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ON THE SYNTACTICO-SEMANTICAL COMPLETENESS OF FIRST-ORDER FUZZY LOGIC

Part II. Main Results

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This paper is a continuation of Part I [9]. First, the extension of fuzzy theories is studied and the important deduction theorem is proved which is a generalisation of the corresponding classical one. In Section 7.3, algebraic properties of the set of formulae are studied. This section serves as a preparation for the proofs of the most important theorems in this paper which are the completeness theorems being generalisation of the famous Gödel's ones in classical logic. At last, some theorems on completion of fuzzy theories are proved.

7. EXTENSION, COMPLETENESS AND COMPLETION OF FUZZY THEORIES

In this section we continue the study of the properties of fuzzy theories of first order. Some important notions of classical logic are generalised and well behaviour of them is demonstrated. We use the notions and notation introduced in [9].

7.1 Extension of fuzzy theories

A language J' is an *extension of* J if $J \subseteq J'$. Obviously, in this case $F_J \subseteq F_{J'}$ and $T = \langle A_L, A_S, R\rangle$, $T' = \langle A'_L, A'_S, R\rangle$ be theories in the respective languages. Put $\bar{A}_S(A) = A_S(A)$ if $A \in F_J$ and $\bar{A}_S(A) = \mathbf{0}$ otherwise. If

$$\bar{A}_S \subseteq A'_S$$

then T' is an *extension* of T. To simplify the notation, we will write A_S instead of $\bar{A}_S$ and understand that $A_S(A) = \mathbf{0}$ for all $A \in F_{J'} - F_J$.

The extension T' is a *conservative extension of* T if $T' \vdash_b A$ and $T \vdash_a A$ implies $a = b$ for every formula $A \in F_J$. The extension T' is a *simple extension* of T if $J(T') = J(T)$.

Lemma 18. Let T' be an extension of T. If $T \vdash_a A$ and $T' \vdash_b A$ then $a \leqq b$. If T' is consistent then T is consistent as well.

Proof. Obvious. □

Lemma 19. Let T' be an extension of T and T' has a model $\mathscr{D}'$. Then the restriction of $\mathscr{D}'$ to $J(T)$ is a model $\mathscr{D}$ of the theory T and

$$\mathscr{D}'(A) = \mathscr{D}(A)$$

holds for every formula $A \in F_{J(T)}$.

Proof. The structure $\mathscr{D}$ for the language $J(T)$ originates from $\mathscr{D}'$ by excluding some fuzzy relations corresponding to predicates which are not in $J(T)$ and, hence, in A. Moreover, $D = D'$ and so $J(T)$ contains all the names $\mathbf{d}$ of all the elements $d \in D'$. We prove that $\mathscr{D}(A) = \mathscr{D}'(A)$ holds for every formula $A \in F_{J(T)}$. Let $t \in J(T)$ be a constant. Then $\mathscr{D}(t) = \mathscr{D}'(t)$. Let $p(t_1, \ldots, t_n) \in F_{J(T)}$ be an atomic formula. Then

$$\mathscr{D}p(t_1, \ldots, t_n)) = p_D(\mathscr{D}(t_1), \ldots, \mathscr{D}(t_n)) = p_{D'}(\mathscr{D}'(t_1), \ldots, \mathscr{D}'(t_n) = \\ = \mathscr{D}'(p(t_1, \ldots, t_n)) .$$

Clearly,

$$\mathscr{D}'(\boldsymbol{a}) = \mathscr{D}(\boldsymbol{a})$$

for every $a \in L$ by the definition. Let $A := B \Rightarrow C$, $A \in F_{J(T)}$ and assume the proposition holds for B and C. Then

$$\mathscr{D}(A) = \mathscr{D}(B) \rightarrow \mathscr{D}'(C) = \mathscr{D}'(B) \rightarrow \mathscr{D}'(C) = \mathscr{D}'(B \Rightarrow C) = \mathscr{D}'(A)$$

by the inductive assumption.

Let $A := (\forall x)\, B$, $A \in F_{J(T)}$. Then

$$\mathscr{D}(A) = \bigwedge_{d \in D} \mathscr{D}(A_x[\mathbf{d}]) = \bigwedge_{d \in D'} \mathscr{D}'(A_x[\mathbf{d}]) = \mathscr{D}'((\forall x)\, A)$$

by the inductive assumption. The proof proceeds analogously if A contains some free variables. We conclude that $\mathscr{D}(A) = \mathscr{D}'(A)$ for every $A \in F_{J(T)}$. Since for all $A \in F_{J(T)}$, $A_S(A) \leqq A'_S$ and $\mathscr{D}'$ is a model T', we have

$$A_S(A) \leqq A'_S(A) \leqq \mathscr{D}'(A) = \mathscr{D}(A) ,$$

i.e. $\mathscr{D}$ is a model of T. □

Let $E \subsetapprox F_{J(T)}$ be a fuzzy set of formulae and T a theory. Then the fuzzy theory

$$T' = \langle A_L, A_S \cup E, R \rangle$$

is an extension of the theory T and we write $T' = T \cup E$.

Theorem 8 (on constants). Let T be a theory in the language J. We enrich J by new constants $\mathbf{v} \in V$, i.e. $J' = J \cup V$ and put $A'_S(A) = A_S(A)$ if $A \in F_J$ and $A'_S(A) = \mathbf{0}$ for $A \in F_{J'} - F_J$. Let $T' = \langle A'_L, A'_S, R \rangle$ be a theory in J'. Then

$$T' \vdash_a A_{x_1 \ldots x_n}[\mathbf{v}_1, \ldots, \mathbf{v}_n] \quad \text{iff} \quad T \vdash_a A$$

holds for every formula $A \in F_J$, where $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$.

Proof. Let $T' \vdash_a A_{x_1 \ldots x_n}[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ and w be a proof of this formula, $\mathrm{Val}_{T'}(w) = a'$. Since special axioms B for which $A'_S(B) \neq \mathbf{0}$ do not contain constants from V we can replace all the constants $\mathbf{v}_i$, $i = 1, \ldots, n$ occurring in w by some variables y_i which do not occur in w and thus obtain a proof w' of $A_{x_1 \ldots x_n}[y_1, \ldots, y_n]$ whose value is greater than that of w. But $A_{x_1 \ldots x_n}[y_1, \ldots, y_n] \in F_J$ and we conclude that

$$T \vdash_b A_{x_1 \ldots x_n}[y_1, \ldots, y_n], \quad a \leqq b .$$

Due to corollary of Lemma 14, $T \vdash_b A$.

Conversely, let $T \vdash_c A$. Then $T' \vdash_{d'} A$, $c \leqq d'$ and due to Lemma 14

$$T' \vdash_d A_{x_1 \ldots x_n}[\mathbf{v}_1, \ldots, \mathbf{v}_n], \quad c \leqq d' \leqq d .$$

The proposition then follows from Lemma 8. □

Corollary. The theory T' is a conservative extension of T.

The following three lemmas are of technical character.

Lemma 20. Let T and T' be theories and

$$T \vdash_a A \quad \text{iff} \quad T' \vdash_a B .$$

The to every proof w of A in T there is a set M of proofs of B in T' such that

$$\mathrm{Val}_T(w) \leqq \bigvee\{\mathrm{Val}_{T'}(w');\ w' \in M\} .$$

Proof. It follows immediately from Theorem 1. □

Lemma 21. Let $\{X_i \subsetapprox F_{J(T)};\ i < q\}$ be a chain in a partially ordered set of fuzzy sets $\langle \mathscr{F}(F_{J(T)}),\ \subsetapprox \rangle$. Put

$$X = \bigcup_{i<q} X_i$$

Then

$$\mathrm{Val}_X(w) = \bigvee\{\mathrm{Val}_{X_i}(w);\ i < q\}$$

holds for every proof w of the formula $A \in F_{J(T)}$.

