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Jiří Anděl; Ivan Netuka; Karel Zvára

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ON THRESHOLD AUTOREGRESSIVE PROCESSES

JIŘÍ ANDĚL, IVAN NETUKA, KAREL ZVÁRA

A threshold autoregressive process of the first order with Gaussian innovations is analyzed in the paper. Four methods are used for finding its stationary distribution: (a) the exact solution is derived in some special cases, (b) a Markov approximation is proposed, (c) a numerical method for solving an integral equation is applied, (d) simulations are produced. In the cases (b) and (d) also estimates of the correlation function and of the spectral density are compared.

1. INTRODUCTION

The classical autoregressive (AR) process $\{X_t\}$ is defined by

$$X_t = \sum_{k=1}^{n} a_k X_{t-k} + e_t \,, \tag{1.1}$$

where a_k are constants and $\{e_t\}$ is a white noise, i.e. a sequence of uncorrelated random variables with $\mathsf{E}e_t = 0$, $\mathsf{Var}\, e_t = \sigma^2 > 0$. Other simple models used in the time series analysis are moving average (MA) process

$$X_t = \sum_{j=0}^{m} b_j e_{t-j} \tag{1.2}$$

and autoregressive-moving average (ARMA) process

$$X_t = \sum_{k=1}^{n} a_k X_{t-k} + \sum_{j=0}^{m} b_j e_{t-j} \,, \tag{1.3}$$

where b_j are also constants. In the last years also some nonlinear models have been proposed. A bilinear model is defined by

$$X_t = \sum_{k=1}^{n} a_k X_{t-k} + \sum_{j=0}^{m} b_j e_{t-j} + \sum_{p=1}^{P} \sum_{q=1}^{Q} \beta_{pq} X_{t-p} e_{t-q} \,,$$

where e_t are supposed to be not only uncorrelated, but also independent (Granger and Anderson [6]). Ozaki and Oda [14] consider models of the type

$$X_t = c_1 X_{t-1} + \ldots + c_p X_{t-p} + d_1 X_{t-1}^2 + \ldots + d_p X_{t-p}^2 + d_{p+1} X_{t-1} X_{t-2} + \ldots$$
$$\ldots + d_{p+p(p+1)/2} X_{t-p+1} X_{t-p} + \ldots + z_r X_{t-1}^{s_1} \ldots X_{t-m}^{s_m} + e_t .$$

They use the Akaike's Information Criterion AIC (Akaike [1]; Anděl [2]) for estimating the dimension of the model. Tong [17] and [18] introduced a non-linear model in the following way. Let $B_1, \ldots, B_h$ be disjoint subsets of the real line $\mathbb{R}$ such that $B_1 \cup \ldots \cup B_h = \mathbb{R}$. Let $\{Y_t\}$ be a time series. Define

$$X_t = a_{0k} + \sum_{j=1}^{n_k} a_{jk} X_{t-j} + e_t \quad \text{if} \quad Y_{t-1} \in B_k ,$$

where a_{jk} are some constants. Then $\{Y_t\}$ is called an instrumental time series. Usually, $B_0 = (-\infty, r_1)$, $B_k = [r_k, r_{k+1})$ for $k = 1, \ldots, h$, where $r_{h+1} = \infty$. The points $r_1, \ldots, r_h$ are then called the thresholds and $\{X_t\}$ is said to be threshold autoregressive (TAR) process. A special case arises when $Y_t = X_t$. Such process $\{X_t\}$ is sometimes called a self-exciting threshold autoregressive (SETAR) process. It has been shown by Pemberton and Tong [16] that TAR models exhibit features known in non-linear vibrations like jump resonance, amplitude-frequency dependency, limit cycles etc. Tong [19] generalized the TAR model to threshold autoregressive-moving average (TARMA) processes.

