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Algorithms for Solution of Equations $\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} = -\mathbf{Q}$ and $\mathbf{M}^{\mathrm{T}}\mathbf{PM} - \mathbf{P} = -\mathbf{Q}$ Resulting in Lyapunov Stability Analysis of Linear Systems

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The article deals with stability solution of the linear continuous and discrete systems by second method of Lyapunov. Two algorithms for solution of equations $\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} = -\mathbf{Q}$ or $\mathbf{M}^{\mathrm{T}}\mathbf{PM} - \mathbf{P} = -\mathbf{Q}$ are asserted. The first algorithm is accepted from [4], the second one is derived. Both algorithms are written as open programs in the language ALGOL. They can be used also as procedures. In conclusion the possibility of utilization of these algorithms for optimalization of dynamic system by quadratic cost function is shown.

1. INTRODUCTION

The article deals with algorithms for the solution of linear matrix equations which are the results of a continuous and discrete linear system stability analysis. The results can be also used to compute the cost function value of a given system and for optimal control system synthesis.

The given linear continuous system is defined by equations of state

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)\,, \qquad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t)\,, \tag{1.1}$$

where $\mathbf{x}(t)$ is the n-vector,
$\mathbf{y}(t)$ – the output m-vector,
$\mathbf{u}(t)$ – the input r-vector, and
$\mathbf{A}$ – $n \times n$-dimensional system matrix,
$\mathbf{B}$, $\mathbf{C}$ – input or output matrices, respectively.

For the purpose of investigating the stability analysis we are only interested in homogeneous equation

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{A}\,\mathbf{x}(t)\,. \tag{1.2}$$

This means that the stability of the systems (1.1) or (1.2) depends only on the form of the state matrix $\mathbf{A}$.

The linear discrete system is defined by equation of state

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{M}\,\mathbf{x}(k) + \mathbf{N}\,u(k)\,, \\ \mathbf{y}(k) &= \mathbf{C}\,\mathbf{x}(k) \end{aligned} \tag{1.3}$$

where $\mathbf{x}(k)$, $\mathbf{y}(k)$, $\mathbf{u}(k)$ are n-, m-, r-dimensional vectors of state, output and input, respectively,

$\mathbf{M}$ – $n \times n$-dimensional system matrix,
$\mathbf{N}$, $\mathbf{C}$ – input or output matrices, respectively, and
k – a sampling constant.

For the purpose of investigating the stability we are also interested in homogeneous equation

$$\mathbf{x}(k+1) = \mathbf{M}\,\mathbf{x}(k)\,. \tag{1.4}$$

It is known that the continuous systems (1.1) and (1.2) are asymptotically stable if and only if the real parts of all eigenvalues of the state matrix $\mathbf{A}$ are negative.

The discrete systems (1.3) or (1.4) are asymptotically stable if and only if the absolute value of all eigenvalues of the state matrix $\mathbf{M}$ is less than 1.

It is no simple matter to compute the eigenvalues of square matrices $\mathbf{A}$ and $\mathbf{M}$, and this is why different criteria or methods to test their stability are used. One of most important methods is the Lyapunov method which is most suitable for the non-linear systems stability analysis. Without giving a proof, we shall now introduce the Lyapunov theorems on the stability of systems. For the purpose of investigating the stability of the linear system we shall use the theorems as specified in Section 2.

Theorem 1. – Lyapunov stability of a continuous system [1]. *Let us take a system whose equation of state is in the form*

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, t) \tag{1.5}$$

and let $\mathbf{f}(0, t) = 0$ *for all values of* t.

Let us assume that a scalar function $V(\mathbf{x}, t)$ *exists with its continuous first partial derivatives. Let* $V(\mathbf{x}, t)$ *meet the following conditions*:

a) $V(\mathbf{x}, t)$ *is positively definite and* $V(0, t) = 0$;

b) $V(\mathbf{x}, t) \geqq \alpha(\|\mathbf{x}\|) > 0$ *for all instances where* $\mathbf{x} \neq 0$ *and all values of* t, *where* α *is a continuous non-decreasing function and* $\alpha(0) = 0$;

c) *derivative* $\dot{V}$ *is negative for all instances where* $\mathbf{x} \neq 0$ *and all values of* t, *i.e.* $\dot{V}(\mathbf{x}, t) \leqq -\gamma(\|\mathbf{x}\|) < 0$; *for all instances where* $\mathbf{x} \neq 0$ *and all values of* t, *where* γ *is a continuous non-decreasing scalar function and* $\gamma(0) = 0$,

d) *there exists a continuous non-decreasing function* $\beta(\|\mathbf{x}\|)$, *so that* $\beta(0) = 0$ *and so that an inequality of* $V(\mathbf{x}, t) > \beta(\|\mathbf{x}\|)$ *is valid for all values of t,*

e) *the function* $\alpha(\|\mathbf{x}\|)$ *meets condition*

$$\lim_{\mathbf{x} \to \infty} \alpha(\|\mathbf{x}\|) = \infty .$$

Then the origin $\mathbf{x} = 0$ *is asymptotically stable in the large.*

Theorem 2. – Laypunov discrete system stability [1]. *Let the discrete system be defined by equation of state*

$$\mathbf{x}(k + 1) = \mathbf{f}(\mathbf{x}(k)) \tag{1.6}$$

and let an equilibrium state be in the origin of coordinates, i.e. $\mathbf{f}(0) = 0$.

Let us suppose that there exists a scalar function $V(\mathbf{x})$, *continuous in* $\mathbf{x}$, *which meets*

a) $V(\mathbf{x}) > 0$ *for all instances where* $\mathbf{x} \neq 0$;

b) $\Delta V(\mathbf{x}) < 0$ *for all instances where* $\mathbf{x} \neq 0$;

c) $V(0) = 0$;

d) $\lim_{\|\mathbf{x}\| \to \infty} V(\mathbf{x}) = \infty$.

Then the equilibrium state, $\mathbf{x} = 0$, *is asymptotically stable in the large. The function* $V(\mathbf{x})$ *is called the Lyapunov function.*

2. STABILITY THEOREMS OF THE CONTINUOUS AND DISCRETE LINEAR SYSTEMS

Theorems 1 and 2 are only sufficient conditions but not the necessary ones. Without Lyapunov function $V(\mathbf{x})$, meeting theorems 1 or 2, we know nothing about the system's stability. The quadratic form of the Lyapunov function is the simpliest one. For a given stable system there does not exist only one Lyapunov function. If a continuous system (1.5) is linear then its equation of state is in the form of (1.1). The following theorem is valid for the Lyapunov stability analysis of this system:

Theorem 3. – Stability of linear continuous system [1, 2]. *A linear continuous system* (1.2) *is asymptotically stable in its equilibrium state if and only if a positively definite real symetrical matrix* $\mathbf{P}$ *meeting equation*

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \tag{2.1}$$

exists for arbitrary positively definite real symmetrical matrix $\mathbf{Q}$. *The quadratic form* $\langle \mathbf{x}, \mathbf{P}\mathbf{x} \rangle$ *is the Lyapunov function.*

As matrix $\mathbf{P}$ is symmetrical, the equation (2.1) is a matrix type equation for $\frac{1}{2}n(n + 1)$ unknown coefficients p_{ij}, $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $i \leqq j$. The equation (2.1) has a solution only if eigenvalues of matrix $\boldsymbol{A}$ are not equal to zero or when the sum of any two matrix $\boldsymbol{A}$ eigenvalues is not equal to zero. The simpliest form of matrix $\mathbf{Q}$ is the unit matrix $\mathbf{E}$. In section 3 we shall convert the matrix-type equation (2.1) to a system of $\frac{1}{2}n(n + 1)$ linear algebraic equations which are very easy to solve.

For the stability analysis of linear discrete systems (1.3) or (1.4) the following theorem can be used:

Theorem 4. – Linear discrete system stability [1, 2]. *The linear discrete system (1.4) is asymptotically stable in its equilibrium state* $\mathbf{x} = 0$ *if and only if a real positively definite symmetrical matrix* $\mathbf{P}$ *meeting equation*

$$\boldsymbol{M}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{M} - \mathbf{P} = -\mathbf{Q} \tag{2.2}$$

exists for an arbitrary positively definite real symmetrical matrix $\mathbf{Q}$.

The Lyapunov function for the given system is

$$V(\mathbf{x}(k)) = \langle \mathbf{x}(k), \mathbf{P}\,\mathbf{x}(k)\rangle \tag{2.3}$$

and its first difference is

$$\Delta V(\mathbf{x}(k)) = -\langle \mathbf{x}(k), \mathbf{Q}\,\mathbf{x}(k)\rangle\,. \tag{2.4}$$

The equation (2.2) is also a matrix-type equation for unknown coefficients, p_{ij}, of the matrix $\mathbf{P}$. In the next paragraph, we shall convert equation (2.2) into a system of $n(n + 1)\,\mathbf{r}$ linear algebraic equations. Investigation about the positive definiteness of matrix $\boldsymbol{P}$ can be done by use of the well-known Sylvester criterion of positive definiteness [1, 2]. A point to be mentioned is that theorems 3 and 4 provide the necessary and sufficient conditions for the linear system stability.

3. ALGORITHMS FOR THE SOLUTION OF EQUATIONS (2.1) AND (2.2)

For low-order systems, matrix-type equations (2.1) and (2.2) can be manually calculated elaborating equations for all the elements. For the purpose of solving the equation with the aid of a digital computer, algorithms must be elaborated which are suitable for converting the matrix-type equation into common linear algebraic equations. The task is to convert equation (2.1) into the linear algebraic equations

$$\boldsymbol{S}\boldsymbol{x} = \mathbf{v}\,, \tag{3.1}$$

where the vector $\mathbf{v}$ is h-dimensional, $h = \frac{1}{2}n(n + 1)$. Its elements are equal to the elements q_{ij} of the matrix $\mathbf{Q}$ above the main diagonal:

$$\mathbf{v} = [q_{11}, q_{12}, \ldots, q_{1n}, q_{22}, \ldots, q_{2n}, \ldots, q_{nn}]^{\mathrm{T}}\,.$$

The elements of matrix $\boldsymbol{S}$ depend on the elements of the systems' matrix $\boldsymbol{A}$.

Algorithms suitable for computing matrix **S** were published in [4]. We shall introduce the algorithm for its computation in Algol, which is slightly different from the one given in [4].

```
                  ik := 0; h := n * (n + 1)/2;
for k := 1 step 1 until n do for l := k step 1 until n do
   begin ij := 0; ik := ik + 1;
      for i := 1 step 1 until n do for j := i step 1 until n do
         begin ij := ij + 1;
         s[ij, ik] := if k = i and l ≠ j then a[l, j] else
                      if k ≠ i and l = j then a[k, i] else
                      if k ≠ i and l ≠ j and k = j and l ≠ i then a[l, i] else
                      if k ≠ i and l ≠ j and k ≠ j and l = i then a[k, j] else
                      if k = i and l = j and k = j and l = i then a[k, i] else
                      if k = i and l = j and k ≠ j and l ≠ i then a[k, i] + a[l, j]
                      else 0;
         end;
   end;
i := 1; j := 0;
sem: if i ≦ h then begin
                    for k := 1 step 1 until h do
                      s[i, k] := 2 * s[i, k];
                      i := i + n − j; j := j + 1;
                      go to sem;
                  end;
```

Note. In this algoritm n is dimension of matrix **A**, $a[i, j]$ are elements of matrix **A**, $s[i, j]$ are elements of matrix **S**.

Vector $\boldsymbol{x}$ is the solution of equation (3.1). Elements of the vector $\boldsymbol{x}$ are equal to the sought coefficients p_{ij} in matrix **P**.

$$\boldsymbol{x} = (p_{11}, p_{12}, p_{13}, \ldots, p_{1n}, p_{21}, \ldots, p_{nn})^{\mathsf{T}} .$$

A similar algorithm can be used to solve equation (2.2). Let's now derive this algorithm.

In equation (2.2) we shall first compute the product of matrices $\boldsymbol{M}^{\mathsf{T}}\boldsymbol{P} = \boldsymbol{B}$. For the elements of the matrix **B** hold

$$b_{lk} = \sum_{i=1}^{k-1} m_{il} p_{ik} + \sum_{i=k}^{u} m_{il} p_{ki} \tag{3.2}$$

supposing that $\sum_a^b(.) = 0$ when $b < a$.

For the elements of matrix $\boldsymbol{C} = \boldsymbol{M}^{\mathsf{T}}\boldsymbol{P}\boldsymbol{M}$ hold

$$c_{lk} = \sum_{j=1}^{n} b_{lj} m_{jk} \tag{3.3}$$

and substituting equation (3.2) into the equation (3.3) we obtain

$$c_{lk} = \sum_{j=1}^{j=n} m_{jk}\Big(\sum_{i=1}^{j-1} m_{il}p_{ij} + \sum_{i=j}^{n} m_{il}p_{ji}\Big) . \tag{3.4}$$

This means that for equation (2.2) we have a system of linear algebraic equations

$$c_{lk} - p_{lk} = q_{lk} , \quad 1 \leqq l \leqq n , \quad 1 \leqq k \leqq n , \quad l \leqq k , \tag{3.5}$$

where c_{lk} is defined in equation (3.4),

p_{lk} are unknown coefficients of matrix $\boldsymbol{P}$,

q_{lk} are known coefficients of matrix $\boldsymbol{Q}$.

Equation (3.5) can be rewritten into the system of linear equations

$$\begin{aligned}
&\sum_{j=1}^{n} m_{j1}\Big(\sum_{i=1}^{j-1} m_{i1}p_{ij} + \sum_{i=j}^{n} m_{i1}p_{ji}\Big) - p_{11} = q_{11} ,\\
&\sum_{j=1}^{n} m_{j2}\Big(\sum_{i=1}^{j-1} m_{i1}p_{ij} + \sum_{i=j}^{n} m_{i1}p_{ji}\Big) - p_{12} = q_{12} ,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\sum_{j=1}^{n} m_{jn}\Big(\sum_{i=1}^{i-1} m_{i1}p_{ij} + \sum_{i=j}^{n} m_{i1}p_{ji}\Big) - p_{1n} = q_{1n} ,\\
&\sum_{j=1}^{n} m_{j2}\Big(\sum_{i=1}^{j-1} m_{i2}p_{ij} + \sum_{i=j}^{n} m_{i2}p_{ji}\Big) - p_{22} = q_{22} ,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\sum_{j=1}^{n} m_{jn}\Big(\sum_{i=1}^{j-1} m_{i2}p_{ij} + \sum_{i=j}^{n} m_{i2}p_{ji}\Big) - p_{2n} = q_{2n} ,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\sum_{j=1}^{n} m_{jn}\Big(\sum_{i=1}^{j-1} m_{in}p_{ij} + \sum_{i=j}^{n} m_{in}p_{ji}\Big) - p_{nn} = q_{nn} .
\end{aligned} \tag{3.6}$$

The vector-like form of the previous equations is

$$\boldsymbol{D}\boldsymbol{p} = \boldsymbol{q} , \tag{3.7}$$

where

$$\boldsymbol{p} = [p_{11}, p_{12}, \ldots, p_{1n}, p_{22}, \ldots, P_{2n}, \ldots, p_{nn}]^{\mathsf{T}} ,$$

$$\boldsymbol{q} = [q_{11}, q_{12}, \ldots, q_{1n}, q_{22}, \ldots, q_{2n}, \ldots, q_{nn}]^{\mathsf{T}} .$$

It is obvious from (3.6) that for $n = 3$ $(\frac{1}{2}n(n + 1) = 6)$ the matrix $\boldsymbol{D}$ takes the form

$$
\mathbf{D} = \left[\begin{array}{ccc|cc|c}
m_{11}m_{11} - 1 & m_{11}m_{21} + m_{21}m_{11} & m_{11}m_{31} + m_{31}m_{11} & m_{21}m_{21} & m_{21}m_{31} + m_{31}m_{21} & m_{31}m_{31} \\
m_{12}m_{11} & m_{12}m_{21} + m_{22}m_{11} - 1 & m_{12}m_{31} + m_{32}m_{11} & m_{22}m_{21} & m_{22}m_{31} + m_{32}m_{21} & m_{32}m_{31} \\
m_{13}m_{11} & m_{13}m_{21} + m_{23}m_{11} & m_{13}m_{31} + m_{33}m_{11} - 1 & m_{23}m_{21} & m_{23}m_{31} + m_{33}m_{21} & m_{33}m_{31} \\
\hline
m_{12}m_{12} & m_{12}m_{22} + m_{22}m_{12} & m_{12}m_{32} + m_{32}m_{12} & m_{22}m_{22} - 1 & m_{22}m_{32} + m_{32}m_{22} & m_{32}m_{32} \\
m_{13}m_{12} & m_{13}m_{22} + m_{23}m_{12} & m_{13}m_{32} + m_{33}m_{12} & m_{23}m_{22} & m_{23}m_{32} + m_{33}m_{22} - 1 & m_{33}m_{32} \\
\hline
m_{13}m_{13} & m_{13}m_{23} + m_{33}m_{13} & m_{13}m_{33} + m_{33}m_{13} & m_{23}m_{23} & m_{23}m_{33} + m_{33}m_{23} & m_{33}m_{33} - 1
\end{array}\right]
$$

From the form of matrix $\mathbf{D}$, it is evident that it consist of row and column blocks of dimensions n, $n - 1$, $n - 2, \ldots, 1$. The number of blocks equals n^2.

By designating

s – the column dimension of the block,

q – the row dimension of the block,

r – the column index of the last element in the block of higher dimension $(s + 1)$, and

p – the row index of the last element in the block of higher dimension $(q + 1)$

we obtain the equation

$$
(3.8) \qquad d[i, j] = m[n - s + 1, i - p + n - q]\, m[j - r + n - s, n - q + 1] + m[j - r + n - s, i - p + n - q]\, m[n - s + 1, n - q + 1]
$$

for the elements of matrix $\mathbf{D}$ in equation (7) and for $i = 1, 2, \ldots, h$, $i \neq j$, $j \neq r + 1$ and $h = n(n + 1)\frac{1}{2}$.

For $r + 1 = j$, then,

$$
(3.9) \qquad d[i, j] = \tfrac{1}{2}[m[n - s + 1, i - p + n - q]\, m[j - r + n - s, n - q + 1] + m[j - r + n - s, i - p + n - q]\, m[n - s + 1, n - q + 1]],
$$

for $i = j$

$$
(3.10) \qquad d[i, j] = m[n - s + 1, i - p + n - q]\, m[j - r + n - s, n - q + 1] + m[j - r + n - s, i - p + n - q]\, m[n - s + 1, n - q + 1] - 1 .
$$

The complete interpretation of the algorithm in Algol is:

```
p := r := 0; q := s := n; h := n*(n + 1);
for j := 1 step 1 until h do
    begin for i := 1 step 1 until h do
        begin d[i, j] := m[n - s + 1, i - p + n - q] * m[j - r + n - s, n - q + 1] +
                + m[j - r + n - s, i - p + n - q] * m[n - s + 1, n - q + 1];
                if j = r + 1 then d[i, j] := d[i, j]/2;
                if i = j then d[i, j] := d[i, j] - 1;
                if i - p = q then
                                begin p := p + q; q := q - 1;
                                end;
                if i = h then
                                begin p := 0; q := n;
                                end;
        end;
        if j - r = s then
                begin r := r + s; s := s - 1;
                end;
    end;
```

4. CONCLUSION

The article states the algorithms for computing the matrix-type algebraic equations which result in stability analysis of linear continuous and discrete systems by the Lyapunov method. Algorithms were used for assembling programes for MINSK 22 and NE 503 digital computers.

The advantage of this method lies in the fact that the algorithm can be used not only for stability analysis of a given system but also for the synthesis of the linear system and computing quadratic performance measure. This fact is expressed in the two theorems below:

Theorem 5. – The quadratic cost function of a linear continuous system [1]. *Let us have the cost function for the given linear continuous system* (1.2) *in the form of*

$$J = \int_{t_0}^{\infty} \langle \mathbf{x}(t), \mathbf{Q}\, \mathbf{x}(t) \rangle \, \mathrm{d}t \tag{4.1}$$

where $\mathbf{Q}$ *is a positively definite or semidefinite real symmetrical matrix. The cost function* (4.1) *can be expressed in the form of equation*

$$J = \langle \mathbf{x}(t_0), \mathbf{P}\, \mathbf{x}(t_0) \rangle \tag{4.2}$$

where $\mathbf{P}$ *is a positively definite matrix and is the solution of the equation* (2.1).

Assuming the initial state $x(t_0)$ be a random variable uniformly distributed on the surface of the n-dimensional unit sphere, the expected value of the performance criterion for the set of such n initial state vectors $x_i(t_0)$, $i = 1, 2, \ldots, n$ is then

(4.3) $$J = \operatorname{tr} \mathbf{P} = \sum_{i=1}^{n} p_{ii} \,.$$

Quadratic cost function value of the discrete system can be expressed by means of the theorem below:

Theorem 6. – The linear discrete system, quadratic cost function [1]. *Let us assume that the cost function of the discrete system* (1.4) *is*

(4.4) $$J = \sum_{k=0}^{\infty} \langle \mathbf{x}(k), \mathbf{Q}\, \mathbf{x}(k) \rangle$$

where $\mathbf{Q}$ *is positively definite or semidefinite real symmetrical matrix. The cost function* (4.4) *can be expressed in the form of equation*

(4.5) $$J = \langle \mathbf{x}(0), \mathbf{P}\, \mathbf{x}(0) \rangle$$

where $\mathbf{P}$ *is a real positively definite matrix and is the solution of equation* (2.2).

Solution of equation (2.2) can be utilized for syntesis optimal discrete controler [7].

Let us choose for linear discrete system 1.4 cost function in the form

(4.6) $$J = \sum_{k=1}^{\infty} \langle \mathbf{x}(k), \mathbf{Q}\, \mathbf{x}(k) \rangle + \langle \mathbf{u}(k), \mathbf{R}\, \mathbf{u}(k) \rangle$$

where $\mathbf{Q}$, $\mathbf{R}$ are positively semidefinite real symmetrical matrices.

It can be derived by simple way [8] that optimal discrete control is determined by linear discrete feedback. Optimal discrete controller is determined by relation

(4.7) $$\mathbf{u}(k) = \mathbf{L}\, \mathbf{x}(k)$$

where real constante matrix $\mathbf{L}$ can be calculated by iterative relations [7]

(4.8) $$\mathbf{V}_k = \mathbf{M}_k^{\mathrm{T}} \mathbf{V}_k \mathbf{M}_k + \mathbf{L}_k^{\mathrm{T}} \mathbf{R} \mathbf{L}_k + \mathbf{Q} \,,$$

(4.9) $$\mathbf{L}_k = (\mathbf{R} + \mathbf{N}^{\mathrm{T}} \mathbf{V}_{k-1} \mathbf{N})^{-1}\, \mathbf{N} \mathbf{V}_{k-1} \mathbf{M} \,,$$

(4.10) $$\mathbf{M}_k = \mathbf{M} - \mathbf{N} \mathbf{L}_k \,.$$

These iterative procedures origine choosing $\mathbf{L}_0$ such that matrix $\mathbf{M}_0$ by (4.10) were stable. Equation (4.8) for calculation real symmetric matrix $\mathbf{V}_k$ has the form (2.2). Optimal linear feedback matrix $\mathbf{L}$ is determined

(4.11) $$\mathbf{L} = \lim_{k \to \infty} \mathbf{L}_k \,.$$

This algorithm has quadratic convergence. The similar procedure for continuous system is published in [3].

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