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A LOCAL STRUCTURE OF STATIONARY PERFECTLY NOISELESS CODES BETWEEN STATIONARY NON-ERGODIC SOURCES

III: Relative Isomorphism of Non-Ergodic Transformations

ŠTEFAN ŠUJAN

The general results from Part I are applied in order to extend Thouvenot's relative isomorphism theorem to aperiodic non-ergodic transformations and processes, and to classes of processes.

1. INTRODUCTION

This paper is a continuation of the first two parts under the same title, hereafter referred to as [I] and [II]. Throughout the paper references to these papers are indicated by writing, e.g., Lemma I.1, formula (II.9), etc. References [1–19] are listed at the end of [I], [20–38] at the end of [II], remaining ones, starting with [39], are listed at the end of this paper. The bibliographical data of [34–36] are completed in the list of references as well, these papers being already published.

In [I, II, 35] we investigated the local structure of (mod 0) isomorphisms between stationary non-ergodic processes and applied the general results to a number of concrete isomorphism problems. In this paper we address the same problem within the setup of the relative isomorphism theory developed by Thouvenot [44].

Thouvenot studied the following generalization of Ornstein's isomorphism problem. Let (X, Y) be a pair process where X is a stationary and ergodic process with a finite state space and Y is a stationary independent process statistically independent of X. The problem is to determine all pair processes (U, V) for which there is a process Z such that

(a) (X, Z) and (X, Y) are almost surely stationary codings of each other, and

(b) $\operatorname{dist}(X, Z) = \operatorname{dist}(U, V)$.

In this case the pair processes (X, Y) and (U, V) are said to be *relatively isomorphic* (this formulation is due to Kieffer [41]; we shall use a slightly more general one). Thouvenot proved that the conditions $H(X, Y) = H(U, V)$, $\operatorname{dist} X = \operatorname{dist} U$, plus a relativized version of Ornstein's finitely determined property (FD) of V relative to U completely characterize such processes.

Unlike in the preceding papers we incline here to the setup of automorphisms of Lebesgue probability spaces and processes induced by finite partitions (this setup has been developed systematically in [12], see also [15, 42], and the recent survey [43]) rather than the processes on sequence spaces. This allows us for avoiding complicated notations, and the reader familiar with the "translation rules" from [43] can without problems pass from one setup to the other.

2. ERGODIC DECOMPOSITION

In order to help an information theorist to clarify the relations between the two frames mentioned above, we describe here the ergodic decomposition (cf. [I, Section 2]) in the abstract setup of Lebesgue spaces. We assume that all spaces $(X, \mathscr{F}, \mu)$ in this paper are Lebesgue spaces with continuous (non-atomic) measures. $T: X \to X$ will always be an automorphism. Partitions measurable in the sense of [15] will be called Rokhlin measurable, while a measurable partition will always mean an at most countable partition whose atoms are measurable sets.

Let ζ be a Rokhlin measurable partition of X and $X \mid \zeta$ the corresponding factor space. We convert $X \mid \zeta$ into a probability space with the aid of the natural factor mapping $H_\zeta : X \to X \mid \zeta$ which assigns to each $x \in X$ that element $C \in X \mid \zeta$ containing x. We put $\mathscr{F}_\zeta = \{E \subset X : H_\zeta^{-1} E \in \mathscr{F}\}$ and $\mu_\zeta(E) = \mu(H_\zeta^{-1} E)$, $E \in \mathscr{F}_\zeta$. The space thus obtained is again a Lebesgue space. A family $(\mu_C; C \in X \mid \zeta)$ is said to be a *canonical system of measures* associated with ζ if

(a) for almost any C there exists a σ-field $\mathscr{F}_C$ such that $(C, \mathscr{F}_C, \mu_C)$ is a Lebesgue space, and

(b) for each $F \in \mathscr{F}$ the mapping $C \mapsto \mu_C(C \cap F)$ is almost everywhere defined, measurable, and

$$\mu(F) = \int_{X|\xi} \mu_C(C \cap F) \, \mathrm{d}\mu_\zeta$$

(see (I.21) for an analogous formula). Let ζ be, in addition, also invariant in the sense that any of its elements is invariant. Thus, if $C \in X \mid \zeta$ the T acts only inside C so that we can think of $T_C = T \mid C$ as of an automorphism of the Lebesgue space $(C, \mathscr{F}_C, \mu_C)$. We denote by ζ_0 the (mod 0) coarsest partition within the family of all invariant, Rokhlin measurable partitions ζ such that T_C is ergodic for all $C \in X \mid \zeta$. The family $(T_C; C \in X \mid \zeta_0)$ is said to be the *ergodic decomposition* of the automorphism T.

Let $P = (P_1, \ldots, P_k)$ be a measurable partition of X. We denote by (T, P) the corresponding process [12]. Any process (T, P) determines a factor of T, namely, the action of T on the measure space $(X, (P)_T, \mu)$, where $(P)_T$ is the invariant σ-field $\bigvee_{i=-\infty}^{\infty} T^i P$. We denote that factor by $(T, (P)_T)$, and call the process (T, P) ergodic if $(T, (P)_T)$ is ergodic, i.e., if $(P)_T \cap \{F \in \mathscr{F}: T^{-1} F = F\} = \{\emptyset, X\}$ mod 0. In particular, if (T, P) is ergodic and P is a generating partition (that is, $(P)_T = \mathscr{F}$ mod 0; here we allow also for countable partitions) then T itself is ergodic. In fact, its ergodic decomposition is then mod 0 trivial.

If $C \in F$ and $P = (P_1, \ldots, P_k)$ we put $P_C = (P_1 \cap C, \ldots, P_k \cap C)$. In particular, if $C \in X \mid \zeta_0$ then the process (T_C, P_C) defined on $(C, \mathscr{F}_C, \mu_C)$ is said to be an ergodic component of the process (T, P). (Note: any $C \in X \mid \zeta$ can be considered also as a subset of X). Since μ_C is concentrated on C, the processes (T_C, P_C) and (T, P_C) are identical mod μ_C, however, they may be quite different mod μ.

If P and Q are finite measurable partitions of X, the process $(T, P \vee Q)$ is called a pair process. If (X, Y) is a stationary process such that X and Y each have a finite state space, then (X, Y) can be identified with $(T, P_x \vee P_y)$, where P_x and P_y are the state space partitions. Thus, the two descriptions of pair processes are essentially equivalent. Of course, the ergodic decomposition result applies to stationary pair processes as well.

3. THE STRUCTURE OF RELATIVE ISOMORPHISMS

If $P = (P_1, \ldots, P_k)$ is an ordered partition of X, we denote by $|P|$ $(=k)$ its cardinality and by $\operatorname{dist}(P)$ the ordered k-tuple $(\mu(P_1, \ldots, \mu(P_k))$. Given two processes (T, P) and $(\bar{T}, \bar{P})$ (on possibly different probability spaces) we write $(T, P) \sim (\bar{T}, \bar{P})$ if $\operatorname{dist}\left(\bigvee_{i=1}^{n} T^{-i}P\right) = \operatorname{dist}\left(\bigvee_{i=1}^{n} \bar{T}^{-i}\bar{P}\right)$ for all $n \geqq 1$.