Haggan and Ozaki [8] proposed and analyzed exponential autoregressive (EAR) process defined by

$$X_t = \sum_{k=1}^{p} (a_k + b_k \, e^{-cX_{t-1}^2}) X_{t-k} + e_t ,$$

where a_k, b_k and c are constants. Ozaki [13] deals with processes of the type

$$X_t = \begin{cases} \sum_{k=1}^{p} a_k X_{t-k} + e_t & \text{if} \quad |X_{t-1}| \geqq r , \\ \sum_{k=1}^{p} \left(\sum_{j=0}^{q_k} b_{kj} X_{t-1}^j \right) X_{t-k} + e_t & \text{if} \quad |X_{t-1}| < r , \end{cases}$$

where r is a threshold and the parameters of the model satisfy the condition

$$\sum_{j=0}^{q_k} b_{kj} r^j = a_k , \quad k = 1, 2, \ldots, p .$$

One of the important problems is to find conditions, under which the processes are stationary. All MA processes (1.2) are always stationary. The AR process (1.1) and the ARMA process (1.3) are stationary if and only if

$$z^n - a_1 z^{n-1} - \ldots - a_n \neq 0 \quad \text{for} \quad |z| \geqq 1$$

(of course, it is understood that X_t belongs to the Hilbert space generated by the variables $e_t, e_{t-1}, \ldots$). However, little is known about stationarity conditions for non-linear processes. For example, Jones [10] investigated the process

$$X_t = \lambda(X_{t-1}) + e_t, \tag{1.4}$$

where λ is an autoregressive function and e_t are independent identically distributed random variable (i.i.d.r.v.'s). Some "negative conditions" are derived in a quite explicit form. E.g., if the distribution of e_t has infinite tails with exponential-like behaviour, then no polynomial autoregressive function of degree greater than one can lead to a stationary process $\{X_t\}$. Jones does give conditions ensuring stationarity, but their verification seems to be rather difficult in any non-trivial case. (Let us mention that it can happen that λ depends on a parameter θ and the distribution of e_t depends on a parameter ψ; Jones [11] uses the maximum likelihood method for estimating θ and ψ and derives some asymptotic properties of the estimators.)

The situation is even worse if we need to compute a stationary distribution of the process. There are two main problems:

(i) The distribution of e_t is given. Calculate stationary distribution of X_t (if it exists).

(ii) A distribution is given which should serve as a stationary distribution of X_t. Find a distribution of e_t (if it exists) which implies such a stationary distribution.

Generally, these problems are hard enough even for the stationary AR process (1.1) with $n = 1$ (see Anděl [2] and [3]).

As for the non-linear processes, Pemberton and Tong [16] analyzed the properties of a stationary distribution in the model

$$X_t = \lambda(X_{t-1}, \ldots, X_{t-p-1}) + e_t .$$

Their main result can be formulated as follows: If e_t has a symmetric density and if λ is an odd function, then all the finite stationary joint densities of $X_{t_1}, \ldots, X_{t_k}$ are symmetric about $(0, \ldots, 0)$ for all $t_1, \ldots, t_k$. However. they showed by an example that this condition is not necessary for the symmetry of stationary densities.

Jones [9] and [12] proposed three methods for evaluating stationary distributions of the model (1.4). All the procedures are based on some power-series expansions. They seem to be too complicated for a practical use. The problem is very difficult also from the theoretical point of view. The results concerning stationary measures (existence, uniqueness) are available under Doob's Condition D ([7], pp. 190–218). This condition is not satisfied even for Gaussian linear processes.

In the present paper we propose some other methods for computing the stationary marginal densities in non-linear processes. For simplicity, they will be explained in the case of a TAR model of the first order with one threshold r, although they can be used for any model (1.4). In a special case an explicit analytical solution is derived, which enables to compare the accuracy of other methods. The results are also checked by simulations.

2. PRELIMINARIES

In this section we calculate some integrals needed for derivations of main results in further parts of the paper.

Denote

$$\varphi(x) = (2\pi)^{-1/2} \exp\{-x^2/2\} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \varphi(t)\,\mathrm{d}t$$

the density and the distribution function of $N(0, 1)$, respectively. Let

$$\mu_n = \int_{-\infty}^{\infty} x^n \varphi(x)\,\mathrm{d}x\,, \quad n \geqq 0\,,$$

be the moment of the nth order of $N(0, 1)$.