Proof. By induction on the length of the proof. Let $w := A[X(A); \mathrm{SA}]$.

Then

$$\mathrm{Val}_X(w) = X(A) = \Big(\bigcup_{i<q} X_i\Big) A = \bigvee\{X_i(A);\ i < q\} = \bigvee\{\mathrm{Val}_{X_i}(w);\ i < q\}$$

due to the definition of the union of fuzzy sets. If A is a logical axiom then

$$\mathrm{Val}_{X_i}(w) = A_L(A)$$

holds for each i. Let

$$A = r^{\mathrm{syn}}(A_{j_1}, \ldots, A_{j_n})$$

where A_{j_k} are results of proofs $w_{(j_k)}$ of the length shorter than that of w, $k = 1, \ldots, n$. Using the induction assumption and the semicontinuity of rules we obtain

$$r^{\mathrm{sem}}(\mathrm{Val}_X(w_{(j_1)}), \ldots, \mathrm{Val}_X(w_{(j_n)})) =$$
$$= \bigvee\{r^{\mathrm{sem}}(\mathrm{Val}_{X_{i_1}}(w_{(j_1)}), \ldots, \mathrm{Val}_{X_{i_n}}(w_{(j_n)});\quad i_1, \ldots, i_n < q\} := A .$$

We show that this formula is equal to

$$B := \bigvee\{r^{\mathrm{sem}}(\mathrm{Val}_{X_i}(w_{(j_1)}), \dots \mathrm{Val}_{X_i}(w_{(j_n)}); \quad i < q\} .$$

By the isotonicity of r^{sem}

$$r^{\mathrm{sem}}(\mathrm{Val}_{X_i}(w_{(j_1)}), \dots, \mathrm{Val}_{X_i}(w_{(j_n)})) \leqq A , \quad i < q$$

whence $B \leqq A$. The inequality $A \leqq B$ follows from the assumption that $\{X_i; i < q\}$ is a chain. Then

$$\mathrm{Val}(X) = \bigvee\{r^{\mathrm{sem}}(\mathrm{Val}_{X_i}(w_{(j_1)}), \dots, \mathrm{Val}_{X_i}(w_{(j_n)}); \quad i < q\} =$$
$$= \bigvee\{\mathrm{Val}_{X_i}(w); i < q\} .$$

□

Lemma 22. Let T be a consistent theory and $\{E_i \subsetneqq F_{J(T)}; i < q\}$ a chain in a partially ordered set $\langle \mathscr{F}(F_{J(T)}), \subseteq \rangle$ such that $T_O = T$, $E_O = A_S$ and

$$T_{i+1} = T_i \cup E_{i+1}$$

is a consistent theory, $i + 1 < q$. Then

$$T' = T \cup \bigcup_{i \in q} E_i$$

is a consistent extension of the theory T.

Proof. Let us denote $E = \bigcup_{i \in q} E_i$, $A \in F_{J(T)}$ and M, M' be sets of all the proofs of A and $\neg A$ respectively. Then it follows from Lemma 21 that

$$(C^{\mathrm{syn}}E)\, A \otimes (C^{\mathrm{syn}}E)\, \neg A = \bigvee\{\mathrm{Val}_E(w); w \in M\} \otimes \bigvee\{\mathrm{Val}_E(w') ;$$
$$w' \in M'\} = \bigvee\{(C^{\mathrm{syn}}E_i)\, A \otimes (C^{\mathrm{syn}}E_j)\, \neg A; i, j < q\} = \mathbf{0}$$

since

$$(C^{\mathrm{syn}}E_i)\, A \otimes ({}^{\mathrm{syn}}E_j)\, \neg A = \mathbf{0}$$

holds for all the couples of fuzzy sets E_i, E_j from the considered chain. □

We now state one of the most important theorems of this paper. It demonstrates the relation between a given theory and its extrension by a new formula.

Theorem 9 (deduction theorem). Let A be a closed formula and $T' = T \cup \{\mathbf{1}/A\}$.

(a) If $T \vdash_a A^n \Rightarrow B$ and $T' \vdash_b B$ for some n then $a \leqq b$.

(b) To every proof w' of B in T' there are n and a proof w of $A^n \Rightarrow B$ in T such that

$$\mathrm{Val}_{T'}(w') = \mathrm{Val}_T(w) .$$

Proof. (a) Using theorem (D 8) and r_{MP} we obtain $T' \vdash A^n$ and $T' \vdash_{a'} A^n \Rightarrow B$, $a \leqq a'$, whence $T' \vdash_b B$, $a' \leqq b$.

(b) By induction on the length of w'.

(ba) Let w_1 be a proof of $B \Rightarrow (A^n \Rightarrow B)$ for some n (theorem (D 23)), $\mathrm{Val}(w_1) = \mathbf{1}$ and

$$w' := B[b; P]$$

where P is LA or SA. Then

$$w := B\,[b; P],\ w_1\,[\mathbf{1}],\ A^n \Rightarrow B\,[b; r_{\mathrm{MP}}]$$

is a proof of $A^n \Rightarrow B$.

(bb) Let $B := A$ and

$$w' := A\,[\mathbf{1};\ \mathrm{SA}]\,.$$

Then

$$w := A \Rightarrow A\,[\mathbf{1};\ \mathrm{LA}_{T2}]\,.$$

(bc) Let $B := \boldsymbol{a} \Rightarrow C$, w_1 be a proof of a formula C,

$$\mathrm{Val}_{T'}(w_1) = c \quad \text{and}$$
$$w' := w_1\,[c]\,,\quad \boldsymbol{a} \Rightarrow C\,[a \to c;\ r_{\mathrm{Ra}}]\,.$$

By the inductive assumption there are n and a proof w_1 of $A^n \Rightarrow C$ such that $\mathrm{Val}_T(w_1) = c$. Let w_2 be a proof of

$$(\boldsymbol{a} \Rightarrow (A^n \Rightarrow C)) \Rightarrow (A^n \Rightarrow (\boldsymbol{a} \Rightarrow C))\,,$$

$\mathrm{Val}(w_2) = \mathbf{1}$ (theorem (D 22)). Then

$$w := w_1[c],\ a \Rightarrow (A^n \Rightarrow C)\,[a \to c;\ r_{\mathrm{Ra}}],\ w_2[\mathbf{1}]\,,$$
$$A^n \Rightarrow (a \Rightarrow C)\,[a \to c;\ r_{\mathrm{MP}}]\,.$$

(bd) Let $B := (\forall x)\,C$, w_1 be a proof of a formula C, $\mathrm{Val}_{T'}(w_1) = c$ and

$$w' := w_1[c],\ (\forall x)\,C\,[c;\ r_{\mathrm{G}}]\,.$$

By the inductive assumption there are n and a proof w_1 of $A^n \Rightarrow C$ such that $\mathrm{Val}_T(w_1) = c$. Then

$$w := w_1[c],\ (\forall x)(A^n \Rightarrow C)\,[c;\ r_{\mathrm{G}}],\ (\forall x)(A^n \Rightarrow C) \Rightarrow$$
$$\Rightarrow (A^n \Rightarrow (\forall x)\,C)\,[\mathbf{1};\ \mathrm{LA}_{\mathrm{T}10}],\ A^n \Rightarrow (\forall x)\,C\,[c;\ r_{\mathrm{MP}}]$$

since A^n is a closed formula.