Definition 1. Two automorphisms T and $\bar{T}$ of Lebesgue spaces $(X, \mathscr{F}, \mu)$ and $(\bar{X}, \bar{\mathscr{F}}, \bar{\mu})$ are said to be *relatively isomorphic* if there exist finite measurable partitions P, Q of X, $\bar{P}$, $\bar{Q}$ of $\bar{X}$, and an isomorphism $\varphi\colon (X, \mathscr{F}, \mu) \to (\bar{X}, \bar{\mathscr{F}}, \bar{\mu})$ such that

(a) $(P \vee Q)_T = \mathscr{F}$ mod 0, $(\bar{P} \vee \bar{Q})_T = \bar{\mathscr{F}}$ mod 0,

(b) $\varphi \circ T = \bar{T} \circ \varphi$ mod 0, and

(c) $\varphi(Q) = \bar{Q}$ mod 0.

Remark 1. Thus two automorphisms T and $\bar{T}$ are relatively isomorphic if and only if there exist pair processes $(T, P \vee Q)$ and $(\bar{T}, \bar{P} \vee \bar{Q})$ such that each carries the whole information about the underlying σ-field, and the processes are relatively isomorphic (relative to the factors (T, Q) and $(\bar{T}, \bar{Q})$). Of course, we can define also the relative isomorphism of pair processes instead of the transformations themselves, in which case the assumption that the partitions generate can be removed. It is easy to check that we get a definition of relative isomorphism equivalent to that one introduced in Section 1. More precisely, two pair processes (X, Y) and (U, V) are relatively isomorphic in the sense of Section 1 if and only if the pair processes $(T, P_x \vee P_y)$ and $(\bar{T}, P_u \vee P_v)$ are.

The results of this section constitute a relativized counterpart of [I]. The first result tells us that a relative isomorphism φ between T and $\bar{T}$ splits into a measurable family of "local" relative isomorphisms between the corresponding ergodic components.

Theorem 1. Let T on $(X, \mathscr{F}, \mu)$ and $\bar{T}$ on $(\bar{X}, \bar{\mathscr{F}}, \bar{\mu})$ be relatively isomorphic via an isomorphism $\varphi : X \to \bar{X}$. Then the following assertion are true:

(a) for μ_{ζ_0} almost every $C \in X \mid \zeta_0$ there is a unique $\bar{C} \in \bar{X} \mid \bar{\zeta}_0$ and for $\bar{\mu}_{\bar{\zeta}_0}$ almost every $\bar{C} \in \bar{X} \mid \bar{\zeta}_0$ there is a unique $C \in X \mid \zeta_0$ such that $\varphi(C) = \bar{C}$.

(b) Let us put for $C \in X \mid \zeta_0 (\text{mod } \mu_{\zeta_0})$ and for $x \in C (\text{mod } \mu_C)$ $\varphi_C(x) = \varphi(x)$. If $C \in X \mid \zeta_0$ and if $\bar{C} \in \bar{X} \mid \bar{\zeta}_0$ corresponds to C by assertion (a) (this takes place for almost all C), then φ_C is an isomorphism from $(C, \mathscr{F}_C, \mu_C)$ to $(\bar{C}, \bar{\mathscr{F}}_{\bar{C}}, \bar{\mu}_{\bar{C}})$ which makes T and $\bar{T}$ relatively isomorphic.

(c) Let $F : X \mid \zeta_0 \times X \to \bar{X}$ be mod 0 defined by the properties that $F(C, x) = \varphi_C(x)$. Then F is $(\mathscr{F}_{\zeta_0} \times \mathscr{F}, \bar{\mathscr{F}})$-measurable.

Proof. The "non-relative" isomorphism part can be obtained exactly as in [I] (in fact, the proof there was made within the frame of induced shifts on sequence spaces but it can be translated into the present setup without problems). Hence, for almost all $C \in X \mid \zeta_0$, if $\bar{C} \in \bar{X} \mid \bar{\zeta}_0$ corresponds to C then we have an isomorphism φ_C from $(C, \mathscr{F}_C, \mu_C)$ to $(\bar{C}, \bar{\mathscr{F}}_{\bar{C}}, \bar{\mu}_{\bar{C}})$ such that $\varphi_C \circ T_C = \bar{T}_{\bar{C}} \circ \varphi_C$. According to Definition 1 it remains to prove that

(d) there exist finite partitions P', Q' of C and $\bar{P}'$, $\bar{Q}'$ of $\bar{C}$ such that $(P' \vee Q')_{T_C} = \mathscr{F}_C \text{ mod } 0$, $(\bar{P}' \vee \bar{Q}')_{T_{\bar{C}}} = \bar{\mathscr{F}}_{\bar{C}} \text{ mod } 0$, and

(e) $\varphi_C(Q') = \bar{Q}'$.

In order to prove (d) recall from Section 2 that the processes (T, P_C) and (T_C, P_C) can be considered as identical on $(C, \mathscr{F}_C, \mu_C)$. If $(P \vee Q)_T = \mathscr{F} \text{ mod } \mu$, then for almost all $C \in X \mid \zeta_0$ we have $(P \vee Q)_{T_C} = \mathscr{F} \text{ mod } \mu_C$ (this is easy to check directly or, consult [19]). But $\mathscr{F} = \mathscr{F}_C \text{ mod } \mu_C$ so that $(P \vee Q)_{T_C} = \mathscr{F}_C \text{ mod } \mu_C$. Consequently, $(P \vee Q)_{T_C} = (P_C \vee Q_C)_{T_C} \text{ mod } \mu_C$. In this way we obtain

$$\mu_{\zeta_0}\{C \in X \mid \zeta_0 : (P_C \vee Q_C)_{T_C} = \mathscr{F}_C \text{ mod } \mu_C\} = 1 ,$$

$$\bar{\mu}_{\bar{\zeta}_0}\{\bar{C} \in \bar{X} \mid \bar{\zeta}_0 : (\bar{P}_{\bar{C}} \vee \bar{Q}_{\bar{C}})_{T_{\bar{C}}} = \bar{\mathscr{F}}_{\bar{C}} \text{ mod } \bar{\mu}_{\bar{C}}\} = 1 .$$

It follows that (d) is valid if $P' = P_C$, $Q' = Q_C$, $\bar{P}' = \bar{P}_{\bar{C}}$, and $\bar{Q}' = \bar{Q}_{\bar{C}}$. It remains to prove that with this choice also (e) is valid. Now (e) can be rewritten in the form

(f) $\mu_{\zeta_0}\{C \in X \mid \zeta_0 : \varphi_C(Q_C) = \bar{Q}_{\bar{C}}\} = 1$,

where $\bar{C}$ corresponds to C by assertion (a).