Lemma 2.1. Put

$$J_n = \int_{-\infty}^{\infty} x^n \,\mathrm{e}^{-kx^2/2}\, \Phi(-x)\,\mathrm{d}x\,, \quad n = 0, 1 \ldots,$$

where $k > 0$ is a constant. Then

$$J_0 = [\pi/(2k)]^{1/2}\,, \quad J_1 = -k^{-1}(k+1)^{-1/2}\,, \quad J_2 = k^{-1}J_0\,,$$
$$J_3 = [2/k + 1(k+1)]\, J_1\,, \quad J_4 = (3/k)\, J_2\,.$$

Generally,

$$J_n = \frac{n-1}{k} J_{n-2} - k^{-1}(k+1)^{-n/2} \mu_{n-1}\,, \quad n \geqq 2\,.$$

Proof. Using $\Phi(-x) = 1 - \Phi(x)$, we have

$$J_0 = \int_{-\infty}^{\infty} \mathrm{e}^{-kx^2/2}\,\mathrm{d}x - \int_{-\infty}^{\infty} \mathrm{e}^{-kx^2/2}\, \Phi(x)\,\mathrm{d}x = (2\pi/k)^{1/2} - J_0\,.$$

From here we obtain J_0. The expression for J_1 can be obtained using the integration by parts. The same method gives

$$J_n = -(1/k)\int_{-\infty}^{\infty} (-kx)\,\mathrm{e}^{-kx^2/2} x^{n-1}\, \Phi(-x)\,\mathrm{d}x = \frac{n-1}{k} J_{n-2} - k^{-1}(k+1)^{-n/2}\mu_{n-1}\,.$$

Inserting $n = 2$, 3 and 4 we get the formulas for J_2, J_3 and J_4. □

Lemma 2.2. Let $k > 0$. Denote

$$H(k) = \int_{-\infty}^{0} \mathrm{e}^{-kx^2/2}\, \Phi(x)\,\mathrm{d}x\,, \quad M(k) = \int_{-\infty}^{0} x^2\,\mathrm{e}^{-kx^2/2}\, \Phi(x)\,\mathrm{d}x\,.$$

Then

$$H(k) = (2\pi k)^{-1/2} \operatorname{arctg} \sqrt{k}\,,$$
$$M(k) = (2\pi k)^{-1/2}\, k^{-1} \operatorname{arctg} \sqrt{(k)} - (2\pi)^{-1/2}\, k^{-1}(k+1)^{-1}\,.$$

Proof. Obviously,

$$H'(k) = -\tfrac{1}{2} M(k).$$

Integration by parts gives

$$M(k) = k^{-1} H(k) - (2\pi)^{-1/2} k^{-1}(k+1)^{-1}. \tag{2.1}$$

Therefore, $H(k)$ satisfies the linear differential equation

$$H'(k) + (2k)^{-1} H(k) = 2^{-1}(2\pi)^{-1/2} k^{-1}(k+1)^{-1}.$$

The solution is

$$H(k) = k^{-1/2}[c + (2\pi)^{-1/2} \operatorname{arctg} \sqrt{k}].$$

A direct evaluation of the integral yields $H(1) = \frac{1}{8}(2\pi)^{1/2}$, which implies $c = 0$. The formula for $M(k)$ follows from (2.1). □

Lemma 2.3. Let $r \in \mathbb{R}$, $\mu \in \mathbb{R}$. Then

$$(2\pi)^{-1/2} \int_{-\infty}^{r} x^2 \exp\{-\tfrac{1}{2}(x-\mu)^2\} \, dx =$$
$$= (1+\mu^2)\, \Phi(r-\mu) - (2\pi)^{-1/2}(r+\mu) \exp\{-\tfrac{1}{2}(r-\mu)^2\},$$
$$(2\pi)^{-1/2} \int_{r}^{\infty} x^2 \exp\{-\tfrac{1}{2}(x-\mu)^2\} \, dx = (1+\mu^2)\, \Phi(\mu - r) +$$
$$+ (2\pi)^{-1/2}(r+\mu) \exp\{-\tfrac{1}{2}(r-\mu)^2\}.$$