(be) Let w'_1 be a proof of C, $\mathrm{Val}_{T'}(w'_1) = c_1$ and w'_2 be a proof of $C \Rightarrow B$, $\mathrm{Val}_{T'}(w'_2) = c_2$. Let

$$w' := w'_1[c_1],\ w'_2[c_2],\ B\,[c_1 \otimes c_2;\ r_{\mathrm{MP}}]\,.$$

By the inductive assumption there are n_1, n_2 and proofs w_1 of $A^{n_1} \Rightarrow C$ and w_2 of $A^{n_2} \Rightarrow (C \Rightarrow B)$ such that

$$c_1 = \mathrm{Val}_T(w_1)\,,\quad c_2 = \mathrm{Val}_T(w_2)\,.$$

Let w_3 be a proof of

$$(A^{n_2} \Rightarrow (C \Rightarrow B)) \Rightarrow ((A^{n_1} \Rightarrow C) \Rightarrow (A^{n_1+n_2} \Rightarrow B))\,,$$

$\mathrm{Val}(w_3) = \mathbf{1}$ (theorem (D 24)). Then

$$w := w_1[c_1],\ w_2[c_2],\ w_3[\mathbf{1}],\ (A^{n_1} \Rightarrow C) \Rightarrow (A^{n_1+n_2} \Rightarrow B)\,[c_2;\ r_{\mathrm{MP}}],$$
$$A^{n_1+n_2} \Rightarrow B\,[c_1 \otimes c_2;\ r_{\mathrm{MP}}]$$

Corollary 1. Let L be a finite chain. Then there is n such that

$$T \vdash_a A^n \Rightarrow B \quad \text{iff} \quad T' \vdash_a B .$$

Proof. In [10] – I it is demonstrated that if L is a finite chain then to every formula C there is a proof w of C such that

$$(C^{\text{syn}} A_S)\, C = \mathrm{Val}_{A_S}(w) .$$

Let w' be such a proof of B in T', i.e. $T' \vdash_a B$ and $\mathrm{Val}_{T'}(w') = a$. Then there are n and a proof w of $A^n \Rightarrow B$ such that

$$\mathrm{Val}_{T'}(w') = \mathrm{Val}_T(w) = a .$$

Thence $T \vdash_b A^n \Rightarrow B$ and $a \leqq b$. From Theorem 9(a) follows $b \leqq a$, i.e. $b = a$.

Corollary 2. Let A be a closed formula and $T' = T \cup \{a/A\}$. Then to every proof w' of B in T' there are n and a proof w of $A^n \Rightarrow B$ in T such that

$$\mathrm{Val}_{T'}(w') \leqq \mathrm{Val}_T(w) .$$

Proof. The proof is analogous to the proof of Theorem 9 with the exception of the case (bb):

$$w' := A\,[a \vee A_S(A);\ \mathrm{SA}]$$
$$w := A \Rightarrow A\,[\mathbf{1};\ \mathrm{LA}_{T2}] .$$

Since $a \vee A_S(A) \leqq \mathbf{1}$ we must modify the inductive assumption into inequality instead of equality. □

This theorem is a generalisation of the classical deduction theorem and it is one of the most important theorems necessary in the proof of the completeness property of fuzzy logic.

The following theorem is a generalisation of the C-rule introduced e.g. in [6].

Theorem 10. Let T be a consistent theory, $T \vdash (\exists x)\,(A(x))^n$ for every n and $\mathbf{t} \notin J(T)$ be a new constant. Then the theory

$$T' = T \cup \{\mathbf{1}/A_x[t]\}$$

in the language $J(T) \cup \{\mathbf{t}\}$ is a conservative extension of the theory T.

Proof. Let $\bar{T}$ denote a theory resulting from T by adding $\mathbf{t}$ into $J(T)$. Due to Theorem 8, $\bar{T}$ is a conservative extension of T. Let $B \in F_{J(T)}$ be a closed formula $T' \vdash_b B$ and w' be a proof of B in T'. We will demonstrate that there is a set M_B of proofs of B in T such that

$$\mathrm{Val}_{T'}(w') \leqq \bigvee\{\mathrm{Val}_T(w);\ w \in M_B\} .$$

Due to Theorem 9(b) there are n and a proof w of $A_x[t]^n \Rightarrow B$ in $\bar{T}$ such that

$$\mathrm{Val}_{T'}(w') = \mathrm{Val}_{\bar{T}}(w) .$$

Due to Theorem 8

$$\bar{T} \vdash_c A_x[t]^n \Rightarrow B \quad \text{iff} \quad T \vdash_c A(x)^n \Rightarrow B .$$

Then by Lemma 20 there is a set M of proofs of $A(x)^n \Rightarrow B$ in T such that

$$\mathrm{Val}_T(w) \leqq \bigvee\{\mathrm{Val}_T(\bar{w});\ \bar{w} \in M\}\,.$$

Let $\bar{w} \in M$, w_2 be a proof of $(\exists x)\,A(x)^n$ in T, $\mathrm{Val}_T(w_2) = d$ and w_1 be a proof of $(\forall x)\,(A(x)^n \Rightarrow B) \Rightarrow ((\exists x)\,A(x)^n \Rightarrow B)$, $\mathrm{Val}\,(w_1) = \mathbf{1}$.

We write down a proof

$$\hat{w} := \bar{w}[b],\ (\forall x)\,(A(x)^n \Rightarrow B)\,[b; r_{\mathrm{G}}],\ w_1[\mathbf{1}],$$
$$(\exists x)\,A(x)^n \Rightarrow B[b; r_{\mathrm{MP}}],\ w_2[d],\ B[b \otimes d; r_{\mathrm{MP}}]\,.$$

Let M_B be a set of all the proofs $\hat{w}$. Then

$$\bigvee\{\mathrm{Val}_T(\hat{w});\ \hat{w} \in M_B\} = \bigvee\{b \otimes d;\ w \in M, w_2\} =$$
$$= \bigvee\{\mathrm{Val}_T(\bar{w});\ \bar{w} \in M\} \otimes \mathbf{1} \geqq \mathrm{Val}_T(w) = \mathrm{Val}_{T'}(w')\,.$$

Since such a set M_B exists to every proof w' of B in T' it follows that

$$(C^{\mathrm{syn}} A_S)\,B \geqq (C^{\mathrm{syn}} A'_S)\,B = b\,.$$

Then

$$T \vdash_b B$$

by Lemma 18. □

7.2 Henkin fuzzy theories

Analogously as in classical logic it is possible to introduce Henkin fuzzy theories. A Henkin fuzzy theory is obtained from the given fuzzy theory when adding Henkin axioms

$$A_x[\mathbf{r}] \Rightarrow (\forall x)\,A(x)$$

to the fuzzy set of special axioms with the membership degree $\mathbf{1}$ where $\mathbf{r}$ is a special constant for $(\forall x)\,A(x)$. This leads to the demand that

$$\mathscr{D}(A_x[\mathbf{r}]) = \bigwedge_{d \in D} \mathscr{D}(A_x[\mathbf{d}])$$

must hold in any model of the Henkin theory. In other words, there must exist an element $d_0 \in D$ such that $\mathscr{D}(\mathbf{r}) = d_0$ and

$$\mathscr{D}(A_x[\mathbf{d}_0]) = \bigwedge_{d \in D} \mathscr{D}(A_x[\mathbf{d}])\,.$$

In this section, we study some properties of Henkin fuzzy theories. The results serve us as a preparatory material for the proof of completeness theorems.

Lemma 23. Let T be a Henkin theory and $\mathbf{r}$ a special constant for $(\forall x)\,A$. Then

$$T \vdash_a (\forall x)\,A(x) \quad \text{iff} \quad T \vdash_a A_x[\mathbf{r}]\,.$$

Proof. This is corollary of Lemmas 7 and 12. □

Lemma 24. Let T be a consistent theory and $\mathbf{r} \in J(T)$ a new constant taking the role of a special constant for $(\forall x) A$. Then the theory

$$T' = T \cup \{\mathbf{1}/(A_x[\mathbf{r}] \Rightarrow (\forall x) A)\}$$

is a conservative extension of T.