We prove (f) indirectly. Hence, suppose there is a set $G \in \mathscr{F}_{\zeta_0}$ with $\mu_{\zeta_0}(G) > 0$ and $\varphi_C(Q_C) \neq \bar{Q}_C$ for any $C \in G$. Thus, if $C \in G$ we find $D \in Q_C$ and $\bar{D}_1, \bar{D}_2 \in \bar{Q}_C$ such that

(g) $\varphi_C(D) \cap \bar{D}_1 \neq \emptyset \text{ mod } 0$, $\varphi_C(D) \cap \bar{D}_2 \neq \emptyset \text{ mod } 0$.

Find $Q_i \in Q$ with $Q_i \cap C = D$ and $\bar{Q}_1, \bar{Q}_2 \in \bar{Q}$ with $\bar{D}_1 = \bar{Q}_1 \cap \bar{C}$, $\bar{D}_2 = \bar{Q}_2 \cap \bar{C}$. Since $\varphi(Q) = \bar{Q}$ we can find $Q_1, Q_2 \in Q$ such that $\varphi(Q_1) = \bar{Q}_1$ and $\varphi(Q_2) = \bar{Q}_2$. We easily get

$$\varphi(Q_i) \cap \varphi(Q_1) \neq \emptyset \bmod 0\,, \quad \varphi(Q_i) \cap \varphi(Q_2) \neq \emptyset \bmod 0\,.$$

Because $\varphi(Q)$ is again a partition mod 0, the latter relations imply that $Q_i = Q_1$ mod 0 and $Q_i = Q_2$ mod 0. Consequently, $Q_i = Q_1 \cap Q_2$ mod 0, i.e., $Q_i = \emptyset$ mod 0. But this contradicts (g) so that (f) must be valid proving the theorem. □

Our next aim is to deduce a converse to Theorem 1. The converse for non-relative isomorphisms was also obtained in [I]. However, because of the generating hypothesis (a) in Definition 1 we cannot immediately use that result. Rather, we must combine it with ideas used to get an extension of Krieger's finite generator theorem to aperiodic non-ergodic transformations (see [19] or [35]). To this end observe the following. There exists an additional natural necessary condition for the relative isomorphism of transformations T and $\bar{T}$. If T and $\bar{T}$ are such then (a) of Definition 1 says there exist pairs (P, Q) and $(\bar{P}, \bar{Q})$ of finite generating partitions. But if P partitions X and if $C \in X \mid \zeta_0$ is given, then at most $|P|$ atoms of P intersect C so that $|P_C| \leqq |P|$. Since the partitions (P_C, Q_C) and $(\bar{P}_{\bar{C}}, \bar{Q}_{\bar{C}})$ have been shown to fulfil the requirements for the relative isomorphism of T_C and $\bar{T}_{\bar{C}}$, a natural necessary condition reads:

$$\operatorname{ess\,sup} \{|P_C \vee Q_C| : C \in X \mid \zeta_0 \bmod \mu_{\zeta_0}\} \leqq K < \infty\,,$$

and similarly for the second pair of partitions. Now we shall prove that if these conditions are added to those ones obtained in Theorem 1, we get a relative isomorphism between T and $\bar{T}$ themselves.

Theorem 2. Let T on $(X, \mathscr{F}, \mu)$ and $\bar{T}$ on $(\bar{X}, \bar{\mathscr{F}}, \bar{\mu})$ be given. Suppose that

(a) for μ_{ζ_0} almost all $C \in X \mid \zeta_0$ there is an isomorphism $\varphi_C : (C, \mathscr{F}_C, \mu_C) \to (\bar{C}, \bar{\mathscr{F}}_{\bar{C}}, \bar{\mu}_{\bar{C}})$ $(\bar{C} = \varphi_C(C))$ which makes T_C and $\bar{T}_{\bar{C}}$ relatively isomorphic,

(b) for any $\bar{F} \in \bar{\mathscr{F}}$,

$$\bar{\mu}(\bar{F}) = \int_{X|\xi_0} \bar{\mu}_{\bar{C}}(\bar{C} \cap \bar{F})\, \mathrm{d}\mu_{\zeta_0}(C)$$

(in the formula, $\bar{C} = \varphi_C(C)$ so that the expression on its right hand side makes sense),

(c) the mapping $(C, x) \mapsto \varphi_C(x) : X \mid \zeta_0 \times X \to \bar{X}$ is $(\mathscr{F}_{\zeta_0} \times \mathscr{F}, \bar{\mathscr{F}})$-measurable, and

(d) if $P(C)$, $Q(C)$, $\bar{P}(\bar{C})$, and $\bar{Q}(\bar{C})$ are finite partitions which correspond to φ_C according to Definition 1, then the functions $C \mapsto |P(C) \vee Q(C)|$ and $\bar{C} \mapsto |\bar{P}(\bar{C}) \vee \bar{Q}(\bar{C})|$ are essentially bounded (relative to the measures μ_{ζ_0} on $X \mid \zeta_0$ and $\bar{\mu}_{\bar{\zeta}_0}$ on $\bar{X} \mid \bar{\zeta}_0$).

Then T and $\bar{T}$ are relatively isomorphic.

Remark 2 (Correction to [I]). By a misprint, a condition like (b) above was forgotten in the formulation of Theorem I.2, though it was used in the proof. The point is that without (b) we merely can compose the local isomorphisms into a single stationary code, but do not know whether its range is almost all of $\bar{X}$, for without (b) there may exists additional ergodic components in $(\bar{X}, \bar{\mathscr{F}}, \bar{\mu}, \bar{T})$. Thus, (a), (c), and (d) just entail $\bar{T}$ is a factor of T.

Proof of Theorem 2. First consider the mappings φ_C as non-relative isomorphisms. Then (a), (b) and (c) imply as in [1] (cf. the preceding remark) the existence of a global isomorphism $\varphi : (X, \mathscr{F}, \mu) \to (\bar{X}, \bar{\mathscr{F}}, \bar{\mu})$ for which $\varphi \circ T = \bar{T} \circ \varphi \bmod 0$.

Condition (d) ensures that the non-ergodic extension of Krieger's finite generator theorems works [35, 19]. Consequently, there exist finite generating partitions R of X and $\bar{R}$ of $\bar{X}$. Since $|P| . |Q| \geqq |P \vee Q|$ and since only non-degenerate ($|Q| \geqq 2$) partitions Q are of interest, it may happen that R is not expressible in the form $P \vee Q$, where $|P| \geqq 2$, $|Q| \geqq 2$. However, we can refine R if necessary and thus assume that $R = P \vee Q$. Similar remarks apply to $\bar{R}$ which from now on is also assumed to take on the form $\bar{P} \vee \bar{Q}$.