Proof. The formulas can be derived by direct calculations. □

3. THE MODEL

In all the remaining sections of this paper we shall deal with the process $\{X_t\}$ defined by

$$X_s = \begin{cases} \alpha X_{s-1} + e_s, & \text{if } X_{s-1} \leqq r, \\ \beta X_{s-1} + e_s, & \text{if } X_{s-1} > r, \end{cases} \tag{3.1}$$

where e_s are independent $N(0, \sigma^2)$ variables and α, β, r, σ are given constants such that $-1 < \alpha, \beta < 1$, $\sigma > 0$. This is a TAR process of the first order with normally distributed residuals. We shall assume that X_s and e_t are independent for $s < t$.

The main aim of our paper is to find a stationary distribution of $\{X_s\}$, its statistical characteristics, the correlation function and the spectral density. In a special case we were able to derive an explicit result. The approximate methods can be applied also to more general models than (3.1).

Obviously, the conditional distribution of X_s given $X_{s-1} = y$ is

$$\mathscr{L}(X_s \mid X_{s-1} = y) = \begin{cases} N(\alpha y, \sigma^2), & \text{if } y \leqq r, \\ N(\beta y, \sigma^2), & \text{if } y > r. \end{cases}$$

Let $q_1(x \mid y)$ and $q_2(x \mid y)$ be the density of $N(\alpha y, \sigma^2)$ and $N(\beta y, \sigma^2)$, respectively. Assume that X_{s-1} has a density $f(y)$. Then the simultaneous density of X_s and X_{s-1} is

$$h(x, y) = \begin{cases} q_1(x \mid y) f(y), & \text{if } y \leqq r, \\ q_2(x \mid y) f(y), & \text{if } y > r. \end{cases}$$

The density f is stationary if and only if the marginal density of X_s is also f. This gives the condition

$$f(x) = \int h(x, y)\, \mathrm{d}y = \int_{-\infty}^{r} q_1(x \mid y) f(y)\, \mathrm{d}y + \int_r^{\infty} q_2(x \mid y) f(y)\, \mathrm{d}y .$$

Inserting for q_1 and q_2 we have the equation

$$(3.2) \qquad f(x) = \frac{1}{\sqrt{(2\pi)}\,\sigma} \int_{-\infty}^{r} \exp\left\{-\frac{(x - \alpha y)^2}{2\sigma^2}\right\} f(y)\, \mathrm{d}y + \\ + \frac{1}{\sqrt{(2\pi)}\,\sigma} \int_r^{\infty} \exp\left\{-\frac{(x - \beta y)^2}{2\sigma^2}\right\} f(y)\, \mathrm{d}y .$$

Without any loss of generality we shall assume that $\sigma = 1$.

4. AN EXPLICIT SOLUTION

In this section we shall assume that $r = 0$, $\beta = -\alpha$, $0 < \alpha < 1$. Under such conditions we prove that there exists a stationary density f satisfying (3.2) and that the solution of (3.2) is unique apart from a multiplying factor.

Theorem 4.1. Consider the equation

$$(4.1) \qquad f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{0} \exp\{-\tfrac{1}{2}(x - \alpha y)^2\} f(y)\, \mathrm{d}y + \\ + \frac{1}{\sqrt{(2\pi)}} \int_0^{\infty} \exp\{-\tfrac{1}{2}(x + \alpha y)^2\} f(y)\, \mathrm{d}y .$$

Then

$$(4.2) \qquad f(x) = [2(1 - \alpha^2)/\pi]^{1/2} \exp\{-\tfrac{1}{2}(1 - \alpha^2)\, x^2\}\, \Phi(-\alpha x)$$

is a solution of (4.1) and f is a probability density.