Proof. This is a corollary of Theorem 10 and axiom (T 11). □

Theorem 11. Let T be a consistent theory, K a set of special constants for all the closed formulae $(\forall x) A$ and let A_H be a fuzzy set of Henkin axioms B_H defined by membership function

$$A_H(B_H) = \mathbf{1}$$

and $A_H(C) = \mathbf{0}$ for $C \neq B_H$. Then the theory

$$T' = T \cup A_H$$

in the language $J(T') = J(T) \cup K$ is a conservative extension of the theory T.

Proof. Similarly as in the classical proof of the analogous theorem we construct sets of special constants $K_1, K_2, \ldots$ of given level. Set $T_O = T$ and

$$T_{i+1} = T_i \cup \{\mathbf{1}/(A_x[\mathbf{r}] \Rightarrow (\forall x) A(x))\}$$

where $\mathbf{r} \in K_{i+1}$ is a special constant for $(\forall x) A(x)$. Then, by Lemma 24, T_{i+1} is a conservative extension of T_i for every i. From this and Lemma 22 follows the proposition of this theorem. □

Let T be a complete Henkin theory. Put

$$D_O = M_V .$$

We define functions f_O assigned to function symbols f in the same way as in [9], Section 4.1, and fuzzy relations $p_O \lessapprox D_O^n$ assigned to predicate symbols p as follows:

$$p_O(t_1, \ldots, t_n) = a \quad \text{iff} \quad T \vdash_a p(t_1, \ldots, t_n) , \; t_1, \ldots, t_n \in D_O .$$

Then

$$\mathscr{D}_O = \langle D_O, p_O, \ldots, f_O, \ldots \rangle$$

is a canonical structure for T.

Theorem 12. Let T be a complete theory. Then the canonical structure $\mathscr{D}_O$ is a model of T such that

$$T \vdash_a A \quad \text{iff} \quad \mathscr{D}_O(A) = a$$

holds for every formula $A \in F_{J(T)}$.

Proof. Let T' be a Henkin extension of T which is conservative due to Theorem 11. If $A := p(t_1, \ldots, t_n)$ is a closed atomic formula then the proposition holds by the definition of p_O.

Let $A := \boldsymbol{a}$, $a \in L$. Then $T' \vdash_a \boldsymbol{a}$ (in any theory) and, hence, $\mathscr{D}_O(A) = a$.

Assume that the proposition holds for all the formulae shorter than A. Let $A :=$

$:= B \Rightarrow C$ and $\mathscr{D}_O(B) = b$, $\mathscr{D}_O(C) = c$. Then

$$\mathscr{D}_O(A) = b \rightarrow c \quad \text{iff} \quad T' \vdash_b B \quad \text{and} \quad T' \vdash_d A\,, \quad d = b \rightarrow c$$

due to Theorem 7 and the inductive assumption.

Let $A := (\forall x)\, B$. Then

$$T' \vdash_a A \quad \text{iff} \quad T' \vdash_a B_x[\mathbf{r}] \quad \text{iff} \quad \mathscr{D}_O(B_x[\mathbf{r}]) = a \quad \text{iff} \quad \mathscr{D}_O((\forall x)\, B) = a$$

due to Lemma 23, inductive assumption and the fact the T' is a Henkin theory. It follows from Theorem 5 that this equivalence holds for every formula $A \in F_{J(T)}$. Clearly, $\mathscr{D}_O$ is a model of T' and due to Lemma 19, a model of T as well. □

7.3 Algebraic properties of the set of formulae

The methods used in this section are adaptation of the methods taken from [10] and [11] and they serve us as a preparatory tool for the proofs of the completeness theorems, which are presented in the next section.

Theorem 13. Let T be a consistent Henkin theory. Then

(a) $A \leqq B$ iff $T \vdash A \Rightarrow B$

is a preorder on $F_{J(T)}$.

(b) $A \approx B$ iff $A \leqq B$ and $B \leqq A$

is a congruence on $F_{J(T)}$.

(c) Let

$$F(T) = \{F_{J(T)}| \approx, \{\|\mathbf{a}\|;\, a \in L\}, \vee, \wedge\, \otimes, \rightarrow, \bigvee, \bigwedge\rangle$$

be a factoralgebra on $F_{J(T)}$ with respect to the congruence $\approx$. Then

$$\mathscr{L}(T) = \langle F_{J(T)}| \approx, \vee, \wedge, \otimes, \rightarrow, \bigvee, \bigwedge, \bar{1}, \bar{0}\rangle$$

is a generalised residuated lattice.

(d) The mapping $h\colon a \rightarrow \|\mathbf{a}\|$ where $\|\mathbf{a}\|$ is an equivalence class with respect to $\approx$ is a homomorphism from $\mathscr{L}$ into $\mathscr{L}(T)$.

Proof. (a) can be obtained immediately using axioms (T 2) and (T 6).

(b) and (c): Obviously, $\approx$ is the equivalence. Put

$$\|A\| \leqq_T \|B\| \quad \text{iff} \quad A \leqq B$$
$$\bar{1} := \|\mathbf{1}\|$$
$$\bar{0} := \|\mathbf{0}\|$$
$$\|A\| \wedge \|B\| := \|A \bigwedge B\|$$
$$\|A\| \vee \|B\| := \|A \bigvee B\|$$
$$\|A\| \otimes \|B\| := \|A \,\&\, B\|$$
$$\|A\| \rightarrow \|B\| := \|A \Rightarrow B\|$$
$$\bigwedge_{t \in M_V} \|A_x[t]\| := \|(\forall x)\, A\|$$
$$\bigvee_{t \in M_V} \|A_x[t]\| := \|(\exists x)\, A\|\,.$$

It follows from (a) and the definition of $\approx$ that $\leqq_T$ is a partial ordering on $F_{J(T)}|\approx$. Tautology (T 3) gives $\|A\| \leqq_T \|\mathbf{1}\|$ and the rule r_{RO} gives $\|\mathbf{0}\| \leqq_T \|A\|$ for every formula $A \in F_{J(T)}$ which means that $\bar{0}$ and $\bar{1}$ are the smallest and the greatest elements respectively.

Now we prove that $\otimes$ and $\rightarrow$ are adjoint operations, $\rightarrow$ is residuum and $\otimes$ product. Using tautology (T 4) we immediately obtain the adjointness condition. Tautology (T5) gives commutativity of $\otimes$. Antitonicity of $\rightarrow$ in the first variable follows from (T 6) and its isotonicity in the second one from theorem (D 1).

Let $A \leqq_T B$. Using the tautology

$$\models (A \Rightarrow B) \Rightarrow ((B \Rightarrow (C \Rightarrow B \,\&\, C)) \Rightarrow (A \Rightarrow (C \Rightarrow (B \,\&\, C)))$$

(T 6), theorem (D 8) and (T 4) we obtain isotonicity of $\otimes$.

Using the theorem

$$\vdash \mathbf{1} \Rightarrow (A \Rightarrow (A \,\&\, \mathbf{1}))$$

(D 8) we obtain

$$A \leqq_T A \otimes \bar{1}$$

and using the tautology

$$\models (\mathbf{1} \Rightarrow (A \Rightarrow A)) \otimes ((\mathbf{1} \,\&\, A) \Rightarrow A)$$

we obtain that $\bar{1}$ is a unit with respect to $\otimes$.