We are not sure that Q and $\bar{Q}$ just chosen are appropriate. But we claim that we always can find appropriate ones. For, suppose that for any pairs (P, Q) and $(\bar{P}, \bar{Q})$ of generating partitions the isomorphism φ has the property that $\varphi(Q) \neq \bar{Q}$. Since $\mathscr{F}_C = \mathscr{F} \cap C$, $\bar{\mathscr{F}}_{\bar{C}} = \bar{\mathscr{F}} \cap \bar{C}$, $T_C = T \mid C$, and $\bar{T}_{\bar{C}} = \bar{T} \mid \bar{C}$, the conditions that

$$(P(C) \vee Q(C))_{T_C} = \mathscr{F}_C \bmod 0\,, \quad (\bar{P}(\bar{C}) \vee \bar{Q}(\bar{C}))_{\bar{T}_{\bar{C}}} = \bar{\mathscr{F}}_{\bar{C}} \bmod 0$$

entail that there exist generating pairs (P, Q) and $(\bar{P}, \bar{Q})$ such that $P(C) = P_C$, $Q(C) = Q_C$, $\bar{P}(\bar{C}) = \bar{P}_{\bar{C}}$, and $\bar{Q}(\bar{C}) = \bar{Q}_{\bar{C}}$. From the construction of φ in [1] it follows that

$$\mu_{\zeta_0}\{C \in X \mid \zeta_0 : \varphi \mid C = \varphi_C\} = 1\,.$$

Summarizing all this we get the conclusion

$$\mu_{\zeta_0}\{C \in X \mid \zeta_0 : \varphi_C \text{ is not a relative isomorphism}\} > 0\,,$$

and this contradicts (a). □

In particular, we get the following relative isomorphism theorem for processes (see Remark 1).

Corollary 3. Two aperiodic pair processes $(T, P \vee Q)$ on $(X, \mathscr{F}, \mu)$ and $(\bar{T}, \bar{P} \vee \bar{Q})$ on $(\bar{X}, \bar{\mathscr{F}}, \bar{\mu})$ are relatively isomorphic relative to factor processes (T, Q) and $(\bar{T}, \bar{Q})$ if and only if the sets of their ergodic components $((T_C, P_C \vee Q_C) : C \in X \mid \zeta_0)$ and $((\bar{T}_{\bar{C}}, \bar{P}_{\bar{C}} \vee \bar{Q}_{\bar{C}}) : \bar{C} \in \bar{X} \mid \bar{\zeta}_0)$ can be mod 0 decomposed into pairs $(T_C, P_C \vee Q_C)$, $(\bar{T}_{\bar{C}}, \bar{P}_{\bar{C}} \vee \bar{Q}_{\bar{C}})$ of pair processes which are relatively isomorphic relative to factor processes (T_C, Q_C) and $(\bar{T}_{\bar{C}}, \bar{Q}_{\bar{C}})$.

Remark 3. In Theorems 1 and 2 the transformations T and $\bar{T}$ were aperiodic because of our assumption that the measures μ and $\bar{\mu}$ were continuous. However, if T is aperiodic then we cannot assert the same for any process (T, P), and that is why an additional aperiodicity assumption occurs in the preceding corollary.

4. MIXTURES OF PAIR PROCESSES

First let us recall Thouvenot's relative isomorphism theorem. To this end we introduce the following concepts. If P and Q are finite ordered partitions such that $|P| = |Q| = k$, then the strong partition distance is defined by

$$|P - Q| = \sum_{i=1}^{k} \mu(P_i \,\Delta Q_i)\,,$$

where Δ stands for the symmetric difference. Of course, one can extend this to partitions with inequal numbers of atoms.

Definition 2. Let $(T, P \vee Q)$ on $(X, \mathscr{F}, \mu)$ and $(\bar{T}, \bar{P} \vee \bar{Q})$ on $(\bar{X}, \bar{\mathscr{F}}, \bar{\mu})$ be ergodic pair processes such that $|P| = |\bar{P}|$ and $(T, Q) \sim (\bar{T}, \bar{Q})$. The *relative $\bar{d}$-distance* is defined by

$$\bar{d}_{Q,\bar{Q}}[(T, P \vee Q), (\bar{T}, \bar{P} \vee \bar{Q})] = \sup_{n \geqq 1} \bar{d}^n[(T, P \vee Q), (\bar{T}, \bar{P} \vee \bar{Q})]\,,$$

where

$$\bar{d}^n[(T, P \vee Q), (\bar{T}, \bar{P} \vee \bar{Q})] = \inf_{\psi} n^{-1} \sum_{i=0}^{n-1} |\psi(T^i P) - \bar{T}^i \bar{P}|\,,$$

and the infimum is taken over all isomorphisms $\psi : X \to \bar{X}$ such that

$$\psi\Big(\bigvee_{i=0}^{m-1} T^{-i} Q\Big) = \bigvee_{i=0}^{m-1} \bar{T}^{-i} \bar{Q}\,, \quad m = 1, 2, \ldots\,.$$

Definition 3 [44]. An ergodic pair process $(T, P \vee Q)$ is called *Q-relatively finitely determined* (FD) if for each $\varepsilon > 0$ there exists a $\delta > 0$ and a positive integer n such that for any ergodic pair process $(T', P' \vee Q')$ the conditions that $|P| = |P'|$, $(T, Q) \sim$ $\sim (T', Q')$,

$$|H(T, P \vee Q) - H(T', P' \vee Q')| < \delta\,,$$

$$\Big|\mathrm{dist}\Big(\bigvee_{i=0}^{n} T^{-i}(P \vee Q)\Big) - \mathrm{dist}\Big(\bigvee_{i=0}^{n} (T')^{-i}(P' \vee Q')\Big)\Big| < \delta$$

imply

$$\bar{d}_{Q,Q'}[(T, P \vee Q)\,,\ (T', P' \vee Q')] < \varepsilon\,.$$

Here, $|\mathrm{dist}(P) - \mathrm{dist}(Q)| = \sum |\mu(P_i) - \mu'(Q_i)|$ is the weak partition distance. Alternate definitions of relative $\bar{d}$-distance and of relativized FD property in terms of processes considered in the introduction are given in [41]. Our Definition 2 is taken from [39]. Of course, several other approaches to the relative $\bar{d}$-distance are possible paralleling the non-relative case (for the latter consult [29] or [12, Appendix C]).

Here is the result of Thouvenot we wish to generalize to non-ergodic automorphisms and non-ergodic pair processes, respectively.

