# Kybernetika

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One possibility of multidimensional control system design

# ONE POSSIBILITY OF MULTIDIMENSIONAL CONTROL SYSTEM DESIGN

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The aim of this paper is to draw attention to a general approach to multidimensional control system design. The system under consideration is assumed to be linear and discrete. The individual subsystems of the overall system are of the model following type. Control quality is secured by a quadratic cost function. Solution of high dimensional matrix Riccati equations is discussed.

## 1. INTRODUCTION

Let us consider a multivariable system consisting of several subsystems each having its own command variable, in general different from the command variables of the other subsystems, and each being exposed to mutual interaction. The design of optimum control for this kind of systems is not an easy task. In this article we shall discuss an approach using the state space description of the overall system as well as of the subsystems. It is advantageous to apply this kind of mathematical means. First, the structure and the solution of optimum control of the diagonal blocks of the overall system is mostly known respecting the different control aims. Then, the multidimensional system can be set up in such a way that the diagonal blocks and the blocks of interactions represent the building stones. A second advantage of the state space approach lies in the possibility to apply the same mathematical tools and software for single-input-single-output (SISO) as well as for multi-input-multi-output (MIMO) systems.

There is a large number of different aspects which may be considered when designing a control system. Let us mention some of them: the nature of command and disturbing variables, the structure of the control loop fitted for the desired control aim, the selected type of control cost function, estimation of unmeasurable variables, arrangement of the multidimensional control system, sampling strategy in the case of digital computer control and many others. It is not possible to discuss all modifications in one article. Rather it is necessary to restrict our considerations

to one selected example showing the basic concept of the design of the control system mentioned.

The problem discussed here concerns a multidimensional linear system, controlled by a digital computer where the main subsystems correspond to model-in-the-system control representing an unconventional model-following control with known desirable response characteristics facilitating to reach a high quality of control. The cost function for each subsystem is assumed to be quadratic and designed for control error minimalization. The cost function for the overall system may differ with respect to the aim of control of the multidimensional system. It is possible to consider the general structure for a hierarchical arrangement, for optimization of dynamics of the overall system for example satisfying a quadratic cost function, for suboptimal decoupling of diagonal blocks in the sense of quadratic cost function and many others. In each case it is necessary to modify the structure of the multidimensional system according to the actual aim of control. As the overall system may be of a high dimension this article recalls the up-to-date procedures needed for the solution of high dimensional matrix Riccati equations. A numerical example illustrates the control quality of a subsystem compared with a solution without the model-in-the-subsystem.

## 2. DESCRIPTION OF A MODEL-IN-THE-SYSTEM CONTROL

Consider a discrete linear time-invariant process described by the following state equations

$$(1) \qquad x_p(k + 1) = F_p x_p(k) + g_p u(k)$$
$$y_p(k) = c_p^T x_p(k),$$

a simple model of the control process

$$(2) \qquad x_m(k + 1) = F_m x_m(k) + g_m y_w(k)$$
$$y_m(k) = c_m^T x_m(k),$$

the generator of the command variable

$$(3) \qquad x_w(k + 1) = W x_w(k)$$
$$y_w(k) = c_w^T x_w(k),$$

and the structure of the system representing a discrete version of the model-in-the-system control (Fig. 1).

Here $u_p \in \mathbb{R}$ and $u_m \in \mathbb{R}$ are scalar correcting variables, $w \in \mathbb{R}$ is a command variable, $y_p \in \mathbb{R}$ and $y_m \in \mathbb{R}$ are process and control process model outputs and $x_p \in \mathbb{R}$, $x_m \in \mathbb{R}$ and $x_w \in \mathbb{R}$ are the state vectors of dimensions $n_p$, $n_m$ and $n_w$. Hence, this description concerns a single-input-single-output process. Sampling instants

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are $k = 0, 1, 2, \ldots$. Matrices in (1), (2) and (3) have compatible dimensions. In Figure 1, $E^{-1}$ denotes the backward shift operator.