At last, using the tautology

$$\models A \,\&\, (B \,\&\, C) \Rightarrow A \,\&\, (B \,\&\, C)$$

(T 2), then (T 4) and theorem (D 22) we obtain

$$(A \,\&\, B) \,\&\, C \leqq_T A \,\&\, (B \,\&\, C)$$

and analogously the converse implication which yields the associativity of $\otimes$.

Using theorems (D 2), (D 3) and (D 5) we prove that $\wedge$ is infimum and analogously using (D 4), (D 6) and (D 7) we prove that $\vee$ is supremum. We conclude that

$$\langle F_{J(T)}|\approx, \vee, \wedge, \otimes, \rightarrow, \bar{1}, \bar{0}\rangle$$

is a residuated lattice.

As for the generalised operations, it will do to demonstrate the properties of $\bigwedge$. The tautology (T 9) yields

$$\bigwedge_{t \in M_V} \|A_x[t]\| \leqq_T \|A_x[t]\|$$

for every $t \in M_V$. If $\|B\| \leqq_T \|A_x[t]\|$ for all $t \in M_V$ then

$$\|B\| \leqq_T \|A_x[\mathbf{r}]\|$$

as a special case where $\mathbf{r}$ is a special constant. Then $T \vdash B \Rightarrow A_x[\mathbf{r}]$ and using Henkin axiom and tautology (T 6) we obtain

$$T \vdash B \Rightarrow (\forall x)\, A$$

whence

$$\|B\| \leqq_T \bigwedge_{t \in M_V} \|A_x[t]\| .$$

This means that $\bigwedge$ is the infimum in $F_{J(T)}| \approx$.

The fact that $\approx$ is a congruence follows from Theorem 6. (d) The proof that h is homomorphism with respect to the operation $\vee$, $\wedge$, $\otimes$, $\rightarrow$ follows immediately from tautology (T 1) and the assumption that T is consistent. Let us demonstrate e.g.

$$h(a \rightarrow b) = \|\overline{\boldsymbol{a} \Rightarrow \boldsymbol{b}}\| = \|\boldsymbol{a} \Rightarrow \boldsymbol{b}\| = \|\boldsymbol{a}\| \rightarrow \|\boldsymbol{b}\| = h(a) \rightarrow h(b) . \qquad \square$$

Theorem 14. A theory T is contradictory iff $\mathscr{L}(T)$ is a degenerated algebra.

Proof. Let T be contradictory. Then $T \vdash A \Leftrightarrow B$ for any two formulae $A, B \in F_{J(T)}$ due to Theorem 4 and so

$$\|A\| = \|B\| .$$

Conversely, let $\mathscr{L}(T)$ be degenerated and $a < b$. Then

$$\|\overline{\boldsymbol{b} \Rightarrow \boldsymbol{a}}\| = \|\mathbf{1}\|$$

i.e.

$$T \vdash (\overline{\boldsymbol{b} \Rightarrow \boldsymbol{a}}) \Leftrightarrow \mathbf{1}$$

whence

$$T \vdash \overline{\boldsymbol{b} \Rightarrow \boldsymbol{a}} .$$

But $b \rightarrow a < \mathbf{1}$ and from the corrolary (b) of Theorem 4 follows that T is contradictory. $\square$

Lemma 25. Let T be a consistent theory. Then the mapping $h\colon L \rightarrow F_{J(T)}| \approx$ defined by $h(a) = \|\boldsymbol{a}\|$ is an injection, i.e. a monomorphism from $\mathscr{L}$ into $\mathscr{L}(T)$.

Proof. Let h be not an injection. Then there are $a \neq b$ such that $h(a) = h(b)$ and $T \vdash \boldsymbol{a} \Leftrightarrow \boldsymbol{b}$. Let $a < b$. Then

$$T \vdash \boldsymbol{b} \Rightarrow \boldsymbol{a}$$

and tautology (T 1) and corollary (b) of Theorem 4 yield that T is contradictory — a contradiction with the assumption that T is consistent. $\square$

A set

$$H \subseteq F_{J(T)}| \approx$$

is a *filter* if:

1. If $\|A\| \in H$ and $\|A\| \leqq_T \|B\|$ then $\|B\| \in H$.
2. If $\|A\|, \|B\| \in H$ then $\|A\| \otimes \|B\| \in H$.
3. If $\|A\| \in H$ then $\bigwedge_{t \in M_V} \|A_x[t]\| \in H$.

If the filter is maximal then we call it *ultrafilter*.

Lemma 26. Let T be a consistent Henkin theory.

(a) Every filter $H \subseteq F_{J(T)}| \approx$ with the property

(P1) $\qquad H \cap h(L) = h(\mathbf{1})$

can be extended into an ultrafilter with the same property.

(b) Let $G \subseteq F_{J(T)}| \approx$ be an ultrafilter with the property (P 1). Then the property

$$\|A\| \notin G \quad \text{iff there are a closed formula } B\,, \quad \|B\| \in G\,, \quad a \in L \quad \text{and}$$
$$n \geqq 1 \quad \text{such that } a < 1 \text{ and } \quad \|A\|^n \otimes \|B\| \leqq_T h(a)$$

holds for every closed formula $A \in F_{J(T)}$.

Proof. (*a*) Let $\bar{H}$ be a chain of filters fulfilling (P 1). Then

$$\bigcup \bar{H} \cap h(L) = \bigcup \{H \cap h(L);\ H \in \bar{H}\} = h(\mathbf{1})\,.$$

The proposition then follows from Zorn's lemma.

(b) If $\|A\| \in G$ then it follows from the assumption that the only a such that $h(a) \in \in G$ is $a = \mathbf{1}$.

Conversely, let there be no such $\|B\|$, $a < \mathbf{1}$ and n. Then $G' = \{\|C\| \in F_{J(T)} \approx;\ \|A\|^n \otimes \|B\| \leqq_T \|C\|$ for some n and $\|B\| \in G\}$ is a filter fulfilling (P 1). Indeed, the properties 1 and 2 of the filter follow from the algebraic properties of $\mathscr{L}(T)$. Let $\|C\| \in G'$. Then

$$T \vdash A^n \,\&\, B \Rightarrow C$$

and due to Lemma 14

$$T \vdash (A^n \,\&\, B) \Rightarrow C_x[t]$$

for all the $t \in M_V$. This means that

$$\|C_x[t]\| \in G\,, \quad \|A\|^n \otimes \|B\| \leqq_T \|C_x[t]\|$$

which yields

$$\bigwedge_{t \in M_V} \|C_x[t]\| \in G'\,.$$

The property (P 1) follows from the assumption. Moreover, $\|A\|$, $\|B\| \in G'$, i.e. $G \cup \{\|A\|\} \subseteq G'$ and, since G is maximal, we have $G' = G$, i.e. $\|A\| \in G$. □

Lemma 27. Let T be a consistent Henkin theory and G an ultrafilter in $\mathscr{L}(T)$ with the property (P 1). Then

$$\|A\| \vee \|B\| \in G \quad \text{iff} \quad \|A\| \in G \quad \text{or} \quad \|B\| \in G$$

holds for every two closed formulae.

Proof. Let $\|A\| \notin G$, $\|B\| \notin G$. Then, due to Lemma 26, there are C, C', a, b, m, n such that

$$\|A\|^m \otimes \|C\| \leqq_T h(a) \quad \text{and} \quad \|A\|^n \otimes \|C'\| \leqq_T h(b)\,.$$

Put $k = \max(m, n)$. Using tautology (T 8) we obtain

$$(\|A\| \vee \|B\|)^k \otimes (\|C\| \otimes \|C'\|) = (\|A\|^k \vee \|B\|^k) \otimes (\|C\| \otimes \|C'\|) \leqq_T$$
$$\leqq_T (\|A\|^m \otimes \|C\|) \vee (\|B\|^n \otimes \|C'\|) \leqq_T h(a) \vee h(b) = h(a \vee b) .$$

Since $\|C\|, \|C'\| \in G$, we have $\|C\| \otimes \|C'\| \in G$. Since $a \vee b < \mathbf{1}$ it follows from Lemma 26 that $\|A\| \vee \|B\| \notin G$.

