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MINIMAL DEGREE SOLUTIONS OF POLYNOMIAL EQUATIONS

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We study the general Bezout equation $A_1X_1 + \ldots + A_rX_r = C$, for A_i, C in $k[x, \ldots, x_n]$, $k = \mathbb{R}$ or $\mathbb{C}$, and we provide minimal degree solutions for it. The results are also extended to the case of A_i, C distinguished polynomials in spaces of entire functions with growth conditions.

1. INTRODUCTION

The study of the so called Bezout equation $AX + BY = 1$, for A, B given polynomials and X, Y unknown polynomials, has been under intense scrutiny, in recent years, due to the interest it has in concrete problems concerning multidimensional systems and control theory. References to these connections can be found, e.g., in [6], [7], [10]. In the same direction, the use of difference-differential equations in the control of delay-differential systems has led to the study of more sophisticated situations in which the classical Bezout equation is replaced by

$$A_1X_1 + \ldots + A_rX_r = C\,, \tag{1}$$

where A_i, C are now holomorphic function of a given growth at infinity (we will always suppose, in the sequel, that any common factor has already been cancelled from (1)). The importance of the study of such an equation is well discussed, e.g., in [5], to which we refer the interested reader. Solvability conditions, and the explicit construction of solutions to (1), even in the more complex case which arises when the A_i are substituted by matrices of entire functions, are now well known, and studied in [2], [3], [9]; on the other hand, a recent paper of Šebek in this same journal, [11], has attacked the question of finding, in the polynomial case, minimal degree solutions, and these results have provided algorithms to solve (1). More precisely, Šebek has considered polynomials in two variables and has shown that, whenever $r = 2$ and $A_1X_1 + A_2X_2 = C$ has a solution, it also admits a unique minimal degree solution, i.e. a solution whose degree can be given a priori bounds which only depend on $\deg(A)$, $\deg(B)$ and $\deg(C)$.

Purpose of this short note is to extend some of Šebek's ideas to a more general situation, i.e. to the case in which $r > 2$, or the A_i, C are distinguished polynomials with respect to one variable (see Section 3, after Remark 9, for the precise definition) as well as to the case of polynomials in more than two variables.

First, in Section 2, we recall some known results due to Hörmander, [9], which enable us to completely characterize the space of solutions $X_1, \ldots, X_r$ to $A_1X_1 + \ldots + A_rX_r = 0$ in spaces of entire functions with growth conditions; these results are used, in Section 3, to describe the general solutions to $A_1X_1 + \ldots + A_rX_r = C$ (Theorem 5 and Corollary 6). The major applications we are interested in are contained in Theorems 8, 11, and Corollary 12. In Theorem 7 we show the existence of minimal degree solutions to (1), for the simple case of polynomials in one variable; in Theorem 8, this result is proved for polynomials in two variables, but in the subsequent Remark 9, (b), we show how the result can actually be extended to more than two variables, while Theorem 11 and Corollary 12 deal with the case in which the A_i, X_i and C are distinguished polynomials in $\mathscr{H}(\mathbb{C}^n)$ or in $\operatorname{Exp}(\mathbb{C}^n)$, the space of entire functions of exponential type; we would like to notice that this seems particularly interesting in view of some quite concrete applications, and is based on a (relatively little known) global version of the Weierstrass division theorem with bounds, [1]. Several examples are also given, to illustrate the situation.

2. KOSZUL COMPLEX

In this section we will briefly recall some results due to Hörmander, [9], which will be useful to study the space of solutions of the Bezout equation (1).

Even though we will be mainly interested in the case of polynomial equations, we will state the results in a more general situation, in view of their possible applications to control theory.

Let p be a plurisubharmonic function on $\mathbb{C}^n$ (i.e. p is upper semi-continuous and for each complex line L of $\mathbb{C}^n$, $p|_L$ is subharmonic on L) and suppose it satisfies the following technical conditions:

(i) $p(z) \geqq 0$ and $\log(1 + |z|) = O(p(z))$;

(ii) there exist constants $K_1, K_2, K_3, K_4 > 0$ such that if $|z_1 - z_2| \leqq \exp(-K_1p(z_1) - K_2)$, then $p(z_2) \leqq K_3p(z_1) + K_4$.

