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A NON-ERGODIC VERSION OF RUDOLPH'S THEOREM

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The classical Ambrose theorem [1] on the representation of any ergodic flow on a non-atomic complete probability space by a flow under a function indicates opportunities how to pass from the time-continuous dynamical system to the discrete one. For ergodic aperiodic flows with finite entropy, D. Rudolph [10] showed a representation by a flow under a function assuming only two (irrationally related sufficiently small) values, the preimages of which make a generator for the basis-automorphism at the same time. By means of that it was possible to code the whole flow as discrete dynamical system over a two-element alphabet. In the present paper, such a representation is constructed for aperiodic, in general non-ergodic flows; the necessary and sufficient condition for it is the finality of the asymptotic rate of a flow.

1. INTRODUCTION AND NOTATIONS

By a flow on a probability space $(\Omega, \mathscr{F}, \mu)$ we mean any group $\{T_t\}_{t\in\mathbf{R}}$ of its automorphisms (1 : 1 measure-preserving transformations of Ω onto itself, indexed by reals) such that $T_{s+t} = T_s \circ T_t$ for $s, t \in \mathbf{R}$, and that the mapping $(\omega, t) \to T_t\omega$ is $\overline{\mathscr{F} \times \mathscr{L}} - \mathscr{F}$-measurable ($\mathscr{L}$ denotes the Lebesgue-measurable subsets of the real line, the completeness of μ is presupposed, and $\overline{\mathscr{F} \times \mathscr{L}}$ denotes the completed product σ-algebra with respect to μ and the usual Lebesgue measure λ).

As an example serves a flow built under a function (denoted by $(B, \mathscr{B}, \nu, S, f)$): $(B, \mathscr{B}, \nu)$ is a basis complete probability space with an automorphism S, and f is a measurable real function on B with $\int f \,\mathrm{d}\nu < \infty$, bounded from below by some positive constant. Put $^*B = \{(\beta, s): \beta \in B, 0 \leqq s < f(\beta)\}$, $^*\mathscr{B} = {}^*B \cap \overline{\mathscr{B} \times \mathscr{L}}$ and $^*\nu = c.\overline{\nu\times\lambda} \mid {}^*\mathscr{B}$ (the restriction to $^*\mathscr{B}$ of the completed product-measure normalized by a constant $c = 1/\int f \,\mathrm{d}\nu$). The flow is prescribed by

$$S_t(\beta, s) = (S^i\beta, s + t - \sum_{j=0}^{i-1} f(S^j\beta)) \tag{1}$$

where $t \geqq 0$, $(\beta, s) \in {}^*B$, and where $i \in \mathbf{I}$ (the integers) is uniquely determined by

$\sum_{j=0}^{i-1} f(S^j\beta) \leqq s + t < \sum_{j=0}^{i} f(S^j\beta)$ (the empty sum is taken as zero, and for negative t we put $S_t = S_{-t}^{-1}$; for the measurability of $S_t(\beta, s)$ see [1]).

Following [5], we call a flow $\{T_t\}_{t\in\mathbf{R}}$ aperiodic, if there are no $0 < s_1 < s_2$ and $F \in \mathscr{F}$ with $\mu(F) > 0$ such that, for all $\mathscr{F}$-measurable $E \subset F$, $\mu(\cup\{T_t E: s_1 < t < s_2, t \text{ rational}\} \setminus E) = 0$; if $(\Omega, \mathscr{F}, \mu)$ is a Lebesgue space, this means the same as the pointwise definition of aperiodicity. As known, the concept of the flow under a function is an underlying one since, among others, any aperiodic flow can be represented in such a form (by the representation we mean the usual point-isomorphism after deleting some invariant nullsets). Further, in case of the flow under a function, the condition of aperiodicity is equivalent to the aperiodicity of the basis-automorphism S, i.e. there is no $F \in \mathscr{B}$ with $\nu(F) > 0$ such that, for some $n \in \mathbf{N}$, $\nu(E \,\Delta S^n E) = 0$ whenever $E \subset F, E \in \mathscr{B}$ (see [5]; Δ denotes the symmetrical difference, $\mathbf{N} = \{1, 2, \ldots\}$). Below, the indicator of some set X will be denoted by χ_X.

Proposition 1. Let p, q be two positive real numbers with p/q irrational, and $\varrho \in (0, 1)$. If a flow $\{T_t\}_{t\in\mathbf{R}}$ is aperiodic, then there is a flow under a function $(B, \mathscr{B}, \nu, S, f)$ with $f = p\chi_X + q\chi_{B\setminus X}$ $(X \subset B)$ which is isomorphic to $\{T_t\}_{t\in\mathbf{R}}$ and, moreover, for which the limit

$$\chi_X^*(\beta) = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \chi_X(S^j\beta)$$

exists and equals ϱ for ν-almost all $\beta \in B$.

For the proof see [7], Theorem 2.1 and Remarks. D. Rudolph [10] constructed such a representation before, in case of an aperiodic ergodic flow on a Lebesgue space with finite entropy. Though it satisfied the relation $\chi_X^*(\beta) = \varrho$ only approximately, the partition $\xi = \{X, B \setminus X\}$, on the other hand, was a generator for S, i.e. $\sigma(\bigvee_{j\in\mathbf{I}} S^j\xi) = \mathscr{B}$ up to symmetrical differences of measure zero. (We use the customary symbols of the entropic ergodic theory. By a partition of B we mean any collection $\xi = \{X_\alpha, \alpha \in A\}$ of pairwise disjoint sets with $B = \bigcup_{\alpha\in A} X_\alpha$, $\bigvee$ means the operation of the roughest common refinement, and for an arbitrary system $\{\xi_d, d \in D\}$ of at most countable $\mathscr{B}$-measurable partitions, i.e. consisting of $\mathscr{B}$-measurable sets only, $\sigma(\bigvee_{d\in D} \xi_d)$ denotes the σ-algebra generated by their sets.)

Our aim is to generalize Rudolph's result to aperiodic (non-ergodic) flows. For the existence and the minimal generator cardinality, the crucial characteristic in the non-ergodic case is the asymptotic rate (see Proposition 4), instead of the entropy. This metrical invariant was introduced by K. Winkelbauer, see [11], [12], and we shall recall an equivalent definition of it (cf. [9]).

2. MAIN RESULTS

From now on, let us suppose that the probability space $(\Omega, \mathscr{F}, \mu)$ is countably generated, i.e. $\mathscr{F}$ is generated up to symmetrical differences of measure zero by countably many sets. Let T be an automorphism of $(\Omega, \mathscr{F}, \mu)$. If $F \in \mathscr{F}$ with $\mu(F) > 0$, we define a probability measure μ_F on $(\Omega, \mathscr{F})$ by $\mu_F(E) = \mu(E \cap F)/\mu(F)$, $E \in \mathscr{F}$. If, besides, $F \in \mathscr{I}_T = \mathscr{I}_T(\mathscr{F}) = \{F \in \mathscr{F}: TF = F\}$, T also is an automorphism of μ_F. Let $h_\mu(T)$ mean the usual entropy of T (see e.g. [3], but for the sake of simplicity let the logarithms from its definition be taken to the base 2). The asymptotic rate $H_\mu(T)$ of T is defined by

$$H_\mu(T) = \sup\{h_{\mu_F}(T): F \in \mathscr{I}_T, \mu(F) > 0\}.$$

For a flow $\{T_t\}_{t\in\mathbf{R}}$, it holds that $H_\mu(T_t) = |t| . H_\mu(T_1)$ for all $t \neq 0$ (cf. [8]), and the asymptotic rate of $\{T_t\}_{t\in\mathbf{R}}$ is defined as

$$H_\mu(\{T_t\}_{t\in\mathbf{R}}) = H_\mu(T_1).$$

According to [9],

$$H_\mu(\{T_t\}_{t\in\mathbf{R}}) = \sup\{h_{\mu_F}(T_1): F \in \bigcap_{t\in\mathbf{R}} \mathscr{I}_{T_t}, \mu(F) > 0\}.$$

Put $h(\varrho) = -\varrho \log \varrho - (1 - \varrho) \log (1 - \varrho)$ for $\varrho \in (0, 1)$; $\log = \log_2$.

Theorem 1. Let $\{T_t\}_{t\in\mathbf{R}}$ be an aperiodic flow on a complete countably generated probability space $(\Omega, \mathscr{F}, \mu)$ with finite asymptotic rate, and $\varrho \in (0, 1)$. Then, for any two $p, q > 0$ with p/q irrational and satisfying $(p\varrho + q(1 - \varrho)) . h_{\mu_F}(T_1) < h(\varrho)$ for every $F \in \bigcap_{t\in\mathbf{R}} \mathscr{I}_{T_t}(\mathscr{F})$ with $\mu(F) > 0$, there is a flow under a function $(B, \mathscr{B}, \nu, S, p\chi_Z + q\chi_{B\setminus Z})$, $Z \subset B$, which is isomorphic to $\{T_t\}_{t\in\mathbf{R}}$, and such that

(a) $\chi_Z^*(\beta) = \varrho$ for each $\beta \in B$,

(b) $\zeta = \{Z, B \setminus Z\}$ is a generator for the basis-automorphism S.

In case of infinite asymptotic rate such a representation does not exist.

Note that, according to (a) and due to the ergodic theorem, there is $\nu'(Z) = \varrho$ for every S-invariant probability ν' on $(B, \mathscr{B})$. Thus Theorem 3 in [10] is a consequence of this one. In the sense of Theorem 1, write $\overline{Z} = \{(\beta, s): \beta \in Z, 0 \leqq s < p\}$, $\overline{B \setminus Z} = \{(\beta, s): \beta \in B \setminus Z, 0 \leqq s < q\}$.

Theorem 2. Under the conditions of Theorem 1, the flow $(B, \mathscr{B}, \nu, S, p\chi_Z + q\chi_{B\setminus Z})$ can be chosen so that, moreover, the partition $\bar{\zeta} = \{\overline{Z}, \overline{B \setminus Z}\}$ is a generator for each S_t with $0 < |t| < \min\{p, q\}$.

3. PROOFS

3.1 The scheme of the proof of Theorem 1

The proof observes the ergodic case and, following [10], it is based on the construction of a finite reordering and on the assertion about minimal generator cardinality.

Let S be an automorphism of a probability space $(B, \mathscr{B}, \nu)$. If $\mathscr{T}(E, n) = \bigcup_{j=0}^{n-1} S^j E$ is a disjoint union (where $E \in \mathscr{B}$ and $n \in \mathbf{N}$), we call it a tower (with base E and height n). For a partition ξ of B, write ξ_S^n instead of $\bigvee_{j=0}^{n-1} S^{-j}\xi$ in short.