Proof. Denote $C = [2(1 - \alpha^2)/\pi]^{1/2}$. If $f(x)$ is defined by (4.2), then the right-hand side (RHS) of (4.1) is

$$RHS = (2\pi)^{-1/2} \int_0^{\infty} \exp\{-\tfrac{1}{2}(x + \alpha y)^2\}\, [f(y) + f(-y)]\, \mathrm{d}y =$$

$$= (2\pi)^{-1/2}\, C \int_0^{\infty} \exp\{-\tfrac{1}{2}(x + \alpha y)^2\} \exp\{-\tfrac{1}{2}(1 - \alpha^2)\, y^2\}\, \mathrm{d}y ,$$

since $\Phi(-\alpha y) + \Phi(\alpha y) = 1$. Further we have

$$RHS = C \exp\{-\tfrac{1}{2}(1-\alpha^2)\,x^2\}\,(2\pi)^{-1/2}\int_0^\infty \exp\{-\tfrac{1}{2}(y+\alpha x)^2\}\,\mathrm{d}y =$$
$$= C \exp\{-\tfrac{1}{2}(1-\alpha^2)\,x^2\}\,\Phi(-\alpha x) = f(x)\,.$$

Obviously, $f(x) > 0$. From Lemma 2.1 we obtain $\int f(x)\,\mathrm{d}x = 1$. This concludes the proof. □

Theorem 4.2. Let $f(x)$ be the density (4.2) and

$$v'_n = \int_{-\infty}^{\infty} x^n f(x)\,\mathrm{d}x\,.$$

Then

$$v'_n = \alpha^{-n-1}[2(1-\alpha^2)/\pi]^{1/2}\,J_n\,,$$

where J_n is defined in Lemma 2.1 for $k = \alpha^{-2} - 1$. The expectation μ and the variance v^2 belonging to $f(x)$ are

$$\mu = -(2/\pi)^{1/2}\,\alpha(1-\alpha^2)^{-1/2}\,,\quad v^2 = (1-\alpha^2)^{-1}\left(1 - \frac{2\alpha^2}{\pi}\right).$$

Proof. The formula for v'_n follows from Lemma 2.1. Further we have $\mu = v'_1$, $v^2 = v'_2 - (v'_1)^2$. □

The central moments

$$v_n = \int_{-\infty}^{\infty} (x-\mu)^n f(x)\,\mathrm{d}x$$

can be computed for $n = 3$ and $n = 4$ from

$$v_3 = v'_3 - 3v'_2 v'_1 + 2(v'_1)^3\,,\quad v_4 = v'_4 - 4v'_3 v'_1 + 6v'_2(v'_1)^2 - 3(v'_1)^4\,.$$

Then the coefficient of skewness α_2 and the coefficient of excess α_4 are given by

$$\alpha_3 = v_3/v^3\,,\quad \alpha_4 = v_4/v^4\,.$$

Theorem 4.3. Let the variable X_{s-1} in the model (3.1) have the density $f(x)$ given in (4.2). Then the correlation coefficient ϱ of the variables X_s and X_{s-1} is

$$\varrho = (\pi - 2\alpha^2)^{-1}\,\alpha[\pi + 2\alpha(1-\alpha^2)^{1/2} - 2\alpha - 2\,\mathrm{arctg}\,(\alpha^{-2} - 1)^{1/2}]\,.$$

Proof. The simultaneous density $h(x, y)$ of the variables $X = X_s$ and $Y = X_{s-1}$ is

$$h(x,y) = \begin{cases}(2\pi)^{-1/2}\exp\{-(x-\alpha y)^2/2\}\,f(y)\,, & \text{if } y \leqq 0\,,\\ (2\pi)^{-1/2}\exp\{-(x+\alpha y)^2/2\}\,f(y)\,, & \text{if } y > 0\,.\end{cases}$$