According to Hörmander, [9], denote by $A_p = A_p(\mathbb{C}^n)$ the algebra of all entire functions $f \in \mathscr{H}(\mathbb{C}^n)$, for which there are positive constants $A, B > 0$ such that

$$|f(z)| \leqq A \exp(Bp(z)), \quad \text{for all} \quad z \in \mathbb{C}^n .$$

The cases which are more interesting for the applications occur when $p(z) = \log(1 + |z|)$ (A_p is then the space $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$ of polynomials in $\mathbb{C}^n$), when $p(z) = |\operatorname{Im} z| + \log(1 + |z|)$ (A_p is the space of Fourier transforms of compactly supported distributions in $\mathbb{R}^n$) and when $p(z) = |z|$ (here A_p is the space of Fourier-Borel transforms of analytic functionals in $\mathbb{C}^n$).

Let then $F_1, \ldots, F_N$ belong to $A_p(\mathbb{C}^n)$ and suppose they satisfy the "corona condition"

$$(2) \qquad \sum_{i=1}^{N} |F_i(z)| \geqq \varepsilon \exp(-Cp(z)) \text{ for all } z \in \mathbb{C}^n \text{ and some } \varepsilon, C > 0.$$

As it is well known, this condition is both necessary and sufficient for the solvability, in A_p, of the equation $F_1X_1 + \ldots + F_NX_N = 1$, [9]; it will turn out that (2) also plays a role in the study of the solutions for $F_1X_1 + \ldots + F_NX_N = 0$. Consider the map $P_F: (A_p)^N \to A_p$ defined by $P_F(g_1, \ldots, g_N) = F_1g_1 + \ldots + F_Ng_N$. In order to study the kernel of such a map, one introduces its Koszul complex as follows: let L_r^s denote the set of all differential forms h of type $(0, r)$ with values in the exterior algebra $\Lambda^s\mathbb{C}^N$ and such that, for some $K > 0$,

$$\int |h(z)|^2 \exp(-2Kp(z))\, \mathrm{d}\lambda < +\infty ,$$

where $\mathrm{d}\lambda$ denotes the Lebesgue measure.

The Cauchy-Riemann operator $\bar{\partial}$ defines a map from all of L_r^s to L_{r+1}^s ($\bar{\partial}h$ is defined in the sense of distributions on each component of h). Notice that the map $P_F: A_p^N \to A_p$ extends naturally to a map from L_0^1 to L_0^0 and, more generally, to a map (which we will denote again by P_F) from L_r^{s+1} to L_r^s defined by

$$(P_Fg)_I = \sum_{j=1}^{N} g_{Ij}F_j \quad \text{for} \quad g \in L_r^{s+1}, \quad |I| = s .$$

It is clear that P_F and $\bar{\partial}$ make L_r^s into a double complex (i.e. $P_F^2 = \bar{\partial}^2 = 0$ and $P_F\bar{\partial} = \bar{\partial}P_F$), and the main result proved by Hörmander states, [9]:

Theorem 1. For every $g \in L_r^s$ with $\bar{\partial}g = P_Fg = 0$, one can find $h \in L_r^{s+1}$ such that $\bar{\partial}h = 0$ and $P_Fh = g$.

The case we are interested in is when $r = 0$, $s = 1$; in this case $g \in A_p^N$ (since $\bar{\partial}g = 0$) and the theorem gives a simple characterization of the kernel of P_F; in order to clarify the statement of the theorem, let us consider the simple case in which $N = 3$. Then if $g \in L_0^1$, we have $g = (g_1, g_2, g_3)$, with g_i in A_p and $P_Fg = 0$ means $g_1F_1 + g_2F_2 + g_3F_3 = 0$; on the other hand if h in L_0^2 is $\bar{\partial}$-closed, it is $h = (h_{12}, h_{13}, h_{23})$ with h_{ij} in A_p. Finally $P_Fh = (h_{12}F_2 + h_{13}F_3, -h_{12}F_1 + h_{23}F_3, -h_{13}F_1 - h_{23}F_2)$, therefore Theorem 1 says that if $g \in A_p^3$ and $P_Fg = 0$, then