The converse implication is obvious. □

Let A_O be a chosen *closed* formula and $T \vdash_a A_O$. Put

$$\text{(Pa)} \qquad b = \begin{cases} a \text{ if } \mathscr{L} \text{ is a finite chain} \\ c,\, c > a \text{ if } L = \langle 0, 1 \rangle \text{ and } a < 1 \\ \mathbf{1} \text{ if } a = \mathbf{1} . \end{cases}$$

Lemma 28. Let T be a consistent Henkin theory. Then

$$H = \{\|B\|;\ (\|A_O\| \rightarrow \|\boldsymbol{b}\|)^n \leqq_T \|B\| \text{ for some } n > 0\}$$

is a filter with the property (P 1).

Proof. The properties 1 and 2 of the filter follow from tautology (T 6), Theorem 6 and from isotonicity of $\otimes$ in $\mathscr{L}(T)$. Let $A \in H$. Then

$$T \vdash (A_O \Rightarrow \boldsymbol{b})^n \Rightarrow (\forall x)\, A$$

i.e. $\|(\forall x)\, A\| \in H$ which means that

$$\bigwedge_{t \in M_V} \|A_x[t]\| \in H$$

and thus H is a filter.

Now we prove that H has the property (P 1). Let $b > a$ and suppose that there are $c < \mathbf{1}$ and $n > 0$ such that $T \vdash (A_O \Rightarrow \boldsymbol{b})^n \Rightarrow \boldsymbol{c}$. The tautologies (T 7), (T 8) and theorem (D 8) yield

$$T \vdash (A_O \Rightarrow \boldsymbol{b})^n \vee (\boldsymbol{b} \Rightarrow A_O)^n .$$

From the assumption and theorems (D 1), (D 6) we obtain

$$T \vdash (A_O \Rightarrow \boldsymbol{b})^n \Rightarrow (\boldsymbol{c} \vee (\boldsymbol{b} \Rightarrow A_O)^n)$$

and from theorem (D 7) and the fact that

$$T \vdash (\boldsymbol{b} \Rightarrow A_O)^n \Rightarrow (\boldsymbol{c} \vee (\boldsymbol{b} \Rightarrow A_O)^n)$$

we obtain $T \vdash \boldsymbol{c} \vee (\boldsymbol{b} \Rightarrow A_O)^n$. At last, using theorems (D 1), (D 4), (D 6), (D 7) and (D 15) we obtain

$$T \vdash (\boldsymbol{b} \Rightarrow A_O) \vee \boldsymbol{c} .$$

Since c is nilpotent we find m such that $c^m = \mathbf{0}$. Then using theorem (D 8) and tautology (T 8) we obtain

$$T \vdash (\boldsymbol{b} \Rightarrow A_O)^m \vee \boldsymbol{c}^m .$$

Since $\models \mathbf{c}^m \Leftrightarrow \mathbf{0}$ we can use Theorem 6 and after some reasoning we have

$$T \vdash \boldsymbol{b} \Rightarrow A_O$$

on the basis of theorems (D 7) and (D 15). Let w' be a proof of $\boldsymbol{b} \Rightarrow A_O$, $\mathrm{Val}_T(w') = d$. We write down a proof

$$w := \boldsymbol{b}\,[b; \mathrm{LA}]\,, \quad w'\,[d]\,, \quad A_O\,[b \otimes d\,;\, r_{\mathrm{MP}}]\,.$$

Since $\bigvee \{\mathrm{Val}_T(w); w'\} = b$ this implies $T \vdash_{b'} A_O$ where $a < b \leqq b'$ – a contradiction. From it follows that $c = \mathbf{1}$ and for any n

$$T \vdash (A_O \Rightarrow \boldsymbol{b})^n \Rightarrow \mathbf{1}\,,$$

i.e. only $\|\mathbf{1}\| \in H$. □

Using Lemma 26 (a) we can extend H into an ultrafilter with the property (P 1). We denote this ultrafilter by G_O.

Lemma 29. Let T be a consistent Henkin theory. Then to every closed formula $A \in F_{J(T)}$ there is $c \in L$ such that

$$\|A\| \leftrightarrow \|\boldsymbol{c}\| \in G_O\,.$$

Proof. Put

$$D_A = \{c;\ \|\boldsymbol{c}\| \to \|A\| \in G_O\}\,, \quad H_A = \{c;\ \|A\| \to \|\boldsymbol{c}\| \in G_O\}\,.$$

Let $c' \leqq c$. Then $\|\boldsymbol{c}'\| \leqq_T \|\boldsymbol{c}\|$ which implies

$$\|\boldsymbol{c}\| \to \|A\| \leqq_T \|\boldsymbol{c}'\| \to \|A\|\,,$$

i.e. $\|\boldsymbol{c}'\| \to \|A\| \in G_O$. Analogously for H_A. Therefore D_A and H_A are initial and terminal segments respectively. Moreover, with respect to tautology (T 7) it follows from Lemma 27 that $D_A \cup H_A = L$.

Let L be a finite chain. Let a_k be the last element of D_A and $a_k \notin H_A$. Then tautology (TK) implies

$$(\|A\| \to \|\boldsymbol{a}_k\|) \vee (\|\boldsymbol{a}_{k+1}\| \to \|A\|) \in G_O$$

and since $(\|A\| \to \|\boldsymbol{a}_k\|) \notin G_O$ it follows from Lemma 26 that $\|\boldsymbol{a}_{k+1}\| \to \|A\| \in G_O$, i.e. $a_{k+1} \in D_A$ – a contradiction.

Let $L = \langle O, 1\rangle$. We show that $L - D_A$ is an open set. Let $c \notin D_A$ and $c' < c$ be such that $c' \notin D_A$. From Lemma 26 follows that there are a closed formula B, $d \in L$, $d < 1$ and n such that $\|B\| \in G_O$ and

$$(\|\boldsymbol{c}\| \to \|A\|)^n \otimes \|B\| \leqq_T \|\boldsymbol{d}\|\,.$$

Then

$$\|B\| \leqq_T (\|c\| \to \|A\|)^n \to \|\boldsymbol{d}\|$$

and so $(\|\boldsymbol{c}\| \to \|A\|)^n \to \|\boldsymbol{d}\| \in G_A$. Choose d', $d < d' < 1$ and c' such that

$$c > c' > c - \frac{d' - d}{n}\,.$$

Then (T 11) implies

$$T \vdash ((c \Rightarrow A)^n \Rightarrow d) \Rightarrow ((c' \Rightarrow A)^n \Rightarrow d')$$

i.e.

$$(\|c\| \to \|A\|)^n \to \|d\| \in G_O .$$

From Lemma 26(b) we have

$$(\|c\| \to \|A\|)^n \notin G_O ,$$

i.e. $c' \notin D_A$. Therefore D_A is a closed set. Analogously using tautology (T 12) we prove that H_A is closed. Since $D_A \cup H_A = L$ and L is connected, it follows that $D_A \cap H_A \neq \emptyset$.

In both cases there is c such that $\|c\| \to \|A\| \in G_O$ and $\|A\| \to \|c\| \in G_0$, i.e.

$$\|A\| \leftrightarrow \|c\| \in G_O . \qquad \square$$

The relation

$$\|A\| \approx^\wedge \|c\| \in G_O \quad \text{iff} \quad \|A\| \leftrightarrow \|B\| \in G_O$$

is an equivalence on $F_{J(T)}| \approx$. Denote the corresponding factor set by

$$F_{J(T)} \mid G_O .$$

It is possible to construct a factor algebra on this set which is similar to $\mathscr{L}(T)$ and whose operations are defined analogously.