Then

$$\mathrm{E}XY = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,h(x,y)\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \pi^{-1}(1-\alpha^2)^{1/2}\left[\int_{-\infty}^{0} y \exp\{-(1-\alpha^2)y^2/2\}\,\Phi(-\alpha y)\int_{-\infty}^{\infty} x \exp\{-(x-\alpha y)^2/2\}\,dx\,dy + \right.$$
$$\left. + \int_{0}^{\infty} y \exp\{-(1-\alpha^2)y^2/2\}\,\Phi(-\alpha y)\int_{-\infty}^{\infty} x \exp\{-(x+\alpha y)^2/2\}\,dx\,dy\right].$$

From

$$\int_{-\infty}^{\infty} x \exp\{-(x \mp \alpha y)^2/2\}\,dx = \pm(2\pi)^{1/2}\,\alpha y$$

and from Lemma 2.2 we obtain

$$\mathsf{E}XY = (1-\alpha^2)^{-1}\alpha + 2\pi^{-1}(1-\alpha^2)^{-1/2}\alpha^2 - 2\pi^{-1}(1-\alpha^2)^{-1}\operatorname{arctg}(\alpha^{-2}-1)^{1/2}.$$

Since $\mathsf{E}X = \mathsf{E}Y = \mu$ and $\mathsf{Var}\,X = \mathsf{Var}\,Y = v^2$, where μ and v^2 are introduced in Theorem 4.2, it only remains to insert into the formula

$$\varrho = (\mathsf{E}XY - \mu^2)/v^2\,. \qquad \square$$

In the following part of this section we prove that (4.2) is a unique density satisfying (4.1).

Lemma 4.4. To any solution $f \in L^1(\mathbb{R})$ of (4.1) there exists a function $\varphi \in L^1(\mathbb{R})$ such that

$$\varphi(\alpha z) = (\alpha\sqrt{\pi})^{-1}\int_{-\infty}^{\infty} e^{-(z-t)^2}\,\varphi(t)\,dt\,, \quad z \in \mathbb{R}\,, \tag{4.3}$$

and

$$f(x) = (\alpha\sqrt{\pi})^{-1}\int_{0}^{\infty} \exp\left\{-\left(\frac{x}{\sqrt{2}} + t\right)^2\right\}\varphi(t)\,dt\,, \quad x \in \mathbb{R}\,. \tag{4.4}$$

Proof. Let $f \in L^1(\mathbb{R})$ be a solution of (4.1). Put

$$\varphi(t) = f(t\sqrt{(2)}/\alpha) + f(-t\sqrt{(2)}/\alpha)\,. \tag{4.5}$$

Since $\varphi \in L^1(\mathbb{R})$ is an even function, we get

$$\int_{-\infty}^{\infty} e^{-(z-t)^2}\,\varphi(t)\,dt = \int_{0}^{\infty}\left[e^{-(z+t)^2} + e^{-(z-t)^2}\right]\varphi(t)\,dt\,.$$

Inserting from (4.5) we obtain four integrals. After the substitutions $\pm t\sqrt{(2)}/\alpha = s$ we use the fact that f is a solution of (4.1). This yields

$$\int_{-\infty}^{\infty} e^{-(z-t)^2}\,\varphi(t)\,dt = \frac{\alpha}{\sqrt{2}}\left[f(z\sqrt{2}) + f(-z\sqrt{2})\right]\sqrt{(2\pi)} = \alpha\sqrt{(\pi)}\,\varphi(\alpha z)\,.$$

Further, from (4.1),

$$f(x) = (2\pi)^{-1/2}\int_{0}^{\infty} \exp\{-\tfrac{1}{2}(x+\alpha y)^2\}\left[f(-y) + f(y)\right]dy\,.$$

The substitution $y = t\sqrt{(2)}/\alpha$ immediately leads to (4.4). $\square$

Lemma 4.5. Let $\varphi \in L^1(\mathbb{R})$ satisfy (4.3). Then for the Fourier transform

$$\hat{\varphi}(t) = \int_{-\infty}^{\infty} e^{itx} \varphi(x)\, dx$$

of the function φ the relation

$$\hat{\varphi}(t/\alpha) = e^{-t^2/4} \hat{\varphi}(t), \quad t \in \mathbb{R}, \tag{4.6}$$

holds.