Theorem 4 (Thouvenot). Let $(T, P \vee Q)$ on $(X, \mathscr{F}, \mu)$ and $(\bar{T}, \bar{P} \vee \bar{Q})$ on $(\bar{X}, \bar{\mathscr{F}}, \bar{\mu})$ be two ergodic pair processes. Suppose

(a) $(P \vee Q)_T = \mathscr{F} \bmod 0$, $(\bar{P} \vee \bar{Q})_{\bar{T}} = \bar{\mathscr{F}} \bmod 0$,
(b) $(T, P \vee Q)$ is Q-relatively FD, $(\bar{T}, \bar{P} \vee \bar{Q})$ is $\bar{Q}$-relatively FD,
(c) $H(T) = H(\bar{T})$, and
(d) $(T, Q) \sim (\bar{T}, \bar{Q})$.
Then T and $\bar{T}$ are relatively isomorphic relative to the factors (T, Q) and $(\bar{T}, \bar{Q})$.

The following idea is valid generally. Suppose T is a non-ergodic transformation and almost all its ergodic components T_C belong to a class $\mathscr{T}$ of ergodic transformation for which $I(\cdot)$ is a complete numerical invariant, i.e., $T, T' \in \mathscr{T}$ are isomorphic if and only if $I(T) = I(T')$. Then the distribution function

$$t \mapsto \mu_{\zeta_0}\{C \in X \mid \zeta_0 : I(T_C) \leqq t\}$$

is a complete isomorphism invariant for the class of transformations having ergodic components in $\mathscr{T}$. In [II] this was proved when $I(T) = H(T)$ and $\mathscr{T}$ was just a class of Bernoulli shifts. This is easy to generalize to non-ergodic pair processes such that their ergodic components each have the same factor:

Theorem 5. Let T and $\bar{T}$ be non-ergodic automorphisms of $(X, \mathscr{F}, \mu)$ and $(\bar{X}, \bar{\mathscr{F}}, \bar{\mu})$. Suppose
(a) there exist ergodic processes (T, Q) and $(\bar{T}, \bar{Q})$ such that $(T, Q) \sim (\bar{T}, \bar{Q})$,
(b) there exist finite partitions $P(C)$ of C and $\bar{P}(\bar{C})$ of $\bar{C}$ such that

$$(P(C) \vee Q)_{T_C} = \mathscr{F}_C \bmod 0\,, \quad (\bar{P}(\bar{C}) \vee \bar{Q})_{\bar{T}_{\bar{C}}} = \bar{\mathscr{F}}_{\bar{C}} \bmod 0\,,$$

(c) $(T_C, P(C) \vee Q)$ is Q-relatively FD and $(\bar{T}_{\bar{C}}, \bar{P}(\bar{C}) \vee \bar{Q})$ is $\bar{Q}$-relatively FD,
(d) $\operatorname{ess\,sup}\{|P(C)| : C \in X \mid \zeta_0 \bmod \mu_{\zeta_0}\} < \infty$,
$\operatorname{ess\,sup}\{|\bar{P}(\bar{C})| : \bar{C} \in \bar{X} \mid \zeta_0 \bmod \bar{\mu}_{\bar{\zeta}_0}\} < \infty$, and
(e) for any $t \geqq 0$,

$$\mu_{\zeta_0}\{C: H(T_C, P(C) \vee Q) \leqq t\} = \bar{\mu}_{\bar{\zeta}_0}\{\bar{C}: H(\bar{T}_{\bar{C}}, \bar{P}(\bar{C}) \vee \bar{Q}) \leqq t\}\,.$$

Then T and $\bar{T}$ are relatively isomorphic relative to the factors (T, Q) and $(\bar{T}, \bar{Q})$.

Proof. It suffices to verify the assumptions of Theorem 2. Since Q and $\bar{Q}$ are fixed, our assumption (d) is the same as (d) of Theorem 2. It remains to verify (a)–(c). We claim that (e) implies $H(T_C) = H(\bar{T}_{\bar{C}})$ in the following sense: for μ_{ζ_0} almost any $C \in X \mid \zeta_0$ there exists a $\bar{C} \in \bar{X} \mid \bar{\zeta}_0$, and for $\bar{\mu}_{\bar{\zeta}_0}$ almost any $\bar{C} \in \bar{X} \mid \bar{\zeta}_0$ there exists a $C \in X \mid \zeta_0$ such that $H(T_C) = H(\bar{T}_{\bar{C}})$. For suppose not. For the sake of concreteness suppose there is a set $G \subset X \mid \zeta_0$ with $\mu_{\zeta_0}(G) > 0$ such that for any $C \in G$, $H(T_C) \neq H(\bar{T}_{\bar{C}})$ for all $\bar{C} \in \bar{X} \mid \bar{\zeta}_0$. But then (d) is violated.

Apply Theorem 4 to conclude that for μ_{ζ_0} almost all $C \in X \mid \zeta_0$ there is a $\bar{C} \in \bar{X} \mid \bar{\zeta}_0$, and for $\bar{\mu}_{\bar{\zeta}_0}$ almost all $\bar{C} \in \bar{X} \mid \bar{\zeta}_0$ there is a $C \in X \mid \zeta_0$ such that T_C and $\bar{T}_{\bar{C}}$ are relatively isomorphic relative to the factors (T, Q) and $(\bar{T}, \bar{Q})$. This gives immediately (a) and (b) of Theorem 2, while (c) may be verified directly. Thus, an application of Theorem 2 concludes the proof. □

In case when T and $\bar{T}$ are allowed to possess different factors $Q(C)$ and $\bar{Q}(\bar{C})$ for different ergodic components the situation becomes much more complicated. The problem, in light of the discussion preceding Theorem 5, is to characterize the condition that $(T_C, Q(C)) \sim (\bar{T}_{\bar{C}}, \bar{Q}(\bar{C}))$ by means of a numerical-like invariant. Since this condition is determined by a total of countably many conditions placed on finite probability vectors, we can construct a countably-dimensional distribution function characterizing $\sim$. Since different $Q(C')$'s may possess different numbers of atoms, however, it is not clear whether a single distribution function can work. On the other hand, we know that a necessary condition for relative isomorphism is essential boundedness of $C \mapsto |P(C) \vee Q(C)|$ so that the function $C \mapsto |Q(C)|$ must be essentially bounded, too. Let K be the essential upper bound to $C \mapsto |Q(C)|$, and L that to $\bar{C} \mapsto |\bar{Q}(\bar{C})|$. Then the distributions

$$\operatorname{dist}\left(\bigvee_{i=0}^{n-1} T_C^{-i} Q(C)\right), \quad \operatorname{dist}\left(\bigvee_{i=0}^{n-1} \bar{T}_{\bar{C}}^{-i} \bar{Q}(\bar{C})\right)$$

can be considered as K^n, resp. L^n-probability vectors for all $C \in X \mid \zeta_0$ and all $\bar{C} \in \bar{X} \mid \bar{\zeta}_0$. Hence, they are the elements of cubes $[0, 1]^{K^n}$ and $[0, 1]^{L^n}$; $n = 1, 2, \ldots$. There is a natural separable topology on the spaces $[0, 1]^m$, and one can find an ordering $\prec$ such that the order topology is equivalent to the natural one. Let $\Pi_n(K)$ denote the ordered set $([0, 1]^{K^n}, \prec)$. Using the same arguments as in the proof of Theorem 5 we can derive the next assertion:

Proposition 6. Let T and $\bar{T}$ be non-ergodic automorphisms of $(X, \mathscr{F}, \mu)$ and $(\bar{X}, \bar{\mathscr{F}}, \bar{\mu})$. Suppose that

(a) almost all ergodic components T_C and $\bar{T}_{\bar{C}}$ have generating pairs $(P(C), Q(C))$ and $(\bar{P}(\bar{C}), \bar{Q}(\bar{C}))$ of finite partitions such that $(T_C, P(C) \vee Q(C))$ is $Q(C)$-relatively FD, and $(\bar{T}_{\bar{C}}, \bar{P}(\bar{C}) \vee \bar{Q}(\bar{C}))$ is $\bar{Q}(\bar{C})$-relatively FD,

(b) $\operatorname{ess\,sup}\{|P(C) \vee Q(C)| : C \in X \mid \zeta_0 \bmod \mu_{\zeta_0}\} < \infty$,
$\operatorname{ess\,sup}\{|\bar{P}(\bar{C}) \vee \bar{Q}(\bar{C})| : \bar{C} \in \bar{X} \mid \bar{\zeta}_0 \bmod \bar{\mu}_{\bar{\zeta}_0}\} < \infty$, and

(c) if $K = \operatorname{ess\,sup} |Q(C)|$ and $L = \operatorname{ess\,sup} |\bar{Q}(\bar{C})|$, then for all $t \geqq 0$, $\pi_1 \in \Pi_1(\max\{K, L\}), \pi_2 \in \Pi_2(\max\{K, L\}), \ldots, \mu_{\zeta_0}\{C : H(T_C) \leqq t, \operatorname{dist}(\bigvee_{i=0}^{n-1} T_C^{-i} Q(C)) \prec \pi_n; n = 1, 2, \ldots\} = \bar{\mu}_{\bar{\zeta}_0}\{\bar{C} : H(\bar{T}_{\bar{C}}) \leqq t, \operatorname{dist}(\bigvee_{i=0}^{n-1} \bar{T}_{\bar{C}}^{-i} \bar{Q}(\bar{C})) \prec \pi_n; n = 1, 2, \ldots\}$.

Then T and $\bar{T}$ are relatively isomorphic relative to the factors (T, Q) and $(\bar{T}, \bar{Q})$ such that $|Q| = |\bar{Q}| \leqq \min\{K, L\}$.

5. CLASSES OF PAIR PROCESSES

As mentioned in the introduction to [I] (cf. [I], pp. 362–363). Kieffer and Rahe [9] developed another approach to universal coding in ergodic theory. In this section we derive results in the spirit of [9] for the relative isomorphism setup.

Let T and $\bar{T}$ be non-ergodic automorphisms of Lebesgue spaces $(X, \mathscr{F}, \mu)$ and $(\bar{X}, \bar{\mathscr{F}}, \bar{\mu})$. Let $(T, Q) \sim (\bar{T}, \bar{Q})$ be fixed factor processes. Let $\mathscr{C} \in \mathscr{F}_{\zeta_0}$ and $\{T(C) : C \in \mathscr{C}\} \subset \{T_C : C \in X \mid \zeta_0\}$, $\{\bar{T}(C) : C \in \mathscr{C}\} \subset \{\bar{T}_{\bar{C}} : \bar{C} \in \bar{X} \mid \bar{\zeta}_0\}$. We suppose that for each $C \in \mathscr{C}$ there exist partitions $P(C)$ and $\bar{P}(C)$ such that $(T(C), P(C) \vee Q)$ is Q-relatively FD, $(\bar{T}(C), \bar{P}(C) \vee \bar{Q})$ is $\bar{Q}$-relatively FD, and $H(T(C), P(C) \vee Q) = H(\bar{T}(C), \bar{P}(C) \vee \bar{Q})$. In particular, the processes $(T(C), P(C) \vee Q)$ and $(\bar{T}(C), \bar{P}(C) \vee \bar{Q})$ are relatively isomorphic via an isomorphism denoted by $\varphi(C)$.

Let $C \in \mathscr{C}$. By our assumptions there exist sets $C_\sim \in X \mid \zeta_0$ and $\bar{C}_\sim \in \bar{X} \mid \bar{\zeta}_0$ such that $T(C) = T_{C_\sim}$ and $\bar{T}(C) = \bar{T}_{\bar{C}_\sim}$. The problem we address in this section is whether there is a single isomorphism $\varphi : X \to \bar{X}$ such that, if $C \in \mathscr{C}$ and $C_\sim \in X \mid \zeta_0$, $\bar{C}_\sim \in \bar{X} \mid \bar{\zeta}_0$ correspond to it, then $\varphi(C) = \varphi_{C_\sim}$ (i.e., $\varphi_{C_\sim}(C_\sim) = \bar{C}_\sim$).

In other words, we ask whether there is an isomorphism independent of $C \in \mathscr{C}$ such that, for each $C \in \mathscr{C}$, φ establishes the relative isomorphism between the processes $(T(C), P(C) \vee Q)$ and $(\bar{T}(C), \bar{P}(C) \vee \bar{Q})$.

On the first step we shall show there exist "universal" partitions R and $\bar{R}$ of the two spaces such that $(T(C), R \vee Q) = (T(C), P(C) \vee Q) \bmod 0$ on C, and similarly for the second transformation. That is, we construct pair processes for which the given classes constitute their ergodic decompositions. Then we shall apply Theorem 2 and construct from local isomorphisms $\varphi(C)$ a global isomorphism φ which will make the processes $(T, R \vee Q)$ and $(\bar{T}, \bar{R} \vee \bar{Q})$ relatively isomorphic. This will be the desired φ, for its restrictions in the sense described above in the preceding section will be just the prescribed $\varphi(C)$'s.

This way seems to be a little roundabout for one could directly define a probability measure $\tilde{\mu}$ on the space $(\mathscr{C}, \mathscr{F}_{\zeta_0} \cap \mathscr{C})$ which is the mixture of measures corresponding to the given class of pair processes. However, in this case we would have no means of control concerning the relations between $\tilde{\mu}$ and the original measure μ. Consequently, the resulting isomorphism could not be considered, in general, as an isomorphism from $(X, \mathscr{F}, \mu)$ to $(\bar{X}, \bar{\mathscr{F}}, \bar{\mu})$.