Lemma 30.

(a) $\approx^\wedge$ is a congruence on $F_{J(T)} \mid \approx$.

(b) Let $f: F_{J(T)} \mid \approx \to F_{J(T)} \mid G_O$ be a canonical epimorphism. Then

$$f\|A\| \leqq^\wedge \|c\| \in G_O \quad \text{iff} \quad \|A\| \to \|B\| \in G_O$$

holds true where $\leqq^\wedge$ is an ordering in $F_{J(T)} \mid G_O$.

(c) Let $T \vdash A$. Then $\|A\| \in G_O$.

Proof. The proof of (a) and (b) is the same as the proof of 2.8 and 2.10 in [10] – II.

(c) If $T \vdash A$ then $T \vdash A \Leftrightarrow \mathbf{1}$. But

$$T \vdash (A_O \Rightarrow b) \Rightarrow \mathbf{1}$$

and so the proposition follows from Theorem 6. $\square$

Theorem 15. Let T be a consistent Henkin theory. Then to every closed formula A_O and $b \in L$ defined in (Pa) there is a Q-homomorphism

$$T_O: F_{J(T)} \mid \approx \to L$$

such that

$$A_S(B) \leqq T_O(|B|)$$

$$T_O(|A_O|) \leqq b$$

for every $B \in F_{J(T)}$.

Proof. Put

$$I = \{\|A\| \in F_{J(T)} \mid \approx ;\ A \text{ is a closed formula}\}$$

and let

$$g \colon F_{J(T)}|\sim \;\to F_{J(T)}| \approx$$

$$f \colon F_{J(T)}|\approx \;\to F_{J(T)}| \, G_O$$

be canonical epimorphism. Let us denote $f\|A\| := [A]$. Since T is consistent the mapping h is a monomorphism due to Lemma 25. Due to Lemma 28

$$h(L) \cap G_O = \|\mathbf{1}\|$$

and thus

$$\|\boldsymbol{a} \leftrightarrow \|\boldsymbol{b}\| \notin G_O$$

when $\|\boldsymbol{a}\| \neq \|\boldsymbol{b}\|$, i.e. $f\|\boldsymbol{a}\| \neq f\|\boldsymbol{b}\|$. From this follows that $fh \colon L \to F \mid G_O$ is a monomorphism. Lemma 29 implies $(fh)(L) = f(I)$. Indeed, $h(L) \subseteq I$ and, hence, $(fh)(L) \subseteq$ $\subseteq f(I)$.

Conversely, if $f\|A\| \in f(I)$ then there is $a \in L$ such that $\|A\| \in [\boldsymbol{a}] \in (fh)(L)$. Then $[A] = [\boldsymbol{a}]$, i.e. $f\|A\| \in (fh)(L)$. This yields $f(I) \subseteq (fh)(L)$.

The mapping

$$fh \colon L \to f(I)$$

is an isomorphism (but not generally a Q-isomorphism) since $f(I)$ is a subalgebra of $\mathscr{L}(T)$. Set

$$T_O(|A|) = \begin{cases} (fh)^{-1} f\, g(|A|) & \text{if } A \text{ is a closed formula} \\ (fh)^{-1} f\, g(|A'|) & \text{where } A' \text{ is a closure of } A \text{ if } A \\ & \text{is an open formula.} \end{cases}$$

We will demonstrate that T_O is a Q-homomorphism.

$$T_O(|\boldsymbol{a}|) = (fh)^{-1} f\, g(|\boldsymbol{a}|) = (fh)^{-1} f\|\boldsymbol{a}\| = (fh)^{-1}(fh)(a) = a\,.$$

If A, B are closed formulae then

$$T_O(|A \Rightarrow B|) = (fh)^{-1} f\, g(|A \Rightarrow B|) = (fh)^{-1} f(\|A\| \Rightarrow \|B\|) =$$
$$= T_O(|A|) \to T_O(|B|)\,.$$

Let $A(x)$ has the only free variable x. Then

$$T \vdash (\forall x)\, A \Rightarrow A_x[t]\,,$$

i.e.

$$\|(\forall x)\, A\| \leqq_T \|A_x[t]\|$$

for any term $t \in M_V$. Then

$$T_O(|(\forall x) A|) = (fh)^{-1} f \|(\forall x) A\| \leqq (fh)^{-1} f \|A_x[t]\| = T_O(|A_x[t]|)$$

holds for all $t \in M_V$. Using Henkin axiom we, at last, obtain

$$T_O(|A_x[\mathbf{r}]|) \leqq T_O(|(\forall x) A|)$$

which yields the equality

$$T_O(|(\forall x) A|) = \bigwedge_{t \in M_V} T_O(|A_x[t]|) .$$

If $A := A(x_1, \ldots, x_n)$ then $A' = (\forall x_1) \ldots (\forall x_n) A$ and the proof proceeds analogously as above.

Let $T \vdash_b B$. Then $A_S(B) \leqq a$ and $h(a) = \|\boldsymbol{a}\| \leqq_T \|B\|$ since $T \vdash \boldsymbol{a} \Rightarrow B$ (using the rule r_{Ra}). Then

$$T_O(|B|) = (fh)^{-1} f \|\bar{B}\| \geqq (fh)^{-1} f \|\boldsymbol{a}\| = a \geqq A_S(B)$$

where $\bar{B}$ is B or B' if B is a closed or an open formula respectively. In the end we obtain

$$T_O(|A_O|) = (fh)^{-1} f \|A_0\| \leqq (fh)^{-1} f h(b) = b$$

since $\|A_O\| \to h(b) \in G_O$ which yields

$$f \|A_0\| \leqq^{\wedge} f h(b)$$

by Lemma 30. □

7.4 Completeness theorems

In this section we present two completeness theorems which are generalisations of classical Gödel's completeness theorems. They are the consequence of the previous results.

Theorem 16 (completeness Theorem II). A fuzzy theory T is consistent iff it has a model. If T is consistent then to every $A \in F_{J(T)}$ and b defined in (Pa) there is a model $\mathscr{D}$ such that

$$\mathscr{D}(A) \leqq b .$$

Proof. If T has a model then it is consistent by Lemma 13.

Let T be consistent and T' be its Henkin (conservative) extension by Theorem 11. By Theorem 15 there is a Q-homomorphism

$$T: F_{J(T)} \big| \sim \to L$$

using which we can construct a canonical structure $\mathscr{D}_O$ for which

$$A'_S(A) \leqq \mathscr{D}_O(A) , \quad A \in F_{J(T)} .$$

This means that $\mathscr{D}_O$ is a model of T' and therefore of T as well. Since T' is a conservative extension it follows from Theorem 15 that $\mathscr{D}_O$ has also the other properties. □

Theorem 17 (completeness Theorem *I*). Let T' be *a* consistent theory. Then

$$T \vdash_a A \quad \text{iff} \quad T \models_a A$$

holds true for every formula $A \in F_{J(T)}$.

Proof. The model $\mathscr{D}_O$ from Theorem 16 has the property

$$a \leqq \mathscr{D}_O(A) \leqq b$$

for a given A and b defined in (Pa). If L is a finite chain then $b = a = \mathscr{D}_O(A)$. In the opposite case

$$a = \bigwedge\{b; a < b\} \geqq \bigwedge\{\mathscr{D}_O(A) ; \quad \mathscr{D}_O \text{ is a canonical model from Theorem } 16\} \geqq a .$$

Hence,

$$a = (C^{\text{syn}} A_S)\, A = (C^{\text{sem}} A_S)\, A . \qquad \square$$

Theorems 16 and 17 are the most important theorems of this paper. They reflect deep properties of first-order fuzzy logic and they have many serious consequences concerning fuzzy logic as well as its applications.