Proof. The Fourier transform of e^{-x^2} is $\sqrt{(\pi)}\, e^{-t^2/4}$. Calculating the Fourier transforms of both sides of (4.3) we obtain (4.6). □

Lemma 4.6. Let F be a function such $F(t/\alpha) = e^{-t^2/4} F(t)$. If F is continuous at $t = 0$, then there exists a constant c such that

$$F(t) = c \exp\left\{-\frac{1}{4}\frac{\alpha^2}{1-\alpha^2} t^2\right\}.$$

Proof. Put

$$G(t) = F(t) \exp\left\{\frac{1}{4}\frac{\alpha^2}{1-\alpha^2} t^2\right\}.$$

Then

$$G(t/\alpha) = F(t/\alpha) \exp\left\{\frac{1}{4}\frac{1}{1-\alpha^2} t^2\right\} = e^{-t^2/4} \exp\left\{\frac{1}{4}\frac{1}{1-\alpha^2} t^2\right\} F(t) = G(t).$$

The function G is also continuous at $t = 0$. For any $t \in \mathbb{R}$ we have $\alpha^n t \to 0$ and

$$G(t) = G[(\alpha t)/\alpha] = G(\alpha t) = G(\alpha^2 t) = \ldots = G(\alpha^n t) \to G(0) = c.$$ □

Lemma 4.7. If $\varphi \in L^1(\mathbb{R})$ is a solution of (4.3), then there exists $k \in \mathbb{R}$ such that

$$\varphi(t) = k \exp\left\{-\frac{1-\alpha^2}{\alpha^2} t^2\right\}, \quad t \in \mathbb{R}. \tag{4.7}$$

Proof. In view of (4.6) and Lemma 4.6,

$$\hat{\varphi}(t) = c \exp\left\{-\frac{1}{4}\frac{\alpha^2}{1-\alpha^2} t^2\right\}.$$

This implies (4.7). □

Theorem 4.8. The function (4.2) is a unique density satisfying (4.1).

Proof. It was proved in Theorem 4.1 that $f(x)$ given in (4.2) is a solution. From (4.2) and (4.5) we obtain

$$\varphi(t) = [2(1-\alpha^2)/\pi]^{1/2} \exp\left\{-\frac{1-\alpha^2}{\alpha^2} t^2\right\}.$$

However, according to Lemma 4.7 this φ is the only solution of (4.2) leading to a density. Since any solution f must satisfy (4.4), we proved the unicity. □

5. MARKOV APPROXIMATION

The Markov process $\{X_s\}$ defined by (3.1) can be approximated by a Markov chain with a finite number of states. We choose some points $c_1 < c_2 < \ldots < c_{h-1}$ in such a way that one of them coincides with the threshold r. If we formally put $c_h = \infty$, then the real line is divided into intervals $J_1, \ldots, J_h$, where $J_1 = (-\infty, c_1)$, $J_k = [c_{k-1}, c_k)$ for $k = 2, \ldots, h$. Let

$$z_1 = c_1 - \tfrac{1}{2}(c_2 - c_1), \quad z_k = \tfrac{1}{2}(c_{k-1} + c_k) \quad \text{for} \quad k = 2, \ldots, h-1,$$
$$z_h = c_{h-1} + \tfrac{1}{2}(c_{h-1} - c_{h-2}).$$

This is the classical representation of the intervals J_k by their middle-points.

Let i, j be some numbers from the set $\{1, 2, \ldots, h\}$. The probability $\mathrm{P}(X_s \in J_j \mid X_{s-1} \in J_i)$ can be approximated by

$$p_{ij} = \mathrm{P}(X_s \in J_j \mid X_{s-1} = z_i).$$

From the model (3.1) we can see that the conditional distribution of X_s given $X_{s-1} = z_i$ is $N(\alpha z_i, \sigma^2)$ for $z_i < r$ and $N(\beta z_i, \sigma^2)$ for $z_i > r$. This leads to a homogeneous Markov chain with h states represented by $z_1, \ldots, z_h$ and with the matrix of transition probabilities $\mathbf{P} = (p_{ij})$, where

$$p_{ij} = \begin{cases} \Phi[(z_j - \alpha z_i)/\sigma] - \Phi[(z_{j-1} - \alpha z_i)/\sigma] & \text{if} \quad z_i < r, \\ \Phi[(z_j - \beta z_i)/\sigma] - \Phi[(z_{j-1} - \beta z_i)/\sigma] & \text{if} \quad z_i > r. \end{cases}$$