It is necessary to stress that the model of the control process does not represent a simplified model of the real system to be controlled but for a given input it enables
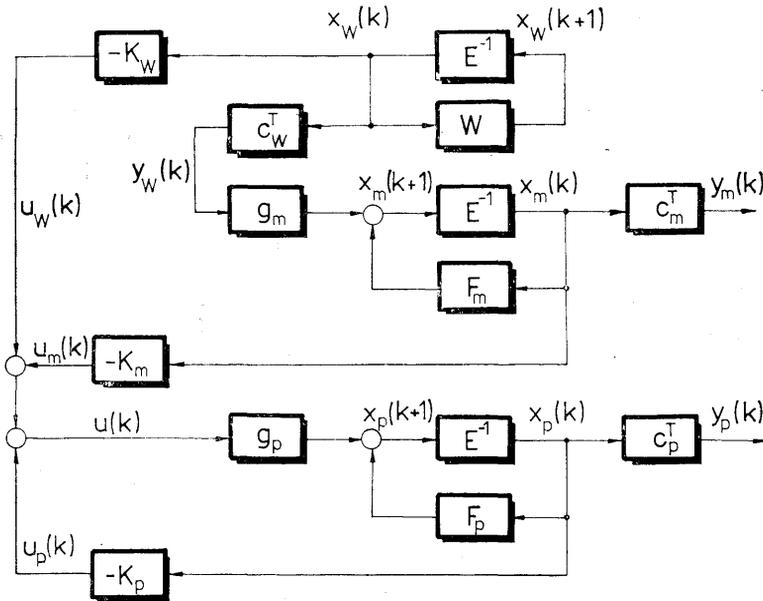


**Fig. 1.** Block diagram of the model-in-the-system control.

the definition of an idealized transient response of the real system. For practical reasons the control process model can be of a low dimension, for example $n_m = 1$ or 2, even if the dimension $n_p$ of the control process itself may be very high. In the case of control problems the model is assumed to be stable.

The structure of the system was derived in [12]. It is evident from Figure 1 that

$$(4) \qquad u(k) = u_p(k) + u_m(k) + u_w(k) = -[K_p\, x_p(k) + K_m\, x_m(k) + K_w\, x_w(k)]$$

Introducing the combined state vector $z(k)$ and output vector $y(k)$

$$(5) \qquad z(k) = \begin{bmatrix} x_p(p) \\ x_m(k) \\ x_w(k) \end{bmatrix} \quad y(k) = \begin{bmatrix} y_p(k) \\ y_m(k) \\ y_w(k) \end{bmatrix}$$

of the dimension

$$n = n_p + n_m + n_w$$

then the overall system is described by the state equations

$$(6) \qquad z(k + 1) = F\, z(k) + g\, u(k)$$
$$y(k) = C^{\mathrm{T}}\, z(k)$$

where

(7)
$$F = \begin{bmatrix} F_p & 0 & 0 \\ 0 & F_m & g_m c_w^T \\ 0 & 0 & W \end{bmatrix} \quad g = \begin{bmatrix} g_p \\ 0 \\ 0 \end{bmatrix} \quad C^T = \begin{bmatrix} c_p^T & 0 & 0 \\ 0 & c_m^T & 0 \\ 0 & 0 & c_w^T \end{bmatrix}.$$

Considering a quadratic cost function of the form

(8)
$$J = \sum_{k=0}^{\infty} [z^T(k) Q z(k) + u^T(k) R u(k)]$$

where $Q = [Q_{ij}]$, $i, j = 1, 2, \ldots, n$, is a symmetric positive semidefinite weighting matrix and $R$ is a symmetric positive definite matrix. In the case of a single-input-single-output system $R$ is a weighting constant. The problem is to find a solution minimalizing the cost function (8) and stabilizing the matrix $(F_p - g_p K_p)$.