$$g = (\alpha F_2 + \beta F_3, -\alpha F_1 + \gamma F_3, -\beta F_1 - \gamma F_2) ,$$

for some α, β, γ in A_p. But

$$(\alpha F_2 + \beta F_3, -\alpha F_1 + \gamma F_3, -\beta F_1 - \gamma F_2) =$$
$$= \alpha(F_2, -F_1, 0) + \beta(F_3, 0, -F_1) + \gamma(0, F_3, -F_2) ,$$

i.e. the kernel of P_F is given by all the combinations (with functions in A_p) of its "obvious elements". Theorem 1 states that this situation occurs for all N.

Remark 2. In the case of polynomials, i.e. when $p(z) = \log(1 + |z|)$, condition (2) is known to be equivalent to the fact that the polynomials F_i have no common zeroes (this is an immediate consequence of Hilbert's Nullstellensatz), so that Theorem 1 describes the kernel of the map P_F for $F_1, \ldots, F_N$ polynomials with complex coefficients and no common zeroes in $\mathbb{C}^n$.

Remark 3. For complex polynomials, weaker conditions are sufficient to imply the conclusion of Theorem 1; indeed if $F_1, \ldots, F_N$ form a *regular sequence* (i.e. F_j is not a zero divisor in $\mathbb{C}[z_1, , \ldots, z_n]/(F_1, \ldots, F_{j-1})$ for all $j = 2, \ldots, N$, [12]), one knows, [8], that the variety $V = \{z \in \mathbb{C}^n: F_1(z) = \ldots = F_N(z) = 0\}$ is a complete intersection and that the kernel of P_F is trivially generated as in Theorem 1; the same holds true for the case of entire functions with no growth conditions: we refer the reader interested in these questions to [8], Chapter 5. On the other hand, if one is interested in entire functions in A_p and condition (2) looks too strong, one can still generalize the notion of regular sequence in such a way that the cohomology of the Koszul complex described before is trivial; this has been done (in order to study some related problems in harmonic analysis) in [4], and has led to the notion of *slowly decreasing* N-tuple of elements in A_p: for such N-tuples (we refer to [4] for the definition, which is quite complicated) the conclusion of Theorem 1 still holds.

For practical applications, it is often useful to consider the Bezout equation for polynomials with real coefficients in $\mathbb{R}^n$. It is therefore of interest the following

Corollary 4. Let $g \in [\mathbb{R}[x_1, \ldots, x_n]]^N$ be such that $F_1 g_1 + \ldots F_N q_N = 0$ for $F_1, \ldots, F_N$ a regular sequence in $\mathbb{R}[x_1, \ldots, x_n]$. Then there is h in $[\mathbb{R}[x_1, \ldots, x_n]]^{\bar{N}}$ such that $P_F h = g$, for $\bar{N} = \binom{N}{2}$.

Proof. It is enough to separate the real and the imaginary part in equation (1), and to look at the case $r = 0, s = 1$ of Theorem 1, in view of Remark 3. □

3. MINIMAL DEGREE SOLUTIONS

In [10], Šebek studies Bezout equations of the form $AX + BY = C$, where A, B, C are given polynomials in $\mathbb{R}[x_1, x_2]$ and X, Y are to be found in $\mathbb{R}[x_1, x_2]$, and his main tool is the knowledge of a simple description of all solutions X, Y of $AX + BY = 0$.

Here we employ the results stated in Section 2, to study the more general Bezout equation

(1) $$A_1 X_1 + \ldots + A_r X_r = C$$

with A_i, X_i, C in $A_p(\mathbb{C}^n)$. and A_i satisfying (for example) condition (2). Because of the linearity of (1), we can easily prove:

Theorem 5. Suppose $A_i \in A_p(\mathbb{C}^n)$, $i = 1, \dots, r$, satisfy (2), and let $X' = (X'_1, \dots \dots, X'_r)$ be any particular solution of (1). Then the general solution of (1) is

$$X = X' + P_A H$$

with H in $(A_p)^{\bar{r}}$, for $\bar{r} = \binom{r}{2}$.