The stationary distribution of the Markov chain is given by such a vector of probabilities $\mathbf{p} = (p_1, \ldots, p_h)'$ which satisfies the equation

$$\mathbf{p} = \mathbf{P}'\mathbf{p}, \quad p_1 + \ldots + p_h = 1.$$

The moments of the stationary distribution are

$$\lambda'_k = \sum_{i=1}^{h} z_i^k p_i.$$

The expectation μ and the variance v^2 are

$$\mu = \sum z_i p_i, \quad v^2 = \sum z_i^2 p_i - \mu^2.$$

The central moments of higher orders as well as the coefficients α_3 and α_4 of this distribution can be calculated in the same way as v_3, v_4, α_3 and α_4 in Section 4.

The matrix of transition probabilities after k steps is $\mathbf{P}^k$. Its elements will be denoted by $p_{ij}^{(k)}$. If $\{X_s\}$ is our stationary Markov chain with the states $z_1, \ldots, z_k$, then

$$\mathrm{E} X_s X_{s+k} = \sum_i \sum_j z_i z_j p_i p_{ij}^{(k)}.$$

The correlation function of the process $\{X_s\}$ is

$$\varrho_k = (\mathsf{E}X_s X_{s+k} - \mu^2)/v^2\,, \quad k = 1, 2, \ldots$$

The spectral density g of this stationary process can be calculated by the formula

$$g(\lambda) = \frac{v^2}{2\pi} \sum_{k=-\infty}^{\infty} \varrho_k \, \mathrm{e}^{-\mathrm{i}k\lambda}\,,$$

where $\varrho_0 = 1$ and $\varrho_k = \varrho_{-k}$ for $k < 0$. In practical cases we shall compute ϱ_k only for $|k| \leqq K$, where K is a bound depending on the rate how ϱ_k tends to zero. Then the spectral density is approximately

$$g(\lambda) = \frac{v^2}{2\pi} \left(1 + 2 \sum_{k=1}^{K} \varrho_k \cos k\lambda\right).$$

6. A NUMERICAL SOLUTION

The equation (3.2) can also be solved numerically. Generally, consider the integral equation

$$f(x) - \lambda \int_{-\infty}^{\infty} q(x \mid y) f(y) \, \mathrm{d}y = 0\,, \tag{6.1}$$

where q is a given function. First, we choose the points $a_1, \ldots, a_{n+1}$ such that the integrals

$$\int_{-\infty}^{a_1} q(x \mid y) f(y) \, \mathrm{d}y \quad \text{and} \quad \int_{a_{n+1}}^{\infty} q(x \mid y) f(y) \, \mathrm{d}y$$

are negligible. One of the points a_i must coincide with r. Put $\Delta_i = a_{i+1} - a_i$, $s_i = (a_{i+1} + a_i)/2$. Then

$$\int_{-\infty}^{\infty} q(x \mid y) f(y) \, \mathrm{d}y \doteq \sum_{i=1}^{n} \int_{a_i}^{a_{i+1}} q(x \mid y) f(y) \, \mathrm{d}y$$

and each integral from a_i to a_{i+1} will be numerically evaluated using the Gauss formula with three knots:

$$\int_{a_i}^{a_{i+1}} q(x \mid y) f(y) \, \mathrm{d}y = \sum_{j=1}^{3} w_{ij} \, q(x \mid y_{ij}) f(y_{ij})\,,$$

$$w_{ij} = \begin{cases} \frac{5}{18}\Delta_i & \text{for } j = 1\,, \