The existence of partitions R and $\bar{R}$ will be deduced from a universal selection theorem of Kieffer and Rahe [9]. To this end define

$$f(C, R) = \bar{d}_{Q,\bar{Q}}[(T(C), R \vee Q), (\bar{T}(C), \bar{P}(C) \vee \bar{Q})] .$$

Proposition 7. Suppose the hypotheses listed in the second paragraph of this section are satisfied. Then the function $f(C, R)$ has the following properties:

(a) for each fixed $C \in \mathscr{C}$, $R \mapsto f(C, R)$ is continuous relative to the strong partition distance, and for each fixed R, $C \mapsto f(C, R)$ is measurable,

(b) for each fixed $C \in \mathscr{C}$, for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $f(C, R) < \delta$ then there exists a partition $\tilde{R}$ with $f(C, \tilde{R}) < \varepsilon$ and $|\tilde{R} - R| < \delta$, and

(c) for each $C \in \mathscr{C}$ there is a partition $R(C)$ such that $f(C, R(C)) = 0$.

Proof. In order to prove (c) put $R(C) = P(C)$. The measurability assertion in (a) can be verified easily. Let us check continuity. By definition,

$$|f(C, R(k)) - f(C, R)| =$$

$$= \Big| \sup_{n \geqq 1} \inf_{\psi} n^{-1} \sum_{i=0}^{n-1} |\psi(T(C)^i R(k)) - \bar{T}(C)^i \bar{P}(C)| -$$

$$- \sup_{n \geqq 1} \inf_{\psi} n^{-1} \sum_{i=0}^{n-1} |\psi(T(C)^i R) - \bar{T}(C)^i \bar{P}(C)| \Big| .$$

If $|R(k) - R| \to 0$ then $|T(C)^i R(k) - T(C)^i R| \to 0$ for each $i \geqq 0$. Hence $|\psi(T(C)^i . R(k)) - \psi(T(C)^i R)| \to 0$, too. But Q and $\bar{Q}$ are common to all $C \in \mathscr{C}$ so that the infima in both expressions are over the same set. Consequently, from

$$|\psi(T(C)^i R(k)) - \bar{T}(C)^i \bar{P}(C)| \to |\psi(T(C)^i R) - \bar{T}(C)^i \bar{P}(C)|$$

it follows that $|f(C, R(k)) - f(C, R)| \to 0$. This proves (a).

Assertion (b) constitutes a relativized version of Ornstein's central copying argument [12]. It follows directly from the strong form of Thouvenot's relative isomorphism theorem due to Kieffer [41]. □

Remark 4. Thouvenot's original proof of (b) (see also Fieldsteel [39] for the flow case) is rather complicated. Kieffer [41] developed a simple proof based essentially on only the Slepian-Wolf theorem of information theory.

Theorem 8. Suppose the hypotheses listed at the beginning of this section are satisfied. Then there exists an isomorphism φ such that if $C \in \mathscr{C}$ and $C_\smile \in X \mid \zeta_0$ corresponds to C, then the processes $(T(C), P(C) \vee Q)$ and $(\bar{T}(C), \bar{P}(C) \vee Q)$ are relatively isomorphic via the isomorphism $\varphi_{C_\smile} : C_\smile \to \bar{C}_\smile$.

Proof. Conclusions (a) and (b) of Proposition 7 say that f is an admissible function in the sense of Kieffer and Rahe [9]. According to one of their universal selection theorems there is a partition R such that $f(C, R) = 0$ for all $C \in \mathscr{C}$, i.e.,

$$\bar{d}_{Q,\bar{Q}}[(T(C), R \vee Q), (\bar{T}(C), \bar{P}(C) \vee \bar{Q})] = 0 .$$

Combine this with the fact that $T(C) = T_{C_\smile}$. Then we have a pair process $(T, R \vee Q)$ such that, for each $C \in \mathscr{C}$, if $C_\smile \in X \mid \zeta_0$ corresponds to C, then the processes $(T_{C_\smile}, R_{C_\smile} \vee Q)$ and $(\bar{T}(C), \bar{P}(C) \vee \bar{Q})$ are relatively isomorphic. Since the argument is asymmetric, we conclude exactly in the same way that there exists a partition $\bar{R}$ such that for each $C \in \mathscr{C}$, the processes $(T(C), P(C) \vee Q)$ and $(\bar{T}_{C_\smile}, \bar{R}_{C_\smile} \vee \bar{Q})$ are relatively isomorphic. By triangle inequality (which can be used for all processes involved have common factors)

$$\bar{d}_{Q,\bar{Q}}[(\bar{T}_{C_\smile}, \bar{R}_{C_\smile} \vee \bar{Q}), (T_{C_\smile}, R_{C_\smile} \vee Q)] \leqq$$

$$\leqq \bar{d}_{Q,\bar{Q}}[(\bar{T}_{C_\smile}, \bar{R}_{C_\smile} \vee \bar{Q}), (T(C), P(C) \vee \bar{Q})] +$$

$$+ \bar{d}_{Q,\bar{Q}}[(T(C), P(C) \vee Q), (\bar{T}(C), \bar{P}(C) \vee \bar{Q})] +$$

$$+ \bar{d}_{Q,\bar{Q}}[(\bar{T}(C), \bar{P}(C) \vee \bar{Q}), (T_{C_\smile}, R_{C_\smile} \vee Q)] = 0 .$$

Consequently, for any $C \in \mathscr{C}$, if $C_{\smile} \in X \mid \zeta_0$ and $\bar{C}_{\smile} \in \bar{X} \mid \bar{\zeta}_0$ correspond to C, then the processes $(T_{C_{\smile}}, R_{C_{\smile}} \vee Q)$ and $(\bar{T}_{C_{\smile}}, \bar{R}_{C_{\smile}} \vee \bar{Q})$ are relatively isomorphic. It remains to apply Corollary 3. □

Remark 5. Our proof does not seem to work for processes having different factors for different $C \in \mathscr{C}$. Indeed, we used the assumption on the existence of common factors in the proof of continuity of the function f (Proposition 7) and in the above proof when using the triangle inequality. However, it seems likely that such a more general result will also be valid.

6. A CORRECTION NOTE

John Kieffer kindly pointed out there is an error in Section I.5, namely, that Lemmas I.4 and I.5 are false and consequently the proofs of Theorems I.4 and I.3 are not correct. In this section we give another proofs without using the quoted lemmas. We shall use notation and concepts from [1] without comments. As mentioned on p. 372 of [1], an isomorphism $\bar{\varphi} : A^Z \to B^Z$ is *finitary* if it has the following properties:

(a) $(\bar{\varphi}u)_0$ is determined by $u^{\tilde{n}(u)}_{-\tilde{n}(u)}$, where $\tilde{n} : A^Z \to N \cup \{\infty\}$ is a measurable function with $\mu\{u : \tilde{n}(u) < \infty\} = 1$, and

(b) $(\varphi^{-1}v)_0$ is determined by $v^{\tilde{m}(v)}_{-\tilde{m}(v)}$, where $\tilde{m} : B^Z \to N \cup \{\infty\}$ is a measurable function with $\varkappa\{v : \tilde{m}(v) < \infty\} = 1$.