Proof. It follows from the linearity of (1) and from Theorem 1. □

Notice that, in Theorem 5, condition 2 could be substituted with the request that the A_i form a slowly decreasing r-tuple, as pointed out in Remark 3.

Corollary 6. Let A_i, $C \in k[x_1, \dots, x_n]$, $i = 1, \dots, r$, $k = \mathbb{R}$ or $\mathbb{C}$, and suppose that the A_i have no common zeroes or that they form a regular sequence. Let $X' = (X'_1, \dots, X'_r)$ be a particular solution of (1). Then the general solution of (1) is

$$X = X' + P_A H$$

with H in $k[x_1, \dots, x_n]^{\bar{r}}$, $\bar{r} = \binom{r}{2}$.

Proof. In case the A_i have no common zeroes, it is an immediate consequence of Theorem 5; if, on the other hand, the A_i form a regular sequence, the result follows from Remark 3 and Corollary 4. □

We now turn to the discussion of minimal degree solutions for equation (1). In the case of $n = 1$ variable, the situation is quite simple, as we can employ the Euclidean division algorithm to prove:

Theorem 7. Consider equation (1) in $k[x]$, $k = \mathbb{R}$ or $\mathbb{C}$, with the A_i without common zeroes. Let $d_r = \deg(A_r) \leqq \deg(A_i) = d_i$ for all $i = 1, \dots, r$. Then there exists a solution $X = (X_1, \dots, X_r)$ to (1) with $\deg(X_i) \leqq d_r - 1$ for all $i = 1, \dots, r - 1$. Moreover if $\deg(C) \leqq \max(d_i) + d_r - 1$, one has that X can be chosen with $\deg(X_r) \leqq \max(d_i) - 1$.

Proof. By Corollary 6, every solution of (1) can be written as $X = X' + P_A H$, for X' a particular solution, and H any element of $k[x]^{\bar{r}}$ (the existence of X' is a consequence of the fact that the A_i do not have any common zero, as we remarked in Remark 2). Let then

$$H = (H_{12}, H_{13}, \dots, H_{1r}, H_{23}, H_{24}, \dots, H_{2r}, H_{34}, \dots, H_{r-1,r}),$$

so that

$$P_A H = (H_{12}A_2 + \dots + H_{1r}A_r, -H_{12}A_1 + \dots + H_{2r}A_r, \dots, -H_{1r}A_1 - \dots \dots -H_{r-1,r}A_{r-1}).$$

Now

$$X_1 = X'_1 + H_{12}A_2 + \dots + H_{1r}A_r;$$

by applying the Euclidean division algorithm to X'_1 we get (if $\deg(X'_1) \leqq d_r$, since otherwise X'_1 is already as required by the thesis)

$$X'_1 = A_r A_r^{11} + A_r^{12}\,, \quad \deg(A_r^{12}) \leqq d_r - 1\,,$$

i.e.

$$X_1 = A_r^{12} + H_{12}A_2 + \ldots + (H_{1r} + A_r^{11})\,A_r\,.$$

Since the H_{ij} are arbitrary polynomials, we get $X_1 = A_r^{12}$ by taking $H_{1j} = 0$ for $j = 2, \ldots, r-1$ and $H_1 r = -A_r^{11}$. With this choice we have (as $H_{12} = 0$)

$$X_2 = X'_2 + H_{23}A_3 + \ldots + H_{2r}A_r\,;$$

repeat the argument dividing X'_2 by A_r:

$$X'_2 = A_r A_r^{21} + A_r^{22}\,, \quad \deg(A_r^{22}) \leqq d_r - 1$$

i.e.

$$X_2 = A_r^{22} + H_{23}A_3 + \ldots + (H_{2r} + A_r^{21})\,A_r\,;$$

now take $H_{2j} = 0$ for $j = 3, \ldots, r-1$ and $H_{2r} = -A_r^{21}$.