Matrix $Q$ may have the special form

$$Q = C \begin{bmatrix} q_1 + q_2 & -q_2 & -q_1 \\ -q_2 & q_2 + q_3 & -q_3 \\ -q_1 & -q_3 & q_1 + q_3 \end{bmatrix} C^T$$

Applying to the individual blocks of the state matrices (7) the canonical form used in Section 4 and derived for example in [13], the weighting matrix $Q$ attains the form

(9)
$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} =$$

$$= \begin{bmatrix} q_1 + q_2 & 0 & \ldots & 0 & -q_2 & 0 & \ldots \ldots & 0 & -q_1 & 0 & \ldots \ldots & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \\ 0 & \cdots \cdots & 0 & 0 & \cdots \cdots & 0 & 0 & \cdots \cdots & 0 \\ -q_2 & 0 & \ldots \ldots & 0 & q_2 + q_3 & 0 & \ldots & 0 & -q_3 & 0 & \ldots \ldots & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \\ 0 & \cdots \cdots & 0 & 0 & \cdots \cdots & 0 & 0 & \cdots \cdots & 0 \\ -q_1 & 0 & \ldots \ldots & 0 & -q_3 & 0 & \ldots \ldots & 0 & q_1 + q_3 & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \\ 0 & \cdots \cdots & 0 & 0 & \cdots \cdots & 0 & 0 & \cdots \cdots & 0 \end{bmatrix}$$

of the dimension $(n; n)$, $n = n_p + n_m + n_w$.

Hence, the first term in the brackets of the cost function (8) can be written as

(10)
$$[x_p^T(k) \ x_m^T(k) \ x_w^T(k)] \ Q \begin{bmatrix} x_p(k) \\ x_m(k) \\ x_w(k) \end{bmatrix} = q_1(y_p - y_w)^2 + q_2(y_p - y_m)^2 + q_3(y_m - y_w)^2$$

By appropriate selection of the three weighting factors $q_1$, $q_2$, $q_3$ and by minimalization of the cost function (8) it is possible to satisfy different requirements of system control. If $q_2 = q_3 = 0$ and $q_1 > 0$ then only the control error $e_{pw}^2 = (y_p - y_w)^2$

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is considered. The cost function has the simplest form used for optimalization of conventional control loops. If $q_1 > 0$, $q_2 > 0$ and $q_3 = 0$ then the process output reflects the control error $e_{pw}^2$ and $e_{pm}^2 = (y_p - y_m)^2$, while the model control error $e_{mw}^2 = (y_m - y_w)^2$ is not involved. If $q_1 = 0$, $q_2 > 0$ and $q_3 > 0$ then both $e_{pm}^2$ and $e_{mw}^2$ are considered. The process output $y_p$ approaches the model output $y_m$ instrumental in following the command variable $y_w$. For all three weighting factors greater than zero we get a combination of the preceding possibilities and the resulting process output $y_p$ depends on the selected values of $q_1, q_2, q_3$.

The controlling variable $u$ can be expressed by the well known formula

$$(11) \qquad u(k) = -[g^T Pg + R]^{-1} g^T PF\, z(k) = -K\, z(k)$$

where $P$ has the general form

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

The individual blocks are of dimensions compatible with vector $z$.

Due to the form of $g$ the scalar value of the term $(g^T Pg + R)$ is

$$(12) \qquad D = g_p^T P_{11} g_p + R > 0$$

and

$$(13) \qquad u(k) = -D^{-1} g_p^T [P_{11} F_p x_p + P_{12} F_m\, x_m(k) + (P_{12} g_m c_w^T + P_{13} W) x_w(k)]$$

The result is in conformity with (4). The matrix $P$ is a solution of the matrix Riccati equation

$$(14) \qquad P = F^T PF - F^T Pg[g^T Pg + R]^{-1} g^T PF + Q$$

but only the blocks $P_{11}$, $P_{12}$ and $P_{13}$ of the matrix $P$ are needed. Substituting all matrices into (14) we obtain for the calculation of $P_{11}$ the matrix Riccati equation

$$(15) \qquad P_{11} = F_p^T P_{11} F_p - F_p^T P_{11} g_p D^{-1} g_p^T P_{11} F_p + Q_{11}$$

and for blocks $P_{12}$ and $P_{13}$ the following Lyapunov equations

$$(16) \qquad F_p^T [I - P_{11} g_p D^{-1} g_p^T] P_{12} F_m - P_{12} = -Q_{12}$$

and

$$(17) \qquad F_p^T [I - P_{11} g_p D^{-1} g_p^T] P_{13} W - P_{13} = -Q_{13} - F_p^T [I - P_{11} g_p D^{-1} g_p^T] \cdot$$
$$\cdot P_{12} g_m c_w^T$$

where $I$ is the identity matrix.