Theorem I.3 asserts that if $[A, \mu]$, $[B, \varkappa]$ are aperiodic and isomorphic via a finitary code $\bar{\varphi} : A^Z \to B^Z$, then the local isomorphisms $\bar{\varphi}_\xi$, $\xi \in Q^A$, are also finitary. This follows easily from the canonical decomposition formula (I.21). Indeed, let $\tilde{E} \in \mathscr{F}^A_m$ correspond to the set $\{u \in A^Z : \tilde{n}(u) < \infty\} \in \mathscr{A}^Z$. In particular, $m^A(\tilde{E}) = 1$. By (I.21)

$$m^A(\tilde{E}) = \int_{Q^A} m_\xi(\tilde{E} \cap \xi)\, m^A_0(d\xi) = 1\,.$$

Since $m_\xi(\tilde{E} \cap \xi) \leqq 1$, the latter conclusion is possible only if

(c) $m^A_0\{\xi \in Q^A : m_\xi(\tilde{E} \cap \xi) = 1\} = 1\,.$

Similarly, if $\tilde{F} \in \mathscr{F}^B_m$ corresponds to the set $\{v \in B^Z : \tilde{m}(v) < \infty\}$ then

(d) $m^B_0\{\eta \in Q^B : m_\eta(\tilde{F} \cap \eta) = 1\} = 1\,.$

Now take $\xi \in Q^A$. Let $\bar{\varphi}_\xi = \bar{\varphi} \mid \xi$, the restriction of $\bar{\varphi}$ onto ξ, and let $\eta = \bar{\varphi}_\xi(\xi)$. By Theorem I.1, with probability one, $\bar{\varphi}_\xi : \xi \to \eta$ is an isomorphism. Since (c) and (d) are true, $\bar{\varphi}_\xi$ is finitary.

Theorem I.4 asserts the converse (see also Remark 2 above). Let Φ denote the set of all isomorphisms $\bar{\varphi}^Z \to B^Z$ which have the local components $(\bar{\varphi}_\xi; \xi \in Q^A)$. We assume that the $\bar{\varphi}_\xi$'s are finitary, i.e., (a) and (b) are satisfied for $\mu = m_\xi$, $\varkappa = m_\eta$ $(\eta = \bar{\varphi}_\xi(\xi))$, $\tilde{n} = \tilde{n}_\xi$, $\tilde{m} = \tilde{m}_\eta$, respectively. By Theorem I.2, $\Phi \neq \emptyset$. Suppose there does not exist a finitary isomorphism $\bar{\varphi} \in \Phi$. Suppose (a) is falsified for each $\bar{\varphi} \in \Phi$ (for (b) we can proceed by symmetry). Thus we find a set $E \in \mathscr{F}$ such that $\mu(E) > 0$ and $\tilde{n}(u) = \infty$ for any $u \in E$. Let $\tilde{E} \in \mathscr{F}_m^A$ correspond to E; in particular, $m^A(\tilde{E}) = \mu(E) > 0$. By (I.21)

(e) $m_0^A\{\xi \in Q^A : m_\xi(\tilde{E} \cap \xi) > 0\} > 0$.

Indeed, if $m_\xi(\tilde{E} \cap \xi) > 0$ only with m_0^A-probability zero, then $m_\xi(\tilde{E} \cap \xi) = 0$ m_0^A-almost everywhere, and this by (I.21) leads to $m^A(\tilde{E}) = 0$; a contradiction. Take ξ from the set in (e), i.e.., $m_\xi(\tilde{E} \cap \xi) > 0$. Since $\bar{\varphi}_\xi$ is finitary, we have $m_\xi\{u : \tilde{n}_\xi(u) < \infty\} = 1$. Hence

(f) $m_\xi[(\tilde{E} \cap \xi) \cap \{u : \tilde{n}_\xi(u) < \infty\}] = m_\xi(\tilde{E} \cap \xi) > 0$.

But if $u \in \tilde{E} \cap \xi$ then $(\bar{\varphi}u)_0 = (\bar{\varphi}_\xi u)_0$ is determined only when knowing the entire sequence u. Thus, we cannot have (f). This implies

$$m_0^A\{\xi \in Q^A : m_\xi[(\tilde{E} \cap \xi) \cap [\tilde{n}_\xi < \infty]] > 0\} = 0,$$

i.e.,

$$m_0^A\{\xi \in Q^A : m_\xi[(\tilde{E} \cap \xi) \cap [\tilde{n}_\xi < \infty]] = 0\} = 1.$$

Since $m_\xi[\tilde{n}_\xi < \infty] = 1$, we get

$$m_0^A\{\xi \in Q^A : m_\xi(\tilde{E} \cap \xi) = 0\} = 1.$$

Hence, again using (I.21)

$$m^A(\tilde{E}) = \int_{Q^A} m_\xi(\tilde{E} \cap \xi) m_0^A(d\xi) = 0,$$

a contradiction showing that $\bar{\varphi}$ is a finitary code. Since the same arguments apply to $\bar{\varphi}^{-1}$, $\bar{\varphi}^{-1}$ also is a finitary code, i.e., $\bar{\varphi}$ is a finitary isomorphism.

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REFERENCES

[I] Š. Šujan: A local structure of stationary perfectly noiseless codes between stationary non-ergodic sources. Part I: General considerations. Kybernetika *18* (1982), 361—375.

[II] Š. Šujan: A local structure of stationary perfectly noiseless codes between stationary non-ergodic sources. Part II: Applications. Kybernetika *18* (1982), 465—484.

[34] Š. Šujan: Epsilon-rates and noiseless fixed-rate block coding for stationary non-ergodic sources. Elektron. Informationsverarb. Kybernet. (EIK) *19* (1983), 375—385.

[35] Š. Šujan: Generators for amenable group actions. Monatsh. für Math. *95* (1983), 67—79.

[36] Š. Šujan: Codes in ergodic theory and information: some examples. In: Ergodic Theory and Related Topics (H. Michel, ed.), Akademie Verlag, Berlin 1982, 181—184.

[39] A. Fieldsteel: The relative isomorphism problem for Bernoulli flows. Israel J. Math. *40* (1981), 197—216.

[40] R. M. Gray and L. D. Davisson: The ergodic decomposition of stationary discrete random processes. IEEE Trans. Inform. Theory *IT-20* (1974), 625—636.

[41] J. C. Kieffer: A simple derivation of Thouvenot's relative isomorphism theorem. Preprint, Univ. of Missouri — Rolla 1982.

[42] V. A. Rokhlin: Lectures on entropy theory of transformations with invariant measure (in Russian). Uspekhi mat. nauk *22* (1967), vyp. 5 (137), 4—56.

[43] Š. Šujan: Ergodic theory, entropy, and coding problems of information theory. Kybernetika *19* (1983), supplement, 68 pp.

[44] J.-P. Thouvenot: Quelques proprietes des systémes dynamiques qui se decomposent en un produit de deux systémes dont l'un est un schema de Bernoulli. Israel J. Math. *21* (1975), 177—207.

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