By repeating this argument r times we get the solution

$$\begin{aligned} X_1 &= A_r^{12} \\ X_2 &= A_r^{22} \\ &\vdots \\ X_{r-1} &= A_r^{r-1,2} \\ X_r &= X'_r + A_r^{11}A_1 + \ldots + A_r^{r-1,1}A_{r-1} \end{aligned}$$

which satisfies the first part of the thesis. The second part is now trivial. □

We now give an example in which this method is worked out explicitly:

Example. Let $r = 3$, and $A_1 = x^2 + x + 1$, $A_2 = (x+1)^3$, $A_3 = (x-1)^2$, $C = -x^4 - 4x^3 + x^2 - 2x + 1$. It is immediate to verify that $X' = (x^3, -x^2, x^2 + 1)$ is a particular solution of the equation $A_1X_1 + A_2X_2 + A_3X_3 = C$. Therefore a general solution of this same equation is given by

$$\begin{aligned} X_1 &= x^3 + H_{12}(x^3 + 3x^2 + 3x + 1) + H_{13}(x^2 - 2x + 1) \\ X_2 &= -x^2 - H_{12}(x^2 + x + 1) + H_{23}(x^2 - 2x + 1) \\ X_3 &= x^2 + 1 - H_{13}(x^2 + x + 1) - H_{23}(x^3 + 3x^2 + 3x + 1)\,, \end{aligned}$$

for H_{12}, H_{13}, H_{23} any polynomials in $\mathbb{R}[x]$.

Divide X'_1 by A_3:

$$x^3 = (x^2 - 2x + 1)(x + 2) + (3x - 2)$$

with $A_3^{11} = x + 2$ and $A_3^{12} = 3x - 2$. Taking $H_{23} = -(x+2)$ and $H_{12} = 0$, we have the solution

$$\begin{aligned} X_1 &= 3x - 2 \\ X_2 &= -x^2 + H_{23}(x^2 - 2x + 1) \\ X_3 &= x^2 + 1 + (x + 2)(x^2 + x + 1) - H_{23}(x^3 + 3x^2 + 3x + 1)\,. \end{aligned}$$

Repeat the argument for $-x^2$, i.e. divide $-x^2$ by A_3 and get

$$-x^2 = (x^2 - 2x + 1)(-1) + (-2x + 1)$$

with $A_3^{21} = -1$, $A_3^{22} = -2x + 1$. Taking now $H_{23} = 1$, we finally obtain

$$X_1 = 3x - 2$$
$$X_2 = -2x + 1$$
$$X_3 = x^2 + 2$$

which is the solution which satisfies the thesis of Theorem 7.

We now turn to the case of $n = 2$ variables.

Theorem 8. Let A_i, $C \in k[u, w]$, $i = 1, \ldots, r$, $k = \mathbb{R}$ or $\mathbb{C}$, and suppose that the A_i's have no common zeroes or that they form a regular sequence. Let the equation (1) be solvable. Consider A_i, C $(i = 1, \ldots, r)$ as polynomials in $k[u][w]$ of w-degrees d_i, d respectively:

$$A_i = a_0^{(i)} + a_1^{(i)} w + \ldots + a_{d_i}^{(i)} w^{d_i} \quad (i = 1, \ldots, r)$$
$$C = c_0 + c_1 w + \ldots + c_d w^d .$$

If

(3) $$\gcd(a_{d_i}^{(i)}, a_{d_j}^{(j)}) = 1$$

for all i, j such that $i \neq j$, and if

$$\bar{d} = \max_{i \neq j} \{d_i + d_j, d\}$$

then there exists a solution $X = (X_1, \ldots, X_r)$ of equation (1), such that

$$\deg_w(X_i) \leqq \bar{d} - d_i \quad (i = 1, \ldots, r) .$$

Proof. By hypothesis, there exists a solution $X' = (X'_1, \ldots, X'_r)$ of equation (1). Set $\deg_w(X'_i) = k_i$ and

$$X'_i = x_0^{(i)} + x_1^{(i)} w + \ldots + x_{k_i}^{(i)} w^{k_i} \quad (i = 1, \ldots, r) .$$