7.5 Completion of theories

This section is based on the completeness theorems.

Lemma 31. Let T be a consistent theory and $T \vdash_a A_O$. Then

$$T' = T \cup \{\neg a / \neg A_O\}$$

is a consistent extension of the theory T.

Proof. Let $(C^{\text{syn}} A_S)(\neg A_O) < \neg a$. Let L be a finite chain. Then there is a model $\mathscr{D}$ of T such that

$$\mathscr{D}(A_O) = a .$$

Since $\mathscr{D}(\neg A_O) = \neg a$, we have

$$A'_S(A) = (A_S \cup \{\neg a / \neg A_O\})\,(A) \geqq \mathscr{D}(A)$$

holds for every $A \in F_{J(T)}$. Because T' is a simple extension, it follows that $\mathscr{D} \models T'$ and thus T' is consistent.

Let $L = \langle 0, 1 \rangle$ and $b > a$. Then there is a model $\mathscr{D}_b \models T$ such that

$$\mathscr{D}_b(A_O) \leqq b .$$

Put

$$T_b = T \cup \{c / \neg A_O\}$$

where $\neg b \leqq c = \mathscr{D}_b(\neg A_O) = \neg \mathscr{D}_b(A_O)$. Clearly, $\mathscr{D}_b \models T_b$ and therefore T_b is a consistent extension of the theory T. At the same time there is a model $\mathscr{D}_b \models T_b$ such that

$$a < \mathscr{D}_{b'}(A_O) \leqq \mathscr{D}_b(A_O) \leqq b .$$

Put

$$T_{b'} = T_b \cup \{c'/\neg A_O\}$$

where $c' = \mathscr{D}_b(\neg A_O) \geqq \mathscr{D}_b(\neg A_O)$. Then $\mathscr{D}_{b'} \models T_{b'}$ and thus $T_{b'}$ is a consistent extension of T_b (and, hence, of T). We obtain a sequence of consistent theories

$$T, T_b, T_{b'}, \ldots$$

such that

$$A_S \subseteq A_{S,b} \subseteq A_{S,b'} \subseteq \ldots$$

is a chain of fuzzy sets in $F_{J(T)}$. Due to Lemma 22

$$T' = T \cup \bigcup_{\neg b \leqq d < \neg a} A_{S,d}$$

is a consistent extension of T. However,

$$\bigcup_{\neg b \leqq d < \neg a} A_{S,d} = A_S \cup \bigcup_{a \leqq d < b} \{\neg d/\neg A_O\} = A_S \cup \{\neg a/\neg A_O\}$$

since

$$\bigvee_{a < d \leqq b} \neg d = \neg \bigwedge_{a < d \leqq b} d = \neg a\,. \qquad \square$$

Corollary. Let T be a consistent theory and $T \vdash_a A$. Then

$$T = T \cup \{\mathbf{1}/(A \Rightarrow \boldsymbol{a})\}$$

is a consistent theory.

Proof. It follows from Lemma 31 that

$$T' = T \cup \{\mathbf{1}/\neg\neg(A \Rightarrow \boldsymbol{a})\}$$

is a consistent theory. Using theorem (D 10) we prove

$$T' \vdash A \Rightarrow \boldsymbol{a}\,.$$

Since C^{syn} is a closure operation, the theory

$$T'' = T' \cup \{\mathbf{1}/(A \Rightarrow \boldsymbol{a})\}$$

is a simple conservative extension and thence T'' is consistent. Since

$$A_S \cup \{\mathbf{1}/(A \Rightarrow \boldsymbol{a})\} \subseteq A''_S\,,$$

T is consistent as well. Moreover,

$$T \vdash A \Leftrightarrow \boldsymbol{a}$$

and thus

$$T \vdash_a A\,.$$

Theorem 18 (completion theorem). Let T be a consistent theory. Then there exists a complete theory T which is a simple extension of T.

Proof. Let $\langle E_i;\, i < q\rangle$ be a chain of fuzzy sets in $F_{J(T)}$ such that $T_O = T$ and

$$T_{i+1} = T_i \cup E_i$$

is consistent. Then

$$E = \bigcup_{i<q} E_i$$

is an upper bound of $\langle E_i;\ i < q\rangle$ and

$$T' = T \cup E$$

is consistent due to Lemma 22. From Zorn's lemma follows that there is a maximal fuzzy set $\bar{A} \subsetneqq F_{J(T)}$ such that $\bar{T} = T \cup \bar{A}$ is a consistent theory which is a simple extension of T.

We show that $\bar{T}$ is complete. Let $\bar{T} \vdash_a A$ and $\bar{T} \vdash_c A \Rightarrow \boldsymbol{a}$, $c < \mathbf{1}$. Then

$$T' = \bar{T} \cup \{\mathbf{1}/(A \Rightarrow \boldsymbol{a})\} = T \cup (\bar{A} \cup \{\mathbf{1}/(A \Rightarrow \boldsymbol{a})\})$$

is a consistent theory. But

$$\bar{A} \subseteq \bar{A} \cup \{\mathbf{1}/A \Rightarrow \boldsymbol{a}\}$$

– a contradiction with maximality of $\bar{A}$. □

Note that the completion of the theory T needs not be conservative.

Let T be a theory. Then the element $c \in L$ is called a *consistency threshold* for A if

$$T \cup \{d/\neg A\}$$

is consistent for all $d \leqq c$ and contradictory otherwise.

Theorem 19. $T \vdash_a A$ iff $\neg a$ is a consistency threshold for A.

Proof. Let $T \vdash_b A$ and $\neg a$ be a consistency threshold. If $b > a$ then $b \otimes \neg a > \mathbf{0}$ and thus the theory

$$T \cup \{\neg a/\neg A\}$$

cannot be consistent. Therefore $b \leqq a$. However, the theory

$$T \cup \{\neg b/\neg A\}$$

is consistent due to Lemma 31. This gives $\neg b \leqq \neg a$, i.e. $a \leqq b \leqq a$. The converse is obvious. □

8. CONCLUSION

We have studied the properties of first-order fuzzy logic based on the set of truth values which formes a residuated lattice. We have confined ourselves only to the case when this set is either a finite chain or the interval $\langle 0, 1\rangle$ since if we assume that the truth values should form a chain then these are the only structures allowing fuzzy logic to be syntactic-semantically complete. Moreover, it seems that they are the only structures allowing this completeness at al.

We have proved the generalisations of Gödel's completeness theorems which are nontrivial theorems having important consequences. Our theorems have been proved only for the basic language of first-order fuzzy logic but some facts concerning the properties of the ultrafilter G_0 make us sure that they will hold also when the language is enriched by a certain set of additional n-ary connectives (cf. [10] - II). The case when the language is enriched also by some generalised quantifiers is still unclear and needs further research.

Let us note that the system of fuzzy logic presented here can serve as a base of most of the systems of many-valued logic studied in the literature (especially in the case it is enriched by the additional connectives mentioned above). This fact has serious consequences both for the theory as well as its applications, e.g. in expert systems. We have a tool at our disposal, which is a sound theory generalising non-trivially classical logic and stepping towards the understanding of the phenomenon of vagueness which is one of the most outstanding features of human regarding of the world. Last but not least is the fact that fuzzy logic presented here is the only system (up to isomorphism) which, under the given assumptions, preserves the completeness property of classical logic. This is an encourage for all the workers in fuzzy set theory justifying the conviction that the latter can be put on theoretically well established basis.

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