Due to the symmetry of the matrix $Q_{11}$, it can be verified by inspection of (15), that $P_{11}$ may be a symmetric matrix of the dimension $(n_p; n_p)$ too.

Let us recall that if $(F_p, g_p)$ is a stabilizable pair and if $(F_p, H_p)$ is a detectable pair, where $H_p^T H_p = Q_{11}$ and rank $(H_p) = $ rank $(Q_{11})$ then these assumptions are necessary and sufficient for the existence of a unique solution $P_{11} \geqq 0$ of (15) such that

$(F_p - g_p D^{-1} g_p^T P_{11} F_p)$ is a stable matrix. Moreover if $(F_p, H_p)$ is reconstructible, then (15) has a unique positive definite solution.

If the conditions related to pairs $(F_p, g_p)$ and $(F_p, H_p)$ are not satisfied then the Riccati equation may have more then one positive semidefinite solutions. For more details the reader is referred to the publication [4] by V. Kučera.

It follows from the preceding that if the problem is split up into three partial equations (15) through (17) then it is necessary to calculate successively $P_{11}$, $P_{12}$ and finally $P_{13}$.

Let us assume that the model is stable and that the command variable is a step function, so that $W$ is of dimension $(1; 1)$. Both these assumptions represent the simplest but realistic variant of the system.

Due to the step function of the command variable the pair $(F, g)$ is semistabilizable only, because one eigenvalue of $(F - gK)$ is equal to $W$. All the other eigenvalues can be made less then one in modulus by an apropriate choice of $K$. Owing to the selected structure of the overall system, it is possible to calculate the matrix $P$ and the controller gain $K$ of (11) by an appropriate numerical procedure for the whole system although there are two feedforward branches influencing the closed control loop.

An example of a numerical solution is given in Section 4.


## 3. EXTENSION TO MULTI-INPUT-MULTI-OUTPUT SYSTEMS

We can apply the principle of model following system described in Section 2 for MIMO systems in very different ways. As examples, let us outline application for multivariable systems, interconnected systems, hierarchical structure of systems and suboptimal decoupling of MIMO systems.

Let us consider the multivariable system first and let us assume that it has $s$ inputs and $s$ outputs. Let the dimension of the overall system be $(N; N)$, where $N = N_p +$ $+ N_m + N_w$. It is possible to apply the structure indicated by Figure 1 in the straightforward way. The single difference with respect to SISO systems refers to the dimension of the individual blocks. dim $F_p$ is $(N_p; N_p)$, dim $F_m$ is $(N_m; N_m)$ and dim $W$ is $(N_w; N_w)$. The input matrices are now $G$ of dimension $(N; s)$, $G_p$ of dimension $(N_p; s)$ and $G_m$ of dimension $(N_m; s)$. The individual output matrices are $C^T$ of dim $(3s; N)$, $C_p^T$ of dim $(s; N_p)$, $C_m^T$ of dim $(s; N_m)$ and $C_w^T$ of dim $(s; N_w)$.

It is evident from (10) that now $q_1$, $q_2$ and $q_3$ in the weighting matrix $Q$ are blocks of the dimension $(s; s)$ and dim $R$ is $(s, s)$, too. All variables are vector variables of compatible dimensions. The main aim of the application of the model following concept is to reach the most convenient dynamic behaviour of the overall system.

The structure introduced in Section 2 generalized for multivariable systems can be applied also for the suboptimal decoupling of a multidimensional system provided that the weighting matrices $Q$ and $R$ strongly accentuate the diagonal elements only.

It is possible to design a process model $F_m$ of dimension $(N_m; N_m)$ as well as a generator $W$ of dimension $(N_w; N_w)$ of the command variables, both yielding $s$ outputs. Zero blocks are set according to the pattern given by equations (7). Hence, the realization $(F_{pi}, g_{pi}, c_{pi}^T)$ follows expressively the outputs $x_{mi}$ and $x_{wi}$ of the block-diagonal matrices $F_m$ and $W$, respectively.

It is clear that the suboptimal decoupling cannot remove completely the influence of the mutual interconnections. The result of this kind of suboptimal decoupling depends on the properties of the MIMO system and on the selected parameters of $Q$ an $R$, respectively.

Interconnected systems represent a special case of MIMO systems allowing to split up the overall system into blocks according to structure (18).

$$
(18) \qquad F = \begin{bmatrix}
F_{p1} & 0 & f_{p12} & & f_{p1s} & \\
 F_{m1}\ g_{m1}c_{w1}^T & 0 & 0 & 0 & 0 & 0 \\
0 & W_1 & 0 & & 0 & \\
\hline
f_{p21} & & F_{p2} & 0 & f_{p2s} & \\
0 & 0 & F_{2m}\ g_{m2}c_{w2}^T & 0 & 0 & 0 \\
0 & & 0 & W_2 & 0 & \\
\hline
f_{ps1} & & f_{ps2} & & F_{ps} & 0 \\
0 & 0 & 0 & 0 & F_{ms}\ g_{ms}c_{ws}^T \\
0 & & 0 & & 0 & W_s
\end{bmatrix}
$$

Provided that, in general, each subsystem has its own command variable $w_i$, $i = 1, 2, \ldots s$ and its own model of the control process, which are determined by technological conditions and are not influenced by the neighbouring subsystems $S_j$, $j = 1, 2, \ldots, s, j \neq i$, then the interaction between the subsystems of a multi-dimensional system can occur only through the state vectors $x_{p,j}$, $j = 1, 2, \ldots, s$, $j \neq i$ and possibly through the inputs $u_{p,j}$, respectively. Besides this the interaction can proceed in such a way that the individual partial subsystems are elements of such a sequence that the state variables $x_{p,j}$ of any partial subsystem can act only on the state variables $x_{p,l}$, $l > j$ of the succeeding partial subsystems of this sequence but cannot influence the state variables of the preceding partial subsystems. In other words, any partial subsystem can only be influenced by the preceding partial subsystems of the sequence but not by the following partial subsystems. This type of interconnected system represents a special version designated as one-way coupling case. A more general form of an interconnected system represents the so called two-way coupling case. Here, mutual couplings arise in both directions and the one-way form may be regarded as a special case of the normal two-way canonical form. For more details the interested reader is referred to reference [13].

Let us consider the latter form of an interconnected system. The possibly non-zero elements of interactions between the matrices $F_{pi}$ in (18) are in the first columns

of the off-diagonal blocks and are denoted as $f_{pij}$, $i, j = 1, 2, \ldots, s$. The model-state variables as well as the command variables of each diagonal block do not act on other diagonal blocks.

The input matrix $G$ may have all its elements non zero except those belonging to models of the control responses and to generators of command variables.

The output matrix has the form

$$(19) \qquad C^{\mathrm{T}} = \begin{bmatrix} C_1^{\mathrm{T}} & 0 & \ldots & 0 \\ 0 & C_2^{\mathrm{T}} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & C_s^{\mathrm{T}} \end{bmatrix}$$

Now, the overall system is described by the state equations

$$(20) \qquad x(k + 1) = F\, x(k) + G\, u(k)$$
$$y(k) = C^{\mathrm{T}}\, x(k)$$

where the dimensions of the individual matrices are $F(N; N)$, $G(N, s)$, $C^{\mathrm{T}}(3s; N)$ and

$$(21) \qquad N = \sum_{i=1}^{s} n_i$$

$n_i$ being the dimensions of the subsystems $S_i$, $i = 1, 2, \ldots, s$.

The vectors $f_{pij}$, $i, j = 1, 2, \ldots, s$, $i \neq j$ describe physical interconnections of the system in question and must be determined by identification or mathematical modeling in the same way as the elements of the diagonal blocks $F_{pi}$. It is possible to design a control for the diagonal blocks individually and save in this way the computer memory, while when solving the control for the interconnected system in the whole, then the requirements on computer memory can exceed the available one.

In the case of interconnections it may happen that the overall system is unstable. Hence, after the design of all controllers of the diagonal blocks it is necessary to verify the stability of the overall system. If instability occurs, the increase of $q_{1i}$ and $q_{2i}$ values, $i = 1, 2, \ldots, s$, may remove this imperfection. In the opposite case the desired solution is not possible.

Suboptimal decoupling of interconnected systems in the sense of quadratic cost function can be achieved by similar means as in the case of multivariable systems.

A different problem is represented by the design of a hierarchical system. Let us neglect the cost function for the overall system which may be formulated in accordance with the technological requirements in very different ways and in general may or need not depend on the cost functions of the systems working at individual levels of the hierarchical control.

The hierarchical structure of $s$ levels of the multidimensional system may be achieved by the one-way coupling of the diagonal blocks, i.e., if the coupling elements are below the main diagonal of $F$ only. Hence, for $j > i$, all $f_{pij}$ in (18) are equal to zero.

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Let us assume step functions of the command variables and the selected canonical form of the state description. Then the outputs of the command variables of the diagonal blocks are $y_{wi} = x_{wi}$, $i = 1, 2, \ldots, s$, and can be influenced by the command variables of the preceding subsystems. It is possible to express the desired command variable $x_{wlo}$ in the form

$$(22) \qquad x_{wlo} = x_{wl} + \sum_i f_{wi} x_{wio}; \quad l > i$$

where $f_{wi}$ are the coupling elements selected by the designer of the hierarchical control according to the technological requirements. Indices $l, i$ denote that the command variable of the diagonal element $l$ is influenced by the command variables of the preceding subsystems $i$. The first diagonal block represents the highest level of the hierarchical structure and is not influenced by other blocks. The global control system design is performed at the subsystem level only. To ensure the consistence between Eq. (22) and the system (18)−(20) only step functions of command variables can be considered when employing the methodology described in Section 2.

It follows from (22) that, in this particuar case, it is necessary to put $x_{wlo}$ in Eq. (4) instead of $x_w$ and to change the system structure of Figure 2 accordingly. The stability of the overall system is guaranteed by the stability of all the diagonal partial systems.
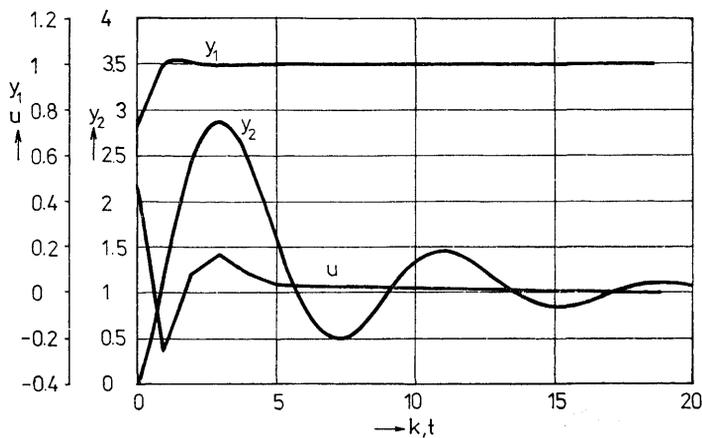


**Fig. 2.** Impulse response $y_2$ of the process, step response $y_1$ of the controlled variable and the respective controlling variable $u$ of the closed control loop.

Of course, it is feasible to have more parallel systems on each level starting by level 2. Extension in this sense is straightforward and the necessary modifications of the basic scheme are obvious.

Because the dimension (21) can reach a very high value, direct procedures elaborated for the solution of the Riccati equation of a low order may fail. Consequently, it is necessary to select a more adequate methods. In the seventies lot of attention was paid to the problem of the solution of high dimensional Riccati equation. It is

one of the most deeply studied nonlinear matrix equation arising in mathematics and engineering. Now, there exist several packages of software that can solve such equations of an order up to a few hundreds. Space limitations preclude an exhaustive description of these up-to-date procedures; references will lead the interested reader to sources of additional detailed information.

The problem in question is closely related to linear least squares and both standard and generalized eigenvalue problem. There are essentially two approaches available, the first related to Potter's method [9] utilizing generalized eigenvectors and the second related to Laub's method [5, 6] utilizing generalized Schur vectors.

As an introduction let us consider the standard problem first concerning a multi-dimensional control. The starting point is the so-called Euler or Hamiltonian symplectic matrix which in the case of discrete systems has the following form derived with respect to the minimum of the quadratic cost function

$$(23) \qquad E = \begin{bmatrix} F + GR^{-1}G^{T}F^{-T}Q & -GR^{-1}G^{T}F^{-T} \\ -F^{-T}Q & F^{-T} \end{bmatrix}$$

It is clear, that the matrix $F$ must be nonsingular. Matrix $E$ can always be represented as

$$(24) \qquad E = VJV^{-1}$$

where $J$ is the Jordan canonical form of $E$ with eigenvalues sorted in increasing order of magnitude so that $n$ stable (or semi-stable) eigenvalues are located at the first $n$ places. The sorting of eigenvalues of $E$ and as matter of course the ordering of eigenvectors accordingly is the presumption for the optimal stabilizing (or semi-stabilizing) solution of the problem. Other solutions of the Riccati equation are omitted provided that such solutions exist. According to this recommendation we may introduce the notation

$$(25) \qquad J = \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix}$$

and the matrix of corresponding eigenvectors

$$(26) \qquad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

The individual blocks in (25) and (26) are of the dimension $(n; n)$. The Riccati matrix is then

$$(27) \qquad P = V_{21}V_{11}^{-1}$$

This is the so-called standard or direct method published for example by D. R. Vaughan [14].

Closely related is the generalized eigenvalue problem [7]. In that case the Euler

matrix is written as the product of two matrices

(28)
$$E = \begin{bmatrix} I & GR^{-1}G^T \\ 0 & F^T \end{bmatrix}^{-1} \begin{bmatrix} F & 0 \\ -Q & I \end{bmatrix} = L^{-1}M$$

where $I$ is the identity matrix. Instead of the standard eigenvalue problem for $E$, rather the generalized eigenvalue problem for $(L, M)$ is solved

(29)
$$Mz = \lambda Lz$$

in order that the roots $\lambda$ satisfy the generalized characteristic equation

(30)
$$\det(M - \lambda L) = 0.$$

For each generalized eigenvalue $\lambda$ a nonzero vector $z$ satisfying (29) is called a generalized eigenvector. If $\lambda$ present a multiplicity $r > 1$, then the set of vectors $z_1, ..., z_l$ satisfying

$$Mz_1 = \lambda Lz_1$$

$$(M - \lambda L)z_k = Lz_{k-1}; \quad k = 2, 3, ..., l; \quad l \leqq r$$

is called a chain of generalized principal vectors.

Let us introduce the matrix $U$ of the dimension $(2n; n)$ of the generalized principal vectors corresponding to $n$ stable eigenvalues. Matrix $U$ can be partitioned into two submatrices $U_1, U_2$ of the dimension $(n; n)$

(31)
$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

and satisfies equation

(32)
$$MU = LUJ$$

where $J$ is the $(n; n)$ Jordan canonical form corresponding to all $\lambda_i \; i = 1, 2, ..., n$, where $|\lambda_i| < 1$.

In this case, the Riccati matrix is

(33)
$$P = U_2 U_1^{-1}.$$

It may be mentioned that the calculation of eigenvectors corresponding to the stable eigenspace of the problem is often highly unsatisfactory from a numerical point of view.

A variant of the classical eigenvector approach to Riccati equations is the application of the so-called Schur vectors as proposed by A. J. Laub [5]. This approach provides a unifying methodology applicable to standard as well as nonstandard Riccati equations including singular control weighting matrices, cross-weighting matrices, singular transition matrices and generalized state space models. Moreover, it yields reliable numerical solutions. In Laub's method the double-Francis-QR algorithm [2, 6, 8, 11] is used providing a sequence of matrices which converge to a so-called quasi-upper-triangular real Schur form (RSF), $S$. It has $1 \times 1$ diagonal blocks corresponding to real eigenvalues of $E$ and $2 \times 2$ diagonal blocks corresponding to complex-conjugate pairs of eigenvalues. All the mentioned transformations

concerning the standard problem can simply be expressed as

$$(34) \qquad\qquad U^T E U = S , \quad U^T U = I$$

Partitioning $U$ and $S$ into four $(n; n)$ blocks, we may write the solution of the Riccati equation as

$$(35) \qquad\qquad P = U_{21} U_{11}^{-1}$$

In the case of generalised Schur vectors instead of the QR algorithm, the so-called QZ algorithm is applied which transforms relation (29) into the form

$$(36) \qquad\qquad QMZx = \lambda QLZx$$

where $Q$ and $Z$ are orthogonal transforming matrices, $A = QMZ$ is quasiupper triangular and $B = QLZ$ upper triangular matrix, respectively.

Defining

$$(37) \qquad\qquad U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = Z \begin{bmatrix} I_n \\ 0_n \end{bmatrix}$$

then solution of (14) is

$$(38) \qquad\qquad P = U_2 U_1^{-1}$$

and the stable eigenvalues of $E$ are

$$(39) \qquad\qquad S = B_{11}^{-1} A_{11}$$

From the above it is evident that the application of real Schur vectors does not need the calculation of eigenvalues or generalized eigenvalues for Riccati matrix determination. Collection of subroutines [3, 10] for the solution of the Riccati equation represents a mathematical software of impressive quality. It may be stressed that the evaluation of the software under consideration is a highly non-trivial task.


## 4. EXAMPLE

The example in this section illustrates the behaviour of one block of a dynamic system designed in accordance with the principle described in Section 2.

Let us have the difference equation of the controlled system

$$\sum_{i=0}^{n_p} a_i y_p(k + n_p - i) = \sum_{i=1}^{n_p} b_i u(k + n_p - i) ; \quad a_0 = 1$$

then, in general, the matrices $F_p$, $g_p$ and $c_p^T$ can have the following expedient canonical form

$$F_p = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ -a_{n_p} & 0 & 0 & \dots & 0 \end{bmatrix} \qquad g_p = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_p} \end{bmatrix}$$

$$c_p^T = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}.$$

The same form can be applied for the system model and for the generator of the command variable.

Matrices (7) have been selected of the following form

$$F = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ -3{\cdot}68 & 0 & 1 & 0 & 0 & 0 \\ 2{\cdot}256 & 0 & 0 & 1 & 0 & 0 \\ -0{\cdot}576 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0{\cdot}2 & 0{\cdot}8 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 1{\cdot}3002 \\ -1{\cdot}4301 \\ 0{\cdot}2339 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the weighting matrices of the quadratic cost function (8)

$$Q = \begin{bmatrix} 6 & 0 & 0 & 0 & -5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 & 6 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \quad R = \begin{bmatrix} 1 \end{bmatrix}$$

Let the initial state vector $z$ be

$$z^T(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

It is evident that the system is exposed to the action of a step input. The controller matrix $K$ calculated according to (11) is of the form

$$K = \begin{bmatrix} 2{\cdot}0785 & 0{\cdot}7718 & 0{\cdot}0427 & -0{\cdot}0330 & -0{\cdot}0896 & -0{\cdot}5359 \end{bmatrix}$$

The step response is shown in Figure 2 and it is denoted as $y_1$. The curve $y_2$ belongs to the impulse response of the process itself. It is clear that it is strongly oscillating while the step response $y_1$ follows the command variable $y_w$ very well.

## 5. CONCLUSION

The described procedure of multidimensional control system design proves the usefulness of the application of the state space approach. It enables a uniform solution of control systems of different complexity. The selected structure of the subsystems represents a model-in-the system control yielding a high control quality illustrated by a numerical example. However, the multidimensional system can be modified for other control problems in a straightforward way when exchanging the individual matrix blocks by the relevant ones. Solution of high dimensional matrix Riccati equations is outlined and basic references are given.

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