If there exists $1 \leqq i \leqq r$ such that $k_i + d_i > \bar{d}$, then it is not possible that $k_i + d_i \geqq \geqq k_j + d_j$ for all $i \neq j$, $1 \leqq j \leqq r$; in fact, if this is the case, we obtain from (1)

$$x_{k_i}^{(i)} a_{d_i}^{(i)} w^{k_i + d_i} \equiv 0$$
$$x_{k_i}^{(i)} a_{d_i}^{(i)} = 0$$

which contradicts the assumption on the degrees of X'_i or A_i. Therefore we can assume that (up to a permutation) there exists $2 \leqq s \leqq r$ such that

$$k_1 + d_1 = \ldots = k_s + d_s = \max_{1 \leqq p \leqq r} \{k_p + d_p\} = M > \bar{d}$$

and that

$$k_q + d_q < M ,$$

for $s < q \leqq r$. Let us consider, in equation (1), the coefficient of w^M. We obtain the equation

$$a_{d_1}^{(1)} x_{k_1}^{(1)} + \ldots + a_{d_s}^{(s)} x_{k_s}^{(s)} = 0$$

i.e.

$$a_{d_1}^{(1)} x_{k_1}^{(1)} = -\left(a_{d_2}^{(2)} x_{k_2}^{(2)} + \ldots + a_{d_s}^{(s)} x_{k_s}^{(s)}\right),$$

and, by (3), we obtain that

$$x_{k_1}^{(1)} = \alpha^{(2)} a_{d_2}^{(2)} + \ldots + \alpha^{(s)} a_{d_s}^{(s)} \tag{4}$$

for some $\alpha^{(i)}$ in $k[u]$, $i = 1, \ldots, s$. Nevertheless we know, by Corollary 6, that the general solution of (1) is $X = X' + P_A H$, with in $[k[u, w]]^{\bar{r}} = [k[u]\,[w]]^{\bar{r}}$. Hence we can write

$$\begin{aligned} X_1 &= X_1' + H_{12}A_2 + H_{13}A_3 + \ldots + H_{1r}A_r \\ X_2 &= X_2' - H_{12}A_1 + H_{23}A_3 + \ldots + H_{2r}A_r \\ &\vdots \\ X_s &= X_s' - H_{1s}A_1 - \ldots - H_{s-1,1}A_{s-1} + H_{s,s+1}A_{s+1} + \ldots + H_{s,r}A_r \\ &\vdots \\ X_r &= X_r' - H_{1r}A_1 - \ldots - H_{r-1,r}A_r\,. \end{aligned} \tag{5}$$

It turns out that $k_i - d_j > 0$ for all $i \neq j$, $1 \leqq i \leqq s$, $1 \leqq j \leqq s$. In fact we have, for such i and j,

$$k_i + d_i > \bar{d} \geqq \max_{s \neq k} \{d_s + d_k\} \geqq d_i + d_j\,,$$

which implies $k_i > d_j$. Let us now choose, in (5), $H_{12} = -\alpha^{(2)} w^{k_1 - d_2}, \ldots, H_{1s} = -\alpha^{(s)} w^{k_1 - d_s}$, and $H_{ij} = 0$ for $i \neq 1$ or $j \neq 2, \ldots, s$. By (4), we obtain a new solution $(X_1'', \ldots, X_r'')$ of (1) with $\deg_w X_1'' \leqq k_1 - 1$. Moreover for $i = 2, \ldots, s$ we obtain that

$$\deg_w (H_{1i}A_1) \leqq k_1 - d_i + d_1 = k_i + d_i - d_i = k_i\,.$$

and therefore that

$$\deg_w (X_i'') \leqq k_i$$

for $i = 2, \ldots, r$. The assertion of the theorem now follows by iterating the argument above. □

Remark 9. (a) Notice that, as Šebek did in the case $r = 2$, [11], hypothesis (3) in the theorem above can be weakened by requiring only conditions similar to (5b), Theorem 2 of [11].

(b) It is, of course, interesting to ask whether Theorem 8 can be extended to $n \geqq 2$ variables. If one follows through the proof, it is clear that the restriction $n = 2$ is only used to obtain (4), so that our theorem actually holds for all n, if one only assumes that all subsets of $a_{d_1}^{(1)}, \ldots, a_{d_s}^{(s)}$ form a regular sequence in the sense of [12].

(c) We finally observe that the hypothesis on the A_i forming a regular sequence, is not necessary since it is used to prove that all solutions of (1) are of the form (5), while all we need is to use the fact that all r-tuples as in (5) are solutions of (1).