Definition. A set $Z \subset B$ is a ν-finite reordering of a set $X \subset B$, if there are pairwise disjoint towers $\mathscr{T}(E_1, n_1), \mathscr{T}(E_2, n_2), \ldots$ such that $\nu(B \setminus \bigcup_{k=1}^{\infty} \mathscr{T}(E_k, n_k)) = 0$, and that

(i) for any $k \in \mathbf{N}$, $E_k \subset F_k$ for some $F_k \in (\xi \vee \zeta)_S^{n_k}$, where $\xi = \{X, B \setminus X\}$ and $\zeta = \{Z, B \setminus Z\}$,

(ii) $\sum_{j=0}^{n_k - 1} \chi_X(S^j\beta) = \sum_{j=0}^{n_k - 1} \chi_Z(S^j\beta)$ for $\beta \in E_k$, $k \in \mathbf{N}$.

Proposition 2. The flows $(B, \mathscr{B}, \nu, S, p\chi_X + q\chi_{B\setminus X})$ and $(B, \mathscr{B}, \nu, S, p\chi_Z + q\chi_{B\setminus Z})$, where $p, q > 0$, $X \subset B$, $Z \subset B$, are isomorphic if Z is a ν-finite reordering of X.

For the proof see [10]. About the next statement (see [9]) recall that the probability space in case of a flow under a function is countably generated, if and only if the basis-space is a countably generated.

Proposition 3. Let $(B, \mathscr{B}, \nu, S, p\chi_X + q\chi_{B\setminus X})$ be a flow under a function $(p, q > 0$, $X \subset B)$ with a countably generated $(B, \mathscr{B}, \nu)$ and let, for some $\varrho \in (0, 1)$, $\chi_X^*(\beta) = \varrho$ ν-a.e. Then, for any $h \in \mathbf{R}$, the following properties are equivalent:

(a) $(p\varrho + q(1 - \varrho))\, h_{(^*\nu)_F}(S_1) < h$ for every $F \in \bigcap_{t \in \mathbf{R}} \mathscr{I}_{S_t}(^*\mathscr{B})$ with $^*\nu(F) > 0$,

(b) $h_{\nu_E}(S) < h$ for every $E \in \mathscr{I}_S(\mathscr{B})$ with $\nu(E) > 0$.

Theorem 3. Let $\varrho \in (0, 1)$, let S be an aperiodic automorphism of a countably generated probability space $(B, \mathscr{B}, \nu)$ such that $h_{\nu_E}(S) < h(\varrho)$ for every $E \in \mathscr{I}_S(\mathscr{B})$ with $\nu(E) > 0$. Let $X \in \mathscr{B}$ and $\chi_X^*(\beta) = \varrho$ for ν-almost all $\beta \in \mathscr{B}$. Then there is a ν-finite reordering $Z \in \mathscr{B}$ of X, such that $\zeta = \{Z, B \setminus Z\}$ is a generator for S and $\chi_Z^*(\beta) = \varrho$ ν-a.e.

Theorem 1 follows by a successive application of Propositions 1 and 3, Theorem 3 and Proposition 2, as the aperiodicity and the fact that $(p\varrho + q(1 - \varrho))\, h_{\mu_F}(T_1) < h(\varrho)$ for every flow-invariant set F with positive measure are preserved by an isomorphism. In case of infinite asymptotic rate of the flow, there does not exist any finite generator for S, according to Proposition 3 and [12].

First, let us notice that it suffices to prove Theorem 3 in case of the shift over a denumerable alphabet only. By $\widetilde{\mathscr{B}}(\nu)$ we will denote the measure-algebra which corresponds to $(B, \mathscr{B}, \nu)$. Due to the aperiodicity of S and $(B, \mathscr{B}, \nu)$ being countably generated, there is a countable generator η for S ([6]), and S is conjugated with the shift $S_{\mathbf{N}}$ in $(\mathbf{N}^{\mathbf{I}}, \mathscr{B}_{\mathbf{N}}, \nu^{\eta})$ (where $(S_{\mathbf{N}}x)_i = x_{i+1}$ for $x = (x_j)_{-\infty}^{\infty} \in \mathbf{N}^{\mathbf{I}}$, $i \in \mathbf{I}$, $\mathscr{B}_{\mathbf{N}}$ is the σ-algebra generated in $\mathbf{N}^{\mathbf{I}}$ by cylinders, and ν^{η} is the probability on $\mathscr{B}_{\mathbf{N}}$ induced by ν, η and S). I.e., there is a measure-algebra isomorphism $\Phi: \widetilde{\mathscr{B}}(\nu) \to \widetilde{\mathscr{B}_{\mathbf{N}}}(\nu^{\eta})$ such that $\Phi(\widetilde{S}\widetilde{E}) = \widetilde{S_{\mathbf{N}}}\Phi\widetilde{E}$, whenever $E \in \mathscr{B}$ ($\widetilde{S}$ and $\widetilde{S_{\mathbf{N}}}$ denote the induced transformations on the equi-alence-classes modulo zero in $\mathscr{B}$ and $\mathscr{B}_{\mathbf{N}}$, respectively, $E \in \widetilde{E} \in \widetilde{\mathscr{B}}(\nu)$). As $(\mathbf{N}^{\mathbf{I}}, \mathscr{B}_{\mathbf{N}})$ is a Polish space (by a suitable metric), there is the ergodic decomposition of ν^{η},

$$\nu^{\eta} = \int \nu_x \, \mathrm{d}\nu^{\eta}(x) ,$$

i.e. the system $(\nu_x, x \in \mathbf{N}^{\mathbf{I}})$ of regular conditional probabilities on $(\mathbf{N}^{\mathbf{I}}, \mathscr{B}_{\mathbf{N}})$, induced by $\mathscr{I}_{S_{\mathbf{N}}}(\mathscr{B}_{\mathbf{N}})$ with respect to ν^{η} (as it is known, ν_x is ergodic ν^{η}-a.e.). For the given X in Theorem 3, choose an arbitrary $Y \in \Phi\widetilde{X}$. For the conjugacy,

$$\chi_Y^*(x) = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \chi_Y(S_{\mathbf{N}}^j x) = \varrho \quad \nu^{\eta}\text{-a.e.},$$

and $h_{(\nu^{\eta})_F}(S_{\mathbf{N}}) < h(\varrho)$ whenever $F \in \mathscr{I}_{S_{\mathbf{N}}}(\mathscr{B}_{\mathbf{N}})$ with $\nu(F) > 0$. Now, if we find a ν^{η}-finite reordering $V \in \mathscr{B}_{\mathbf{N}}$ of Y such that $\{V, \mathbf{N}^{\mathbf{I}} \setminus V\}$ is a generator for $S_{\mathbf{N}}$ and ν^{η} and $\chi_V^*(x) = \varrho$ ν^{η}-a.e., an arbitrary $Z \in \Phi^{-1}\widetilde{V}$ will satisfy all the requirements of Theorem 3.

Proposition 4. Let S be an aperiodic automorphism of a countably generated $(B, \mathscr{B}, \nu)$. If $h_{\nu_E}(S) < 1$ for every $E \in \mathscr{I}_S(\mathscr{B})$ with $\nu(E) > 0$, then there exists a two-element generator for S.

Proof. It suffices to show it for $S = S_{\mathbf{N}}$, $(B, \mathscr{B}, \nu) = (\mathbf{N}^{\mathbf{I}}, \mathscr{B}_{\mathbf{N}}, \nu^{\eta})$. After completion we obtain a Lebesgue space, and the assertion follows from [3], Theorem 30.1. □

Proposition 5. Let S be an aperiodic automorphism of a countably generated $(B, \mathscr{B}, \nu)$, let $\xi = \{X_1, \ldots, X_k\}$ be a finite $\mathscr{B}$-measurable partition of B, $n \in \mathbf{N}$ and $\varepsilon > 0$. Then there exists a tower $\mathscr{T}(E, n)$ with $\nu(\mathscr{T}(E, n)) > 1 - \varepsilon$ such that $\nu(X_j) = \nu_E(X_j)$ for any $j = 1, \ldots, k$. (This fact will be recorded as $\mathscr{T}(E, n; \xi, \nu, \varepsilon)$.)

Proof. It is given in [4] in case of the Lebesgue space. Our situation can be transferred to this case as before. □

3.2 Construction of a ν-finite reordering (Proof of Theorem 3)

We start with a combinatorial assertion (cf. [10]). Let $\varrho_0 \in (0, 1)$ be a rational. A finite zero-one sequence $(b_j)_{j=1}^n \in \{0, 1\}^n$ of length n is called (r, ϱ_0)-allowable $(r \in \mathbf{N})$ if it contains just $\varrho_0 n$-times the unit $(\sum_{j=1}^n b_j = \varrho_0 n)$, and at the same time, no constant r-block (i.e. there is no j, $1 \leqq j \leqq n - r + 1$, with $b_j = b_{j+1} = \ldots = b_{j+r-1}$).

Lemma 1. Let $\delta \in (0, 1)$. There exist $r(\delta, \varrho_0)$ and $n(\delta, \varrho_0)$ such that, for any $r \geqq r(\delta, \varrho_0)$ and any $n \geqq n(\delta, \varrho_0)$ with $\varrho_0 n \in \mathbf{N}$ $(r, n \in \mathbf{N})$, the number of (r, ϱ_0)-allowable sequences of length n is greater than $2^{(1-\delta)nh(\varrho_0)}$.

Proof. According to Stirling's formula, there is $m_0 = m(\delta, \varrho_0) \in \mathbf{N}$ with $\varrho_0 m_0 \in \mathbf{N}$ such that $\binom{m_0}{\varrho_0 m_0} > 2^{(1-\frac{1}{2}\delta)m_0 h(\varrho_0)}$. Put $r(\delta, \varrho_0) = 2m_0$ and $n(\delta, \varrho_0) = [2m_0/\delta]$ (the integer-part). Let $n \geqq n(\delta, \varrho_0)$ with $\varrho_0 n \in \mathbf{N}$. The number of subsets $A \subset \{1, 2, \ldots, n\}$ with $|A| = \varrho_0 n$ and

$$|A \cap \{jm_0 + 1, jm_0 + 2, \ldots, (j+1)\, m_0\}| = \varrho_0 m_0 \quad \text{for all}$$
$$j = 0, 1, \ldots, [n/m_0] - 1\,,$$

is greater than

$$2^{(1-\frac{1}{2}\delta)m_0 h(\varrho_0)[n/m_0]} > 2^{(1-\frac{1}{2}\delta)(n-m_0)h(\varrho_0)} > 2^{(1-\delta)nh(\varrho_0)}\,,$$

and, if we regard A as a zero-one sequence of length n (the ones denote the elements of A), it is $(2m_0, \varrho_0)$-allowable, hence (r, ϱ_0)-allowable for all $r \geqq r(\delta, \varrho_0)$. □

We write in short $\bar{0}_r$ and $\bar{1}_r$ for the list of $0, \ldots, 0$ (r-times) and $1, \ldots, 1$ (r-times), respectively. We say that a finite zero-one sequence $(b_j)_{j=1}^n$ contains no r-double block $(2r \leqq n)$ if there is no $1 \leqq j < n - 2r$ with $(b_j, b_{j+1}, \ldots, b_{j+2r-1})$ equal to $(\bar{0}_r, \bar{1}_r)$ or $(\bar{1}_r, \bar{0}_r)$. It is evident that, in case $r > 1$, we can reorder (i.e. permute) any finite zero-one sequence such that it contains no r-double block.

From now on, let ϱ, $(B, \mathscr{B}, \nu)$, S and X be the same as in the supposition of Theorem 3 and, in accordance with the observation above, we can presuppose the case $S = S_{\mathbf{N}}$, $(B, \mathscr{B}, \nu) = (\mathbf{N}^{\mathbf{I}}, \mathscr{B}_{\mathbf{N}}, \nu^\eta)$. Let $(\nu_\beta, \beta \in B)$ be the ergodic decomposition of ν, and write $\xi = \{X, B \setminus X\}$.

For any $F \subset B$, $\beta \in B$ and for $k \leqq n$, the zero-one sequence $(\chi_F(S^k\beta), \chi_F(S^{k+1}\beta), \ldots \ldots, \chi_F(S^n\beta))$ will be denoted by $[\beta]_k^n(F)$ (call it as $[\beta]_k^n(F)$-name, too), and let $|[\beta]_k^n(F)| = \sum_{j=k}^{n} \chi_F(S^j\beta)$. Further, put $\chi_F^{(n)}(\beta) = (1/n) \sum_{j=0}^{n-1} \chi_F(S^j\beta)$ for $n \in \mathbf{N}$. For $j \in \mathbf{I}$ and $r \in \mathbf{N}$, let

$$M_j^r(F) = \{\beta \in B\colon [\beta]_j^{j+4r-1}(F) = (\bar{0}_r, \bar{1}_r, \bar{0}_r, \bar{1}_r)\}\,.$$

As $h_{\nu_E}(S) < h(\varrho) \leqq \log 2 = 1$ for every $E \in \mathscr{I}_S$ with $\nu(E) > 0$, there is a two-element generator $\gamma = \{C, B \setminus C\}$ for S by Proposition 4. Let $B_0 \in \mathscr{I}_S$ with $\nu(B_0) > 0$; obviously, γ is a generator for S with respect to ν_{B_0}, too. We say that B_0 has the property $\mathfrak{P}((\mathscr{T}(E_i, n_i), \varepsilon_i)_{i=1}^\infty, \varrho_0, \sigma, r)$ if

1° $\varrho_0 \in (0, 1)$ is rational (denote its denominator by l_0), $r \in \mathbf{N}$, $r > 2$, $\sigma \in (0, h(\varrho_0))$; let $\hat{\varrho}_0 = \min\{\varrho_0, 1 - \varrho_0\}$ and $\sigma' = \sigma/(1 + 1/2^7)$;

2° $(\varepsilon_i)_{i=1}^\infty$ is a decreasing sequence in $(0, 1)$ such that, for each i, $\varepsilon_i < \hat{\varrho}_0 \sigma' h(\varrho_0)/2^{i+7}$;

3° $(n_i)_{i=1}^\infty$ is an increasing sequence in $\mathbf{N}$ such that $\varepsilon_1 n_1 \geqq l_0$, $\hat{\varrho}_0 \sigma' n_1/2^3 > r + 1$ and, for each i, $n_i < \varepsilon_{i+1} n_{i+1}$;

4° $|\{E \in \gamma_S^{n_1}\colon E \cap E_1 \neq \varnothing\}| < 2^{(h(\varrho_0) - \sigma + \varepsilon_1)n_1}$;

5° for any $E \in \xi_S^{n_1}$ with $E \cap E_1 \neq \emptyset$, it is $|\chi_X^{(n_1)}(E) - \varrho_0| < \varepsilon_1$;

6° $\mathscr{T}(E_i, n_i)$ is a tower and $\nu_{B_0}(\mathscr{T}(E_i, n_i)) > 1 - \varepsilon_i$ (denote $\mathscr{T}_i = \mathscr{T}(E_i, n_i)$ and $\tau_i = \{\mathscr{T}_i, B \setminus \mathscr{T}_i\}$) for each i;

7° for any $i \in \mathbf{N}$, if $E \in (\tau_i)_S^{n_{i+1}}$ with $E \cap E_{i+1} \neq \emptyset$, then $\chi_{\mathscr{T}_i}^{(n_{i+1})}(E) > 1 - 2\varepsilon_i$;

8° there are $N, N' \in \mathbf{N}$ such that $\varrho_0 N, \varrho_0 N' \in \mathbf{N}$, $(1 - \sigma')\, n_1 \leqq N < (1 - \sigma' + \varepsilon_1)\, n_1$, $\frac{1}{2}\sigma' n_1 \leqq N' < (\frac{1}{2}\sigma' + \varepsilon_1)\, n_1$, and such that the number of (r, ϱ_0)-allowable sequences of length N and N' is greater than $2^{(1-\varepsilon_1)Nh(\varrho_0)}$ and $2^{(1-\varepsilon_1)N'h(\varrho_0)}$, respectively.

Lemma 2. Let B_0 satisfy $\mathfrak{P}((\mathscr{T}(E_i, n_i), \varepsilon_i)_{i=1}^{\infty}, \varrho_0, \sigma, r)$. Then there is a ν_{B_0}-finite reordering $Z_{(0)} \in \mathscr{B}$ of X such that $\zeta_{(0)} = \{Z_{(0)}, B \setminus Z_{(0)}\}$ is a generator for S with respect to ν_{B_0}, and such that $\nu(B_0 \,\Delta B_0') = 0$ for

$$B_0' = \bigcap_{r'=r+1}^{\infty} \bigcap_{i=-\infty}^{\infty} (B \setminus M_i^{r'}(Z_{(0)})) \cap \Big(\bigcup_{j=-\infty}^{\infty} M_j^{r}(Z_{(0)})\Big) .$$

The proof will be given in the next section. For better understanding of Conditions 1°–8°, we show at first that the whole space can be decomposed (up to a nullset) into countably many parts, each of which satisfies the conditions with its own parameters.

At first, let us fix a decreasing sequence $(\sigma_k)_{k=1}^{\infty}$ in $(0, 1)$ with $\lim_k \sigma_k = 0$, and let $G_k = \{\beta\colon h_{\nu_\beta}(S) \leqq h(\varrho) - \sigma_k\}$ for every k (observe that G_k belongs to $\mathscr{I}_S$, having in mind that the probabilities ν_β are induced by $\mathscr{I}_S$). Due to the supposition and the integral representation of the entropy, $\nu(\bigcup_{k=1}^{\infty} G_k) = 1$.

Fix $k \in \mathbf{N}$ such that $\nu(G_k) > 0$ and put $\sigma_k' = \sigma_k/(1 + 1/2^7)$, $\hat{\varrho} = \min\{\varrho, 1 - \varrho\}$. There is $c_k \in \mathbf{N}$ such that $1/2^{c_k} < \hat{\varrho}\sigma_k' \, h(\varrho)/2^7$. For each i, put $\varepsilon_i = 1/2^{i+c_k}$, and let $(\varrho_i)_{i=1}^{\infty}$ be a sequence in $(0, 1)$ of rational numbers such that, for every i, $h(\varrho_i) \geqq h(\varrho)$ and $|\varrho_i - \varrho| < \varepsilon_i$ (hence $\hat{\varrho}_i \geqq \hat{\varrho}$ where $\hat{\varrho}_i = \min\{\varrho_i, 1 - \varrho_i\}$). For each i, choose an arbitrary $r_i = r_{i,k} \in \mathbf{N}$ such that $r_i \geqq r_i(\varepsilon_i, \varrho_i)$ (i.e., in the sense of Lemma 1, by $\delta = \varepsilon_i$ and $\varrho_0 = \varrho_i$, r_i is sufficiently large).

Let $(\alpha_i)_{i=1}^{\infty}$ be a sequence in $(0, 1)$ with $\sum_{i=1}^{\infty} \alpha_i < \infty$. Since, for given $c \in (0, 1)$ and $\delta \in (0, c)$, $\nu_{G_k}(\{\beta\colon \nu_\beta(E) > 1 - c\}) > 1 - \delta/c$ whenever $E \in \mathscr{B}$ with $\nu_{G_k}(E) > 1 - \delta$, there exists a sequence $(\delta_i)_{i=1}^{\infty}$ in $(0, 1)$ such that, for every i,

$$\nu_{G_k}(\{\beta\colon \nu_\beta(E) > 1 - \varepsilon_i\}) > 1 - \alpha_i\,, \tag{2}$$

whenever $\nu_{G_k}(E) > 1 - 4\delta_i$.

It holds that, for any $i \in \mathbf{N}$, there is $m_i \in \mathbf{N}$ which satisfies the following:

1. For every $n \geqq m_i$, there is some $U_{i,n} \subset \gamma_S^n$ such that $\nu_{G_k}(\bigcup(U_{i,n})) > 1 - \delta_i$ and $(1/n) \log |U_{i,n}| < h(\varrho) - \sigma_k + \varepsilon_i$. This is due to the McMillan theorem ([3]).
2. For any $n \geqq m_i$, $\nu_{G_k}(V_{i,n}) > 1 - \delta_i$ for $V_{i,n} = \{\beta\colon |\chi_X^{(n)}(\beta) - \varrho_i| < \varepsilon_i\}$. It is due to the supposition $\chi_X^*(\beta) = \varrho$ ν-a.e. and the choice of ϱ_i near ϱ.

3. Continuing in the sense of Lemma 1,

$$\min\left\{(1-\sigma_k')\,m_i;\ \frac{\sigma_k'}{2}\,m_i\right\} \geqq n(\varepsilon_i, \varrho_i).$$

Besides, $\hat{\varrho}\sigma_k' m_i/2^3 > r_i + 1$ and $\varepsilon_i m_i > l_i$ (the denominator of ϱ_i).

We will construct an auxiliary sequence of towers $\mathscr{T}_i = \mathscr{T}(E_i, n_i)$ now. In the first step, put $n_1 = m_1$. By Proposition 5, there is a tower $\mathscr{T}(E_1', n_1; (\xi \vee \gamma)_S^{n_1}, \nu_{G_k}, \delta_1)$; we put $E_1 = E_1' \cap U_{1,n_1} \cap V_{1,n_1}$. As $\nu_{G_k}(E_1) > (1 - 2\delta_1)\,\nu_{G_k}(E_1')$, $\nu_{G_k}(\mathscr{T}_1) > 1 - 3\delta_1$ for $\mathscr{T}_1 = \mathscr{T}(E_1, n_1)$.

4. In the $(i+1)$-st step, assume that the tower $\mathscr{T}_i = \mathscr{T}(E_i, n_i)$ with $\nu_{G_k}(\mathscr{T}_i) > 1 - 4\delta_i$ has been constructed. Due to the ergodic theorem,

$$\lim_n \int \left|\chi_{\mathscr{T}_i}^{(n)}(\beta) - \nu_\beta(\mathscr{T}_i)\right| \mathrm{d}\nu_{G_k}(\beta) = 0\,.$$

Write $W_{i+1,n} = \{\beta: |\chi_{\mathscr{T}_i}^{(n)}(\beta) - \nu_\beta(\mathscr{T}_i)| < \varepsilon_i\}$ for each $n \geqq 1$. There is m_{i+1}' such that $\nu_{G_k}(W_{i+1,n}) > 1 - \delta_{i+1}$, whenever $n \geqq m_{i+1}'$, and in addition to it, let $\varepsilon_{i+1} m_{i+1}' > n_i$. Let $n_{i+1} = \max\{m_{i+1}, m_{i+1}'\}$, and let $\vartheta_{i+1} = \{W_{i+1,n_{i+1}}, B \setminus W_{i+1,n_{i+1}}\}$. As before, there is a tower

$$\mathscr{T}(E_{i+1}', n_{i+1}; (\xi \vee \gamma)_S^{n_{i+1}} \vee \vartheta_{i+1}, \nu_{G_k}, \delta_{i+1}).$$

We put $E_{i+1} = E_{i+1}' \cap U_{i+1,n_{i+1}} \cap V_{i+1,n_{i+1}} \cap W_{i+1,n_{i+1}}$. There is $\nu_{G_k}(E_{i+1}) > (1 - 3\delta_{i+1})\,\nu_{G_k}(E_{i+1}')$, and so $\nu_{G_k}(\mathscr{T}_{i+1}) > 1 - 4\delta_{i+1}$ for $\mathscr{T}_{i+1} = \mathscr{T}(E_{i+1}, n_{i+1})$.

For $i \geqq 1$, let $A_i = \{\beta \in G_k: \nu_\beta(\mathscr{T}_i) > 1 - \varepsilon_i\}$. It is always $\nu_{G_k}(A_i) > 1 - \alpha_i$ by (2) because $\nu(\mathscr{T}_i) > 1 - 4\delta_i$. Thus $\nu_{G_k}(A) = 1$ for $A = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} A_i$ and, for some $j_k \in \mathbf{N}$, $\nu_{G_k}(\bigcap_{i=j_k}^{\infty} A_i) > 1 - 1/k$. Put $B_k = \bigcap_{i=j_k}^{\infty} A_i$; $B_k \in \mathscr{I}_S$ because so is every A_i.

Lemma 3. B_k has the property $\mathfrak{P}((\mathscr{T}(B_k \cap E_{j_k+i-1}, n_{j_k+i-1}), \varepsilon_{j_k+i-1})_{i=1}^{\infty}, \varrho_{j_k}, \sigma_k, r_{j_k})$.

Proof. As $B_k \subset A_{j_k+i-1}$ for every $i \geqq 1$, we obtain Conditions 6° and 7° seeing that

$$\left|\chi_{\mathscr{T}_{j_k+i-1}}^{(n_{j_k+i})}(\beta) - \nu_\beta(\mathscr{T}_{j_k+i-1})\right| < \varepsilon_{j_k+i-1}$$

for every $\beta \in E_{j_k+i}$ $(\subset W_{j_k+i,n_{j_k+i}})$. Conditions 4° and 5° are satisfied, too, due to the definitions of $U_{j_k,n_{j_k}}$ and $V_{j_k,n_{j_k}}$, respectively. According to Lemma 1 and as m_i is sufficiently large, we obtain 8°, and Conditions 1°–3° are obvious. □

Further, take the next $k' > k$ with $\nu(G_{k'} \setminus B_k) > 0$ and repeat the whole procedure for $G_{k'} \setminus B_k$ and $\sigma_{k'}$ instead of G_k and σ_k. But now, choose the numbers $r_{i,k'}$ such that $r_{i,k'} > r_{j_k,k}$ for every $i \in \mathbf{N}$.

Thus, we decomposed the space B up to a nullset into countably many parts B_k, $k \in K$, where $K \subset \mathbf{N}$. Note that $r_{j_k} < r_{j_{k'}}$ whenever $k < k'(k, k' \in K)$.

Following Lemma 2, for any $k \in K$, there is a generator $\zeta_{(k)} = \{Z_{(k)}, B \setminus Z_{(k)}\}$ for S with respect to ν_{B_k} such that $Z_{(k)}$ is a ν_{B_k}-finite reordering of X and $\nu(B_k \Delta B'_k) = 0$, where

$$B'_k = \bigcap_{r=r_{j_k}+1}^{\infty} \bigcap_{i=-\infty}^{\infty} (B \setminus M_i^r(Z_{(k)})) \cap \Big(\bigcup_{j=-\infty}^{\infty} M_j^{r_{j_k}}(Z_{(k)}) \Big) . \tag{3}$$

Let $Z = \bigcup_{k \in K} Z_{(k)}$. Clearly, Z is a ν-finite reordering of X (this can be verified by the components B_k), and $\zeta = \{Z, B \setminus Z\}$ is a generator for S with respect to ν (this will also be proved by components which we obtain due to (3) up to nullsets).

The relation $\chi_Z^*(\beta) = \varrho$ ν-a.e. follows directly from the definition of ν-finite reordering. Namely, for ν-almost all $\beta \in B$, it is, for every $j \geqq 0$, $S^j\beta \in \bigcup_{k=1}^{\infty} \mathscr{T}(\dot{E}_k, \dot{n}_k)$, where $(\mathscr{T}(\dot{E}_k, \dot{n}_k))_{k=1}^{\infty}$ are the towers from the reordering. For any $n \in \mathbf{N}$, the $[\beta]_0^n(Z)$-name consists only of the blocks $[\beta]_{j'}^{j''}(Z)$ which correspond to the passages through these towers (and are attached to each other; the first and the last block need not be complete). Due to (ii) in the definition, $\sum_{j=j'}^{j''} \chi_Z(S^j\beta) = \sum_{j=j'}^{j''} \chi_X(S^j\beta)$ for each such block, and so $\chi_Z^*(\beta) = \chi_X^*(\beta)$.

Finally, for the proof of Theorem 3 it suffices to prove Lemma 2.

3.3 Proof of Lemma 2

Let $B_0 \in \mathscr{I}_S$, $\nu(B_0) > 0$, which satisfies $\mathfrak{P}((\mathscr{T}(E_i, n_i), \varepsilon_i)_{i=1}^{\infty}, \varrho_0, \sigma, r)$. We will construct the desired generator $\zeta_{(0)}$ by the induction. Put $n'_1 = n_1$ and, for $i > 1$, $n'_i = n_i - 2\varepsilon_{i-1}n_i - 2n_{i-1}$.

As the first approximation to $\zeta_{(0)}$ we will construct a partition $\zeta_1 = \{Z_1, B \setminus Z_1\}$ in the first step. Let φ_1 be a fixed injective map which assigns an (r, ϱ_0) – allowable sequence of length N to each $E \in \gamma_S^{n_1}$ with $E \cap E_1 \neq \emptyset$ (the number of the sequences is greater than

$$2^{(1-\varepsilon_1)Nh(\varrho_0)} > 2^{(h(\varrho_0)-\sigma+\varepsilon_1)n_1} > |\{E \in \gamma_S^{n_1} : E \cap E_1 \neq \emptyset\}|$$

due to 8°, 2°, 1° and 4°). Put $Z_1 \setminus \mathscr{T}_1 = X \setminus \mathscr{T}_1$ (we write $\mathscr{T}_i = \mathscr{T}(E_i, n_i)$ in short) and define $Z_1 \cap \mathscr{T}_1$ in such a way that the $[\beta]_0^{n_1-1}(Z_1)$ – name is prescribed to each $\beta \in E_1$ as some reordering of $[\beta]_0^{n_1-1}(X)$ which in addition, satisfies:

(a) $[\beta]_0^{4r-1}(Z_1) = (\bar{0}_r, \bar{1}_r, \bar{0}_r, \bar{1}_r)$,
(b) $[\beta]_{4r}^{4r+N-1}(Z_1) = \varphi_1(E)$ where $\beta \in E \in \gamma_S^{n_1}$,
(c) $[\beta]_{4r+N}^{4r+N+N'-1}(Z_1) = (\bar{1}_{\varrho_0 N'}, \bar{0}_{(1-\varrho_0)N'})$,
(d) $[\beta]_{4r+N+N'}^{n_1-4}(Z_1)$ contains no r-double block,
(e) $[\beta]_{n_1-3}^{n_1-1}(Z_1) = (1, 0, 1)$.

This is possible (with regard to the reordering requirement $|[\beta]_0^{n_1-1}(X)| = |[\beta]_0^{n_1-1}(Z_1)|$) due to 5°:

$$|[\beta]_0^{4r+N+N'-1}(Z_1)| + |[\beta]_{n_1-3}^{n_1-1}(Z_1)| = 2r + \varrho_0(N + N') + 2 <$$

$$< \hat{\varrho}_0\sigma' n_1/2^2 + \varrho_0(1 - \sigma'/2 + 2\varepsilon_1)\, n_1 \leqq \varrho_0(1 - \sigma'/4 + 2\varepsilon_1)\, n_1 <$$
$$< \varrho_0 n_1(1 - \varepsilon_1/\varrho_0) = n_1(\varrho_0 - \varepsilon_1) < |[\beta]_0^{n_1 - 1}(X)|$$

by the use of 3°, 8°, 1°, 2°, too, and, similarly,

$$|[\beta]_0^{4r+N+N'-1}(B \setminus Z_1)| + |[\beta]_{n_1-3}^{n_1-1}(B \setminus Z_1)| = 2r + (1 - \varrho_0)(N + N') +$$
$$+ 1 < \hat{\varrho}_0\sigma' n_1/2^2 + (1 - \varrho_0)\left(1 - \frac{\sigma'}{2} + 2\varepsilon_1\right) n_1 < (1 - \varrho_0)\, n_1\left(1 - \frac{\varepsilon_1}{1 - \varrho_0}\right) =$$
$$= n_1(1 - \varrho_0 - \varepsilon_1) < |[\beta]_0^{n_1 - 1}(B \setminus X)|\,.$$

In order that $Z_1 \in \mathscr{B}$, we prescribe, for every $E \in (\xi \vee \gamma)_S^{n_1}$, the $[\beta]_0^{n_1-1}(Z_1)$-name identical for each $\beta \in E \cap E_1$.

The induction-supposition

In $(i + 1)$-st step $(i \geqq 1)$ we will construct a partition $\zeta_{i+1} = \{Z_{i+1}, B \setminus Z_{i+1}\}$ under the following supposition. A measurable partition $\zeta_i = \{Z_i, B \setminus Z_i\}$ and $\mathscr{T}'_i \subset \mathscr{T}_i(\mathscr{T}'_i \in \mathscr{B})$ are given. The set $\mathscr{T}'_i$ arose from the tower $\mathscr{T}_i$ by deleting some parts of some lowest and highest storeys (for the purpose of finite reordering; the sense will be clarified later). More precisely, to each $\beta \in E_i$, there are given integers $0 \leqq j_i(\beta) < \bar{j}_i(\beta) < n_i$ (measurable in β), such that $m_i(\beta) = \bar{j}_i(\beta) - j_i(\beta) + 1 \geqq n'_i$, and

$$E'_i = \{S^{j_i(\beta)}\beta \colon \beta \in E_i\}, \quad \mathscr{T}'_i = \bigcup_{\beta \in E_i'} \bigcup_{j=0}^{m_i(\beta)-1} \{S^j\beta\}$$

(where $m_i(S^{j_i(\beta)}\beta) = m_i(\beta)$ for $\beta \in E_i$). Further, for any $\beta \in E'_i$,

$$|[\beta]_0^{m_i(\beta)-1}(Z_i)| = |[\beta]_0^{m_i(\beta)-1}(Z_{i-1})| = |[\beta]_0^{m_i(\beta)-1}(X)| \tag{4}$$

(in case $i = 1$ we put $Z_0 = X$),

$$[\beta]_0^{4r+2i-3}(Z_i) = (\bar{0}_{r+i-1}, \bar{1}_{r+i-1}, \bar{0}_r, \bar{1}_r)\,, \tag{5}$$

$$[\beta]_{m_i(\beta)-3}^{m_i(\beta)-1} = (1, 0, 1)\,, \tag{6}$$

$$\max\{j \colon 0 \leqq j < m_i(\beta), [\beta]_j^{j+4r-1}(Z_i) = (\bar{0}_r, \bar{1}_r, \bar{0}_r, \bar{1}_r)\} = m_i(\beta) - n_1\,, \tag{7}$$

and there are no $0 < j < m_i(\beta)$ and $k \geqq i$, $k' \geqq 1$, such that $[\beta]_j^{j+4r+2k+2k'-5}(Z_i) = (\bar{0}_{r+k-1}, \bar{1}_{r+k-1}, \bar{0}_{r+k'-1}, \bar{1}_{r+k'-1})$ or $[\beta]_j^{j+4r+4k'-1}(Z_i) = (\bar{0}_{r+k'}, \bar{1}_{r+k'}, \bar{0}_{r+k'}, \bar{1}_{r+k'})$.

For

$$J_i(\beta) = \{j \colon 4r + N \leqq j < m_i(\beta), [\beta]_{j-4r-N}^{j-N-1}(Z_i) =$$
$$= (\bar{0}_r, \bar{1}_r, \bar{0}_r, \bar{1}_r), [\beta]_j^{j+N'-1}(Z_i) = (\bar{1}_{\varrho_0 N'}, \bar{0}_{(1-\varrho_0)N'})\}$$

$(\beta \in E'_i)$, there is

$$|J_i(\beta)| \geqq \frac{1}{2^{i-1}} \prod_{k=1}^{i-1} \left(\frac{n'_{k+1} - 2\varepsilon_k n_{k+1}}{n_k} - 5\right) \tag{8}$$

(in case $i = 1$, the right side equals one). The $[\beta]_0^{n_i-1}(C)$-name can be recognized for any $\beta \in E'_i$, by means of $[\beta]_0^{n'_i-1}(Z_i)$ only. Moreover, no positive information is contained in the blocks $[\beta]_j^{j+N'-1}(Z_i)$, $j \in J_i(\beta)$, about $[\beta]_0^{n_i-1}(C)$. More precisely, if we reorder any arbitrary of those into some arbitrary (r, ϱ_0)-allowable sequences of length N' (and, hence, we obtain a new set $\dot{Z}_i$ instead of Z_i), no new four-block will arise having the form $(\bar{0}_{r+k-1}, \bar{1}_{r+k-1}, \bar{0}_{r+k'-1}, \bar{1}_{r+k'-1})$, $k, k' \in \mathbf{N}$, in $[\beta]_0^{m_i(\beta)-1}(\dot{Z}_i)$ compared with $[\beta]_0^{m_i(\beta)-1}(Z_i)$ (while all previous ones are preserved), and the $[\beta]_0^{m_i(\beta)-1}(Z_i)$-name can be recognized from $[\beta]_0^{m_i(\beta)-1}(\dot{Z}_i)$ again (thus the $[\beta]_0^{n_i-1}(C)$-name, too).

The construction of ζ_1 agrees with this supposition (for $i = 1$) if we define $j_1(\beta) = 0$ and $\tilde{j}_1(\beta) = n_1 - 1$ for all $\beta \in E_1$ (thus $m_1(\beta) = n_1$ and $\mathcal{T}_1' = \mathcal{T}_1$). For a preliminary illustration of $J_i(\beta)$, we will inform of the construction that, in any step i, there will be $[\beta]_{4r+N}^{4r+N+N'-1}(Z_i) = [\beta]_{4r+N}^{4r+N+N'-1}(Z_1) = (\bar{1}_{\varrho_0 N'}, \bar{0}_{(1-\varrho_0)N'})$ still for a sufficiently large number of $\beta \in E_1$ (i.e. the information about γ will be coded elsewhere). For $\beta \in E_i$, $J_i(\beta)$ will indicate just the starts of these blocks (which correspond to the passages through $\bigcup_{j=4r+N}^{4r+N+N'-1} S^j E_1$ and have not yet been changed).

The induction-step

We start by the reduction of $\mathcal{T}_{i+1}$ into $\mathcal{T}_{i+1}'$. For $\beta \in E_{i+1}$, let $j_{i+1}(\beta) = \min\{j : j \geqq 0,\ S^j\beta \in E_i'\}$, $\tilde{j}_{i+1}(\beta) = \max\{j : j < n_{i+1},\ S^j\beta \in \mathcal{T}_i',\ S^{j+1}\beta \notin \mathcal{T}_i' \setminus E_i'\}$, and $m_{i+1}(\beta) = m_{i+1}(S^{j_{i+1}(\beta)}\beta) = \tilde{j}_{i+1}(\beta) - j_{i+1}(\beta) + 1$. Due to 7°, $n_{i+1} \geqq m_{i+1}(\beta) > n_{i+1} - 2\varepsilon_i n_{i+1} - 2n_i = n_{i+1}'$, and we put

$$E_{i+1}' = \{S^{j_{i+1}(\beta)}\beta : \beta \in E_{i+1}\}, \quad \mathcal{T}_{i+1}' = \bigcup_{\beta \in E'_{i+1}} \bigcup_{j=0}^{m_{i+1}(\beta)-1} \{S^j\beta\};$$

both the sets belong to $\mathcal{B}$ since the functions $j_{i+1}, \tilde{j}_{i+1}$ are measurable. Thus the segments below the lowest and above the highest complete $\mathcal{T}_i'$-segment, respectively, have been deleted from $\mathcal{T}_{i+1}$. Hence, if $\beta \in E_{i+1}'$ and $S^j\beta \in \mathcal{T}_i'$ for some $0 \leqq j < m_{i+1}(\beta)$, then there is j' such that $S^{j'}\beta \in E_i'$ and $0 \leqq j' \leqq j \leqq j' + m_i(S^{j'}\beta) - 1 < m_{i+1}(\beta)$; $\{S^{j'}\beta, S^{j'+1}\beta, \ldots, S^{j'+m_i(S^{j'}\beta)-1}\beta\}$ is called a complete $\mathcal{T}_i'$-segment. According to 7°, the number $p(\beta)$ of such complete $\mathcal{T}_i'$-segments, with additional restriction $0 \leqq j' \leqq n_{i+1}' - n_i$, satisfies

$$p(\beta) > \frac{n_{i+1}' - 2\varepsilon_i n_{i+1}}{n_i} - 2. \tag{9}$$

At first, we will define an auxiliary partition $\zeta_{i+1}' = \{Z_{i+1}', B \setminus Z_{i+1}'\}$ (which will later be changed into ζ_{i+1}). Let $Z_{i+1}' \setminus \mathcal{T}_{i+1}' = Z_i \setminus \mathcal{T}_{i+1}'$, and let, for each $\beta \in E_{i+1}'$, the $[\beta]_0^{m_{i+1}(\beta)-1}(Z_{i+1}')$-name equals the $[\beta]_0^{m_{i+1}(\beta)-1}(Z_i)$-name, but with an additional reordering of any complete $(\mathcal{T}_i')^c$-block so that it contains no r-double block, and with the reordering of the lowest complete $\mathcal{T}_i'$-block so that

$$[\beta]_0^{4r+2i+1}(Z_{i+1}') = (\bar{0}_{r+i}, \bar{1}_{r+i}, \bar{0}_r, \bar{1}_r), \tag{10}$$

and that $[\beta]_{4r+2i}^{j_1-1}(Z_{i+1}')$ contains no r-double block ($j_1 = \min\{j : j > 0, S^j\beta \in E_i'\}$;

by complete $(\mathscr{T}'_i)^c$-blocks we mean those which correspond to the segments $\{S^{j'}\beta, S^{j'+1}\beta, \ldots, S^{j''}\beta\}$, where $0 \leqq j' \leqq j'' < m_{i+1}(\beta)$, $S^{j'-1}\beta$, $S^{j''+1}\beta \in \mathscr{T}'_i$ and $S^j\beta \notin \mathscr{T}'_i$ for every $j' \leqq j \leqq j''$). To ensure $Z'_{i+1} \in \mathscr{B}$, make every such reordering identical for each $\beta \in E'_{i+1}$ with the same $[\beta]_0^{m_{i+1}(\beta)-1}$ $(Z_i \cap \mathscr{T}'_i)$-name.

Due to (5) and (6), no new four-blocks of the form $(\bar{0}_{r+k-1}, \bar{1}_{r+k-1}, \bar{0}_{r+k'-1}, \bar{1}_{r+k'-1})$, $k, k' \in \mathbf{N}$, have arisen in $[\beta]_0^{m_{i+1}(\beta)-1}$ (Z'_{i+1}) compared with $[\beta]_0^{m_{i+1}(\beta)-1}$ (Z_i) by the reordering (with the exception of (10)), and all previous ones have been preserved (with the exception of all those which occurred in the lowest complete $\mathscr{T}'_i$-block or which started in some complete $(\mathscr{T}'_i)^c$-block). Hence, according to (5), for $\beta \in E'_{i+1}$, we recognize the start of any complete $\mathscr{T}'_i$-block (except for the lowest one) in $[\beta]_0^{m_{i+1}(\beta)-1}$ (Z'_{i+1}) looking for the four-blocks $[\beta]_{j'}^{j'+4r+2i-3}$ $(Z'_{i+1}) = (\bar{0}_{r+i-1}, \bar{1}_{r+i-1}, \bar{0}_r, \bar{1}_r)$ (they do not occur anywhere else). Then, any corresponding $[\beta]_{j'}^{j'+n'_i-1}$ (Z'_{i+1})-name determines the $[\beta]_{j'}^{j'+n_i-1}(C)$-name by the supposition. Let

$$m'_{i+1}(\beta) = \max\{j': 0 \leqq j' \leqq n'_{i+1} - n_i, [\beta]_{j'}^{j'+4r+2i-3} (Z'_{i+1}) = \\ = (\bar{0}_{r+i-1}, \bar{1}_{r+i-1}, \bar{0}_r, \bar{1}_r)\} + n'_i$$

for $\beta \in E'_{i+1}$ (thus $m'_{i+1}(\beta) \leqq m_{i+1}(\beta)$).

If we use the $[\beta]_0^{m'_{i+1}(\beta)-1}$ (Z'_{i+1})-name only, the number of coordinates in which the values of $[\beta]_0^{n_{i+1}-1}$ (C) will not be recognized is, by 3°, less than

$$n_{i+1} - n'_{i+1} + 2\varepsilon_i n_{i+1} + p(\beta)(n_i - n'_i) <$$

$$< 4\varepsilon_i n_{i+1} + 2n_i + \frac{n_{i+1}}{n_i}(2n_i\varepsilon_{i-1} + 2n_{i-1}) < 8\varepsilon_i n_{i+1} + 2\varepsilon_{i-1}n_{i+1}$$

(for $i = 1$ we put $\varepsilon_0 = n_0 = 0$).

Hence, for any $F' \in (\zeta'_{i+1})_S^{n'_{i+1}}$ with $F' \cap E'_{i+1} \neq \varnothing$, the set $F \cap E'_{i+1}$ (where $F' \subset F \in (\zeta'_{i+1})_S^{m'_{i+1}(\beta)}$, $\beta \in F' \cap E'_{i+1}$) could be further divided by means of $\gamma_S^{n_{i+1}}$, but at most into $2^{8\varepsilon_i n_{i+1} + 2\varepsilon_{i-1}n_{i+1}}$ non-empty parts. For $\beta \in F \cap E'_{i+1}$ put

$$J'_{i+1}(\beta) = J'_{i+1}(F) = \\ = \{j: 0 \leqq j - 4r - N \leqq n'_{i+1} - n_1, [\beta]_{j-4r-N}^{j-N-1} (Z'_{i+1}) = \\ = (\bar{0}_r, \bar{1}_r, \bar{0}_r, \bar{1}_r), [\beta]_j^{j+N'-1} (Z'_{i+1}) = (\bar{1}_{\varrho_0 N'}, \bar{0}_{(1-\varrho_0)N'})\},$$

and fix an arbitrary subset $J''_{i+1}(F) \subset J'_{i+1}(F)$ with $|J''_{i+1}(F)| = [|J'_{i+1}(F)|/2]$. By (8) and (9),

$$|J''_{i+1}(F)| > \frac{1}{2^i} \prod_{k=1}^{i} \left(\frac{n'_{k+1} - 2\varepsilon_k n_{k+1}}{n_k} - 5\right). \tag{11}$$

Let φ_F be a fixed injective map which assigns an ordered $|J''_{i+1}(F)|$-tuple of (r, ϱ_0)-allowable sequences of length N' to each $E \in \gamma_S^{n_{i+1}}$ with $E \cap F \cap E'_{i+1} \neq \varnothing$;

the number of the tuples being, by 8°, (11) and 3°, at least

$$2^{(1-\varepsilon_1)N'h(\varrho_0)|J''_{i+1}(F)|} >$$

$$> \exp_2\left\{(1-\varepsilon_1)\frac{\sigma'}{2}\,n_1 h(\varrho_0)\frac{1}{2^i}\prod_{k=1}^{i}\left(\frac{n'_{k+1}-2\varepsilon_k n_{k+1}}{n_k}-5\right)\right\} >$$

$$> \exp_2\left\{(1-\varepsilon_1)\frac{\sigma'}{2}\,n_1 h(\varrho_0)\frac{1}{2^i}\prod_{k=1}^{i}\left(\frac{n_{k+1}-4\varepsilon_k n_{k+1}-7n_k}{n_k}\right)\right\} >$$

$$> \exp_2\left\{(1-\varepsilon_1)\frac{\sigma'}{2}\,h(\varrho_0)\frac{n_{i+1}}{2^i}\prod_{k=1}^{i}(1-4\varepsilon_k-7\varepsilon_{k+1})\right\} >$$

$$> 2^{\sigma' h(\varrho_0)\frac{n_{i+1}}{2^{i+1}}\prod_{k=1}^{\infty}(1-12\varepsilon_k)} > 2^{\sigma' h(\varrho_0)\frac{n_{i+1}}{2^{i+2}}}$$

(there we use also the fact that $\prod_{k=1}^{\infty}(1-12\varepsilon_k) > 1-\sum_{k=1}^{\infty}12\varepsilon_k > \frac{1}{2}$), which is more than $2^{8\varepsilon_i n_{i+1}+2\varepsilon_{i-1}n_{i+1}}$ by 2°.

For $\beta \in F \cap E'_{i+1}$, the blocks $[\beta]_j^{j+N'-1}(Z'_{i+1})$, $j \in J''_{i+1}(F)$, bring no positive information about C in the sense of the induction-supposition and we reorder them into (r, ϱ_0)-allowable sequences of length N' so that, as an ordered $|J''_{i+1}(F)|$-tuple upwards according to j, they equal $\varphi_F(E)$ if $\beta \in E \in \gamma_S^{n_{i+1}}$. Let us call the resulting sequence $[\beta]_0^{m_{i+1}(\beta)-1}(Z_{i+1})$-name. By the construction and by (10), $[\beta]_0^{m_{i+1}(\beta)-1}(Z_{i+1})$ fulfils (4)–(7) (for $i+1$) for $\beta \in E'_{i+1}$. Thus we have defined $Z_{i+1} \cap \mathscr{T}'_{i+1}$, and let us put $Z_{i+1} \setminus \mathscr{T}'_{i+1} = Z'_{i+1} \setminus \mathscr{T}'_{i+1}$.

Notice that the collections of four-blocks of the form $(\bar{0}_{r+k-1}, \bar{1}_{r+k-1}, \bar{0}_{r+k'-1}, \bar{1}_{r+k'-1})$, $k, k' \in \mathbf{N}$, are the same both in $[\beta]_0^{m_{i+1}(\beta)-1}(Z_{i+1})$ and $[\beta]_0^{m_{i+1}(\beta)-1}(Z'_{i+1})$, for $\beta \in E'_{i+1}$; hence we find the start j' of any complete $\mathscr{T}'_i$-block (except for the lowest one) in $[\beta]_0^{m_{i+1}(\beta)-1}(Z_{i+1})$, as before. Moreover, we can also find its end, as

$$m_i(S^{j'}\beta) = n_1 - j' + \max\{j: j < m_{i+1}(\beta), j < j'', [\beta]_j^{j+4r-1}(Z_{i+1}) =$$
$$= (\bar{0}_r, \bar{1}_r, \bar{0}_r, \bar{1}_r)\}\,, \tag{12}$$

where $j'' = \min\{j: j > j', [\beta]_j^{j+4r+2i-3}(Z_{i+1}) = (\bar{0}_{r+i-1}, \bar{1}_{r+i-1}, \bar{0}_r, \bar{1}_r)\}$ ($\min\varnothing = +\infty$), by (7) and the definition of $\mathscr{T}'_{i+1}$. Anyhow, $j'' < n'_{i+1} \leqq m_{i+1}(\beta)$, whenever $j' \leqq n'_{i+1} - n_i$, so we need not know the height $m_{i+1}(\beta)$ in (12) for each such j'. Afterwards, for each such j', we can recognize $[\beta]_{j'}^{j'+m_i(S^{j'}\beta)-1}(Z'_{i+1})$ by means of $[\beta]_{j'}^{j'+m_i(S^{j'}\beta)-1}(Z_{i+1})$ by the induction-supposition.

Thus we can reconstruct $[\beta]_0^{m'_{i+1}(\beta)-1}(Z'_{i+1})$ by means of $[\beta]_0^{n'_{i+1}-1}(Z_{i+1})$ for $\beta \in E'_{i+1}$, and so obtain F, $F \in (\zeta'_{i+1})_S^{m'_{i+1}(\beta)}$, in which β lies. Then we obtain the corresponding $E \in \gamma_S^{n_{i+1}}$ by means of the blocks $[\beta]_j^{j+N'-1}(Z_{i+1})$, $j \in J''_{i+1}(F)$ (by φ_F^{-1}), and so the $[\beta]_0^{n_{i+1}-1}(C)$-name, too. Clearly, it is $J_{i+1}(\beta) \supset J'_{i+1}(F) \setminus J''_{i+1}(F)$ for

$$J_{i+1}(\beta) = \{j: 4r + N \leqq j < m_{i+1}(\beta), [\beta]_{j-4r-N}^{j-N-1}(Z_{i+1}) = (\bar{0}_r, \bar{1}_r, \bar{0}_r, \bar{1}_r),$$
$$[\beta]_j^{j+N'-1}(Z_{i+1}) = (\bar{1}_{\varrho_0 N'}, \bar{0}_{(1-\varrho_0)N'})\}\,,$$

hence $|J_{i+1}(\beta)|$ satisfies (8) for $i + 1$. Now, if we should reorder some of the blocks $[\beta]_j^{j+N'-1}(Z_{i+1})$, $j \in J_{i+1}(\beta)$, into some (r, ϱ_0)-allowable sequences of length N', and obtain a new, say $[\beta]_0^{m_{i+1}(\beta)-1}(\dot{Z}_{i+1})$-name, the $[\beta]_0^{m_{i+1}(\beta)-1}(Z_{i+1})$-name would be reconstructed by means of the new as above. Obviously, the partition ζ_{i+1} satisfies the whole induction-supposition for $i + 1$.

The limit-partition

For each $i > 1$, $\nu_{B_0}(\mathscr{T}'_i) > (1 - \varepsilon_i)(n'_i/n_i) = (1 - \varepsilon_i)(1 - 2\varepsilon_{i-1} - 2n_{i-1}/n_i) > 1 - 5\varepsilon_{i-1}$. If $\beta \in \mathscr{T}'_i$, then the $[\beta]_{-n_i+1}^{n_i-1}(Z_i)$-name determines, whether or not $\beta \in C$. Namely, we look for the greatest non-positive j such that $[\beta]_j^{j+4r+2i-3}(Z_i) = (\bar{0}_{r+i-1}, \bar{1}_{r+i-1}, \bar{0}_r, \bar{1}_r)$, after which the $[\beta]_j^{j+n_i'-1}(Z_i)$-name determines $[\beta]_j^{j+n_i-1}(C)$ because it should be $S^j\beta \in E'_i$. Thus we recognize C with at least $5\varepsilon_{i-1}$-accuracy by means of ζ_i.

We show the convergence ζ_i's now. For $i \in \mathbf{N}$, let $O_i = \bigcup_{\beta \in E_i'} \bigcup_{j \in J_i(\beta)} \bigcup_{k=0}^{N'-1} \{S^{j+k}\beta\}$. There is

$$\bigcup_{j=4r+N}^{4r+N+N'-1} S^j E_1 = O_1 \supset O_2 \supset O_3 \supset \dots$$

For $i > 1$, let

$$Q_i^{(1)} = (\mathscr{T}'_{i-1})^c \cap \mathscr{T}'_i\,,$$

$$Q_i^{(2)} = \bigcup_{\beta \in E_i'} \bigcup_{j=0}^{m_{i-1}(\beta)-1} \{S^j\beta\} \quad \text{(the lowest complete } \mathscr{T}'_{i-1}\text{-segments in } \mathscr{T}'_i\text{)},$$

$$Q_i^{(3)} = (O_{i-1} \cap \mathscr{T}'_i) \setminus (O_i \cup Q_i^{(2)}) \quad \text{(the segments of length } N' \text{ used for the code by the transition from } Z'_i \text{ to } Z_i\text{)}.$$

Obviously, the sets $Q_2^{(3)}, Q_3^{(3)}, Q_4^{(3)}, \dots$ are pairwise disjoint and the partitions ζ_{i-1} and ζ_i can differ trom one another on $Q_i^{(1)} \cup Q_i^{(2)} \cup Q_i^{(3)}$ only, i.e. $Z_i \setminus \bigcup_{k=1}^{3} Q_i^{(k)} = Z_{i-1} \setminus \bigcup_{k=1}^{3} Q_i^{(k)}$. But the sets $Q_i^{(1)}$ and $Q_i^{(2)}$ are small:

$$\nu_{B_0}(Q_i^{(1)} \cup Q_i^{(2)}) < \frac{2\varepsilon_{i-1}n_i + \dfrac{n_i}{n_{i-1}}(2n_{i-1}\,\varepsilon_{i-2} + 2n_{i-2}) + n_{i-1}}{n_i} < 7\varepsilon_{i-2}$$

for $i > 2$, hence $\nu_{B_0}(A) = 1$ for $A = B_0 \cap \bigcup_{k=1}^{\infty} D_k$, where $D_k = \bigcap_{i=k+1}^{\infty} (\mathscr{T}'_i \setminus (Q_i^{(1)} \cup Q_i^{(2)}))$. Thus the limit $\lim_{i\to\infty} \chi_{Z_i}(\beta)$ exists for all $\beta \in A$, because $\chi_{Z_i}(\beta)$ is the same for all but at most finite number of i. Put

$$Z_{(0)} = \{\beta\colon \beta \in A\,, \lim_{i\to\infty} \chi_{Z_i}(\beta) = 1\}\,, \quad \zeta_{(0)} = \{Z_{(0)}, B \setminus Z_{(0)}\}\,.$$

Clearly, $Z_{(0)} \in \mathscr{B}$ because all the Z_i are. As it follows from the construction of each $\mathscr{T}'_i$ (the deleting of the segments below the lowest and above the highest complete

$\mathscr{T}'_{i-1}$-segments), if $S^j\beta \in D_k$ (given $k \in \mathbf{N}$) for some $\beta \in E'_k$ and $0 \leqq j < m_k(\beta)$, then $S^{j'}\beta \in D_k$ for all $0 \leqq j' < m_k(\beta)$, too.

We show that $\zeta_{(0)}$ is the desired partition. If $\beta \in A$, then $\beta \in D_k$ for some k. Then $\beta \in \mathscr{T}'_k$, $\beta = S^j\beta'$ for some $\beta' \in E'_k$ and $0 \leqq j < m_k(\beta')$, and $S^{j'}\beta' \in D_k$ for all $0 \leqq j' < m_k(\beta')$. For all such j', it is $\chi_{Z_k}(S^{j'}\beta') = \chi_{Z_{(0)}}(S^{j'}\beta')$ if, of course, $S^{j'}\beta' \notin \bigcup_{i=k+1}^{\infty} Q_i^{(3)}$. But $\bigcup_{i=k+1}^{\infty} Q_i^{(3)} (\subset O_k)$ means some segments of length N', whose corresponding blocks in $[\beta']_0^{m_k(\beta')-1}(Z_{(0)})$-name should always be an (r, ϱ_0)-allowable sequence. Thus, no new four-block of the form $(\bar{0}_{r+k'-1}, \bar{1}_{r+k'-1}, \bar{0}_r, \bar{1}_r)$, $k' \in \mathbf{N}$, will occur in $[\beta']_0^{m_k(\beta')-1}(Z_{(0)})$ compared with $[\beta']_0^{m_k(\beta')-1}(Z_k)$, by preserving all those in $[\beta']_0^{m_k(\beta')-1}(Z_k)$. It follows that j is the smallest non-negative j' such that $[\beta]_{-j'}^{-j'+4r+2k-3}(Z_{(0)}) = (\bar{0}_{r+k-1}, \bar{1}_{r+k-1}, \bar{0}_r, \bar{1}_r)$, and we reconstruct the $[\beta]_{-j}^{-j+m_k'(\beta')-1}(Z_k)$-name (and thus obtain the $[\beta]_{-j}^{-j+n_k-1}(C)$-name) in the same way as in the induction-step. So we know whether or not $\beta \in C$: $\chi_C(\beta) = \lim_{i\to\infty} \chi_{C,i}(\beta)$, where the index i now means the recognition-procedure by means of Z_i (at all events, $\chi_{C,i}(\beta)$ is the right value for all $i \geqq k$). Hence, $\zeta_{(0)}$ is a generator for S with respect to ν_{B_0} because γ is.

Further, we show that $Z_{(0)}$ is a ν_{B_0}-finite reordering of X. Let $D'_1 = D_1$ and $D'_{k+1} = D_{k+1} \setminus D_k$ for $k \geqq 1$. Each D'_k can be decomposed (up to a nullset) into a finite number of disjoint towers which satisfy (i) and (ii) in the definition. So do the towers $\mathscr{T}(D'_1 \cap E \cap E_1, n_1)$, $E \in (\xi \vee \zeta_{(0)})_S^{n_1}$, for $k = 1$. For $k > 1$, $D'_k = Q_k^{(1)} \cup Q_k^{(2)}$ (disjoint union), and

$$B_0 \cap Q_k^{(1)} = \bigcup_E \mathscr{T}(B_0 \cap D'_k \cap E \cap E'_k, m_{k-1}(E)), \tag{13}$$

$$B_0 \cap Q_k^{(2)} = \bigcup_E \bigcup_{(j',j'')} \mathscr{T}(B_0 \cap D'_k \cap S^{j'}(E \cap E'_k), j'' - j' + 1), \tag{14}$$

where the unions are taken over all $E \in (\xi \vee \zeta_{(0)})_S^{n_k} \vee \eta_{k-1} \vee \eta_k$ (we put, for $i \geqq 1$, $\eta_i = \{\{\beta\colon \beta \in E'_i, m_i(\beta) = j\}\colon j = n'_i, n'_i + 1, \ldots, n_i\}$ and $m_i(F) = m_i(\beta)$, $\beta \in F$, whenever $F \subset F'$ for some $F' \in \eta_i$) and over all pairs (j', j'') with $0 < j' \leqq j'' < m_k(E)$, $S^{j'-1}(E \cap E'_k), S^{j''+1}(E \cap E'_k) \subset \mathscr{T}'_{k-1}$, $S^j(E \cap E'_k) \subset (\mathscr{T}'_{k-1})^c$ whenever $j' \leqq j \leqq j''$, for a given E. Both unions (13), (14) are disjoint and satisfy (i), (ii).

Finally,

$$\bigcap_{j=-\infty}^{\infty} S^j A = \bigcap_{r'=r+1}^{\infty} \bigcap_{i=-\infty}^{\infty} (B \setminus M_i^{r'}(Z_{(0)})) \cap \bigcup_{j=-\infty}^{\infty} M_j^r(Z_{(0)}),$$

because after each step i in the induction, for $\beta \in E'_i$, there are four-blocks in $[\beta]_0^{m_i(\beta)-1}(Z_i)$ of the forms $(\bar{0}_{r+k-1}, \bar{1}_{r+k-1}, \bar{0}_r, \bar{1}_r)$, $k = 1, 2, \ldots, i$, only. This is preserved for $Z_{(0)}$ and $\beta \in B_0 \cap D_i \cap E'_i$, too. The proof is complete.

3.4. The step-coding (Proof of Theorem 2)

Suppose that $q < p$ (the case $q > p$ is the same) and fix an arbitrary $a \in \mathbf{N}$ which satisfies $q < p(a-1)/a$. At first, we shall a bit refine the construction of Z. We will do it in symbols from the proof of Lemma 2. In every step $i \to i+1$, $i \geqq 1$, in the induction, we have defined Z'_{i+1} so that, for $\beta \in E'_{i+1}$, among others, $|[\beta]_0^{m_i(\beta)-1}(Z'_{i+1})| = |[\beta]_0^{m_i(\beta)-1}(Z_i)|$ and $[|\beta]_{m_i(\beta)}^{j_1-1}(Z'_{i+1})| = |[\beta]_{m_i(\beta)}^{j_1-1}(Z_i)|$ where $j_1 = \min\{j : j > 0,\ S^j\beta \in E'_i\}$ (a reordering of the lowest complete $\mathcal{T}'_i$-block, and of the next complete $(\mathcal{T}'_i)^c$-block if $m_i(\beta) \neq j_1$), and so that $[\beta]_{4r+2i}^{j_1-1}(Z'_{i+1})$ contains no r-double block.

In addition to this, we now shall need

$$\begin{aligned} &[\beta]_{4r+2i}^{4r+2i+2M_i-1}(Z'_{i+1}) = (1, 0, 1, 0, \ldots, 1, 0), \\ &[\beta]_{4r+2i+2M_i}^{4r+2i+5M_i-1}(Z'_{i+1}) = (1, 1, 0, 1, 1, 0, \ldots, 1, 1, 0), \ldots \\ &\ldots, [\beta]_{4r+2i+K_i}^{4r+2i+K_i+(a+1)M_i-1}(Z'_{i+1}) = (\bar{1}_a, 0, \bar{1}_a, 0, \ldots, \bar{1}_a, 0), \end{aligned} \tag{15}$$

where $M_i = [(n_i(\hat{\varrho}_0 - \varepsilon_1)/2 - 2r - i)/(1 + 2 + 3 + \ldots + a)]$ and $K_i = (2 + 3 + + \ldots + a)\,M_i$. This is possible because $r > 2$, and because

$$|[\beta]_0^{m_i(\beta)-1}(Z_i)| = |[\beta]_0^{m_i(\beta)-1}(X)| > n_1(\varrho_0 - \varepsilon_1)\prod_{k=1}^{i-1}\left(\frac{n'_{k+1} - 2\varepsilon_k n_{k+1}}{n_k} - 2\right)$$

by 5° and (9), which is greater than

$$n_1(\varrho_0 - \varepsilon_1)\frac{n_i}{n_1}\prod_{k=1}^{\infty}(1 - 8\varepsilon_k) > n_i(\varrho_0 - \varepsilon_1)/2,$$

and analogously,

$$|[\beta]_0^{m_i(\beta)-1}(B \setminus Z_i)| > n_i(1 - \varrho_0 - \varepsilon_1)/2;$$

both $\varrho_0 - \varepsilon_1$ and $1 - \varrho_0 - \varepsilon_1$ are positive by 2° (if i is not sufficiently large, $M_i = 0$, and by this, the requirement (15) will vanish). So we have a new amended Z'_{i+1} and Z_{i+1} in each step, and, owing to this modification, a new set Z (we will use the same symbol for it as before). Clearly, no previously prescribed properties are broken by this change.

Having in mind the whole construction in the proof of Theorem 1, we see that for each $k \in \mathbf{I}$ and for ν-almost all $\beta \in B$, there are arbitrarily long blocks of the forms (15) in $[\beta]_k^{\infty}(Z)$-name $(=(\chi_Z(S^k\beta), \chi_Z(S^{k+1}\beta), \ldots))$ and in $[\beta]_{-\infty}^{k}(Z)$-name, too.

Let us call any interval of a form $[b, b+p)$ and $[b, b+q)$, $b \in \mathbf{R}$, a p-interval and q-interval, respectively. For $\omega \in {}^*B$, $t \in \mathbf{R}$ and $t_1 < t_2$ $(t_1 \in \mathbf{R}, t_2 \in \mathbf{R} \cup \{+\infty\})$, $[\omega, *]_{t_1}^{t_2}$ means the continuous $\bar{Z}$-name of ω, i.e. the function $\chi_{\bar{Z}}(S_t\omega)$ of t on $[t_1, t_2)$. Clearly; $[\omega, *]_{t_1}^{t_2} \in \Theta_{t_1}^{t_2}$, where we define $\Theta_{-\infty}^{\infty}$ as the set of all right-continuous functions θ on $\mathbf{R}$ taking only the values 0 and 1 such that their pre-images $\theta^{-1}\{0\}$ and $\theta^{-1}\{1\}$ are a disjoint union of q-intervals and a disjoint union of p-intervals, respectively, and $\Theta_{t_1}^{t_2}$ consists of their restrictions to $[t_1, t_2)$. We can imagine each

$\theta \in \Theta_{t_1}^{t_2}$ as a sequence (finite or infinite in accordance with t_2) of p-intervals and q-intervals, such that t_1 lies in the first one, say at a distance d, $0 \leqq d < \max\{p,q\}$, from its start. We mean just the list (from the left to the right) of the intervals between jumps which is exactly so long that $[t_1, t_2)$ is covered by them. So a pair (d, Ξ), where $\Xi = (\Xi_i) \in \{p, q\}^{\mathbf{N}} \cup \bigcup_{k=2}^{\infty} \{p, q\}^k$, represents θ in case when $\max\{p, q\} < t_2 - t_1$. If need be, we will write $\Xi = \Xi(\theta)$ or $\Xi = \Xi([\theta]_{t_1}^{t_2})$. For $\theta \in \Theta_{t_1}^{t_2}$ and $0 < t < t_2 - t_1$, let $[\theta, t]_{t_1}^{t_2} = (\theta(jt), \theta((j+1)t), \ldots, \theta(kt))$, where $j, k \in \mathbf{I}$, $(j-1)t < t_1 \leqq jt$ and $kt < t_2 \leqq (k+1)t$ (in case $t_2 = +\infty$ we put $[\theta, t]_{t_1}^{t_2} = (\theta(jt), \theta((j+1)t), \ldots)$). Let us write in short $[\omega, t]_{t_1}^{t_2}$ for $[[\omega, *]_{t_1}^{t_2}, t]_{t_1}^{t_2}$. Theorem 2 is a consequence of the following assertion.

Lemma 4. If $0 < t_0 < \min\{p, q\}$, then, for arbitrary $k \in \mathbf{I}$, $\sigma(\bigvee_{t=kt_0}^{\infty} S_t\zeta) = \sigma(\bigvee_{j=k}^{\infty} S_{t_0}^j\zeta)$ and $\sigma(\bigvee_{t=kt_0}^{\infty} S_{-t}\zeta) = \sigma(\bigvee_{j=k}^{\infty} S_{-t_0}^j\zeta)$ up to symmetrical differences of measure zero.

Proof. We show the first equality only (the proof of the second one being analogical). To this end, it suffices to recognize, for almost all $\omega \in \Omega$, the continuous name $[\omega, *]_{kt_0}^{\infty}$ by means of $[\omega, t_0]_{kt_0}^{\infty}$. If we knew that a jump of $\theta \in \Theta_{kt_0}^{\infty}$ has to lie in $(t, t + \varepsilon)$ for some $t \geqq kt_0$ and $0 < \varepsilon < \min\{p, q\} - t_0$, then we should obtain the whole $\Xi(\theta)$ by means of $[\theta, t_0]_{kt_0}^{\infty}$ only. Indeed, the points kt_0, $(k+1)\,t_0, \ldots$ must hit every interval between jumps and if, e.g. $[\theta, t_0]_{t+\varepsilon}^{t+\varepsilon+jt_0} = (\bar{0}_{j-1}, 1)$ for some $j \geqq 2$, then we know that in $(t, t+\varepsilon)$ a block starts of q-intervals whose number n is uniquely determined by $(j-1)\,t_0 < nq < jt_0 + \varepsilon$. Hence, a start of a p-interval lies $(t + nq, t + \varepsilon + nq)$, and so on in both directions. Thus for the recognition of $[\omega, *]_{kt_0}^{\infty}$ it suffices to find, for arbitrarily small $\varepsilon > 0$, an interval $(t, t + \varepsilon)$, $t \geqq kt_0$, which contains some jump of $[\omega, *]_{kt_0}^{\infty}$.

As p/q is irrational, either $(p + q)/t_0$ is, or $(lp + q)/t_0$ for all $l = 2, 3, \ldots, a$ are. In the former case, for a given $0 < \varepsilon < \min\{p, q\} - t_0$, there is $m \in \mathbf{N}$ such that $\{jt_0 \bmod (p+q)\colon j = 0, 1, \ldots, [(m-1)(p+q)/t_0]\}$ is an $\varepsilon/2$-dense subset in $[0, p+q)$. As it was done above, for almost all $\omega \in \Omega$ there is a sufficiently large block $\Xi' = (p, q, p, q, \ldots, p, q)$ (of length $2n > 8m$) in $\Xi([\omega, *]_{kt_0}^{\infty})$ which corresponds to a segment $[\omega, *]_t^{t+n(p+q)}$, $t \geqq kt_0$, in the continuous name.

Let $\theta \in \Theta_{kt_0}^{\infty}$ so that

$$[\theta, t_0]_{kt_0}^{\infty} = [\omega, t_0]_{kt_0}^{\infty}\,. \tag{16}$$

As $0 < t_0 < \min\{p, q\}$, $\Xi([\theta]_t^{t+n(p+q)})$ must contain Ξ' as a subsequence, and contains one element at most in addition to it. Thus in any case, there is a segment $[t_1, t_1 + 2m(p+q)) \subset [t, t + n(p+q))$ such that $\Xi([\theta]_{t_1}^{t_1+2m(p+q)}) = (p, q, p, q, \ldots \ldots, p, q)$. From the condition concerning m, there must exist points $j't_0$ and $j''t_0$ $(t_1 + t_0 \leqq j't_0 < j''t_0 < t_1 + 2m(p+q), j', j'' \in \mathbf{I})$ lying in a distance less than $\varepsilon/2$ behind the start of some p-interval and in front of the end of some p-interval of θ, respectively. Of course, these $j't_0$ and $j''t_0$ satisfy

$$\theta(j't_0) = \theta(j''t_0) = 1\,, \quad \theta((j'-1)\,t_0) = \theta((j''+1)\,t_0) = 0\,, \tag{17}$$

$$j''t_0 - j't_0 > n'(p + q) + p - \varepsilon\,, \tag{18}$$

for some integer $n' \geqq 0$. On the other hand, by (16) and by $\Xi([\theta]_t^{t+n(p+q)})$ given, the last formulae (17), (18) will determine the point $j't_0$ to the effect that a jump lies in $(j't_0 - \varepsilon, j't_0]$. However, this is sufficient for us because $\Xi([\theta]_{t_1}^{t_1+2m(p+q)}) = \Xi([\omega, *]_{t_1}^{t_1+2m(p+q)})$.

The latter case ($(p + q)/t_0$ is rational) will be solved similarly. Let $m \in \mathbf{N}$ satisfy now, for every $l = 2, 3, \ldots, a$, $\{jt_0 \bmod (lp + q) : j = 0, 1, \ldots, [(m - 1)(lp + q) : t_0]\}$ is an $\varepsilon/2$-dense subset in $[0, lp + q)$. For almost all $\omega \in \Omega$ and every $l = 2, 3, \ldots, a$, there is a sufficiently large block $\Xi_l = (\bar{p}_l, q, \bar{p}_l, q, \ldots, \bar{p}_l, q)$ in $\Xi([\omega, *]_{kt_0}^{\infty})$ (of length $(l + 1)\,n > 4(l + 1)\,m$, $\bar{p}_l$ means the list $p, \ldots, p$ of length l) which corresponds to a segment $[\omega, *]_{t_l}^{t_l+n(lp+q)}$.

Let $\theta \in \Theta_{kt_0}^{\infty}$ so that (16) holds. Due to $0 < t_0 < q$, it is $p(l_0 - 1)/l_0 < t_0 \leqq pl_0/(l_0 + 1)$ for some $l_0 \in \{1, 2, \ldots, a - 1\}$. Then any interval $[b, b + l_0 p)$, $b \geqq kt_0$, contains at least $l_0 + 1$ points of the sequence $(jt_0)_{j=k}^{\infty}$, whereas any interval $[b', b' + (l_0 - 1)\,p)$, $b' \geqq kt_0$, contains l_0 such points at most. Thus $\Xi([\theta]_{\bar{t}}^{\bar{t}+n(l_0p+q)})$ (where $\bar{t} = t_{l_0}$) contains Ξ_{l_0} as a subsequence and has one element at most in addition to it, which implies the existence of a segment $[\bar{t}_1, \bar{t}_1 + 2m(l_0 p + q)) \subset [\bar{t}, \bar{t} + n(l_0 p + q))$ with $\Xi([\theta]_{\bar{t}_1}^{\bar{t}_1+2m(l_0p+q)}) = (\bar{p}_{l_0}, q, \bar{p}_{l_0}, q, \ldots, \bar{p}_{l_0}, q)$. Hence, if $l_0 > 1$, we can determine a point jt_0, $j \in \mathbf{I}$, to the effect that a jump of $[\omega, *]_{kt_0}^{\infty}$ has to lie in $(jt_0 - \varepsilon, jt_0]$, similarly as by means of (17), (18) in the first case. In case $l_0 = 1$ we use this method too, because then the length of any finite block of p-intervals (which has q-intervals on either side) in $\Xi(\theta)$ can be exactly recognized by means of $[\theta, t_0]_{kt_0}^{\infty}$. □

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