boldsymbol{n}_a}{\mathrm{d}t^j} + c^{j+1} \frac{\mathrm{d}^j\boldsymbol{n}_b}{\mathrm{d}t^j} + c^j \frac{\mathrm{d}^j\boldsymbol{g}}{\mathrm{d}t^j} \tag{55}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{\boldsymbol{\theta}}^\mathrm{T} \frac{\partial}{\partial\boldsymbol{\theta}} + \ddot{\boldsymbol{\theta}}^\mathrm{T} \frac{\partial}{\partial\dot{\boldsymbol{\theta}}} + \dddot{\boldsymbol{\theta}}^\mathrm{T} \frac{\partial}{\partial\ddot{\boldsymbol{\theta}}} + \ldots \tag{56}$$

Substituting (54) and (55) into (53) we obtain instead of the quadratic inequality (35) a more complex inequality for the scaling constant value of c (for example in the case of non-negligible inductance of electrical motor we obtain the third degree inequality), but other considerations and procedures are still valid.

APPENDIX 2

If there are more joint variables than trajectory variables, i.e. the kinematic equations (3) have infinite number of solutions, we speak about redundant degrees of freedom by the manipulator. In this case equations (4), (5) are not valid and we cannot use the above described theory. The simpliest way how to obtain equations (4), (5) to be valid in this case is to add further conditions to the equation (3) in order to make the solution of (3) unique.

We must distinguish two cases. Either the number of joint variables of the manipulator is less or equal to 6 or is greater than 6. Because every robot task has its geometric orientation which however can be at first not specified, the first case of redundancy can be easily transferred into unique solution of (3) by adding the information about orientation, i.e. about the orientation of robot gripper. The orientation of the robot is at least given at the both terminal positions and between them the behaviour of orientation can be replaced by the linear or other interpolation function of the geometric parameter p between these terminal orientations.

The truly redundant case is the second one. In this case even the complete information about the robot orientation is not enough for the uniqueness of the solution of (3). In order to obtain this we can either prescribe the motion of other points or bodies of the manipulator in the form (1) or prescribe the behaviour of the redundant degrees of freedom in the form (5) or use these redundant degrees of freedom for the minimization of some performance index.

The first possibility can be very difficult especially if the number of redundant degrees of freedom is not equal to the number of degrees of freedom of whole number of points or bodies. The second possibility is very easy and natural, e.g. we prescribe the complete position of the manipulator in the initial and terminal positions and the behaviour between them is prescribed as a linear or other approximation function of the geometric parameter p as in equation (5). These methods enable us completely to use the described theory of the first part of the paper in the case of redundancy.

There could be difficulties if we look for the synthesis of dynamically realizable time parametrization of geometric trajectory according to the second part of the described theory. The choice of the behaviour of redundant degrees of freedom can make this synthesis unsuccessful however there exists a solution of this synthesis. In this case we must use the third possibility and formulate the task as an optimization one and look for an admissible solution of it. In the whole complexity it is the problem of optimal control [12]. Such optimization problems has infinite number of parameters. But we can formulate a simpler optimization problem only with finite number of parameters which can make use of above described theory. We describe the behaviour of redundant degrees of freedom by an approximation function of the geometric parameter p as in (5) which is determined by a finite number of parameters. Such approximation function can be a spline function the optimization parameters of which are the coordinates of its node points in the space and its own parameter is the geometric parameter p. This spline function can describe the motion of further points or bodies of the manipulator as equation (1) or the behaviour of redundant degrees of freedom as equation (5). By the change of these parameters we change the shape of redundant degrees of freedom behaviour. The results of Section 5 can be applied and are just parametrised by these optimization parameters, i.e. the domains as in Figure 8 or Figure 9 are parametrised by them and we search the solution not in one domain but in many domains according to these parameters. According to them we can search the minimum of some performance index, e.g. energy consumption, or the dynamically realizable time parametrization of the manipulator motion which has other desired property, e.g. zero deceleration, maximum velocity etc.

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Ing. Michael Valášek, CSc., katedra automatického řízení strojní fakulty ČVUT (Department of Automatic Control, Faculty of Mechanical Engineering — Czech Technical University), Suchbátarova 4, 166 07 Praha 6. Czechoslovakia.