Let us now apply our procedure in a concrete situation:

Example. Let $r = 3$, and let $A_1 = 1 + uw$, $A_2 = u^2 + w^2$, $A_3 = 1 + (1 + u)\,w$, $C = 1 + (1 + u + 2u^2)\,w + (1 + u^3 - u^2)\,w^2 + (u - u^3)\,w^3$. The hypotheses of Theorem 8 are obviously verified, and a particular solution of the associated Bezout equation is given by $X_1' = u + u^2w + w^2 + w^4$, $X_2' = (-w^2 - uw^3)$ and $X_3' = 1$. According to Theorem 8 we obtain a new solution by taking

$$X_1 = u + u^2w + w^2 + w^4 + (-w^2)(u^2 + w^2) = u + u^2w + (1 - u^2)w^2$$

$$X_2 = -w^3 - uw^3 + (w^2)(1 + uw) = 0$$

$$X_3 = 1\,.$$

It is now immediate to verify that (X_1, X_2, X_3) is a solution of equation (1) and that $\deg_w(X_i) \leqq \deg(C) - \deg(C) - \deg(A_i)$, $i = 1, 2, 3$.

We now proceed to give a final application of the previous results by considering the equation

$$A_1X_1 + \ldots + A_rX_r = C \tag{6}$$

with A_i, C distinguished polynomials in $\mathscr{H}(\mathbb{C}^n)$, the space of entire functions, or in $\mathrm{Exp}(\mathbb{C}^n) = A_p(\mathbb{C}^n)$, $p(z) = |z|$, the space of entire functions of exponential type (but other spaces A_p of entire functions might as well be considered); by this we mean that if $\zeta = (z, w) \in \mathbb{C}^n$, $z \in \mathbb{C}^{n-1}$, $w \in \mathbb{C}$, there are entire functions (in $\mathbb{C}^{n-1}$) $a_i^{(j)}(z)$, $c_k(z)$, $1 \leqq j \leqq r$, $0 \leqq i \leqq d_j = \deg(A_j)$, $0 \leqq k \leqq t = \deg(C)$, such that

$$A_j(\zeta) = \sum_{i=0}^{d_j} a_i^{(j)}(z)\,w^i\,, \quad C(\zeta) = \sum_{k=0}^{t} c_k(z)\,w^k\,.$$

We are interested in finding solutions X_i to (6) which, too, are distinguished polynomials in $\mathscr{H}(\mathbb{C}^n)$, and for which the degree in w is minimal.

Remark 10. For our next results, we need to use a global version of the Weierstrass division theorem which can be found, e.g., in [1], Lemma 1, page 96. What is most interesting in the treatment of [1] is the fact that very precise bounds are given on the growth of the quotient and the remainder of the division of an entire function by a distingushed polynomial. More precisely, the bounds given in [1], Lemma 3, together with Remark 5 and the proof of Lemma 5, show that if F and P belong to $\mathrm{Exp}(\mathbb{C}^n)$, and P is a monic distinguished polynomial, then $F = GP + R$, with $G, R \in \mathrm{Exp}(\mathbb{C}^n)$, R a distinguished polynomial of degree smaller than P.

Theorem 11. Suppose that (6) has a solution in $\mathscr{H}(\mathbb{C}^n)$, and suppose that $a_{d_r}^{(r)} = 1$. If $d_r \leqq d_j$ for all j, there exists a distinguished polynomial solution to (6) (i.e. in which all X_i are distinguished with respect to w), with $\deg(X_i) \leqq d_r - 1$ for all $i = 1, \ldots$

..., $r-1$. Moreover if $\deg(C) \leqq \max_i (d_i) + d_r - 1$, X_r can be chosen with $\deg(X_r) \leqq \max_i (d_i) - 1$.

Proof. It runs exactly as in Theorem 7, with the Euclidean algorithm replaced by the Weierstrass division theorem. □

Corollary 12. The same result holds true if A_i and C belong to $\mathrm{Exp}(\mathbb{C}^n)$. In this case the distinguished polynomials X_i can be found in $\mathrm{Exp}(\mathbb{C}^n)$ as well.

Proof. As for Theorem 11. We now use the Weierstrass division theorem with bounds as discussed in Remark 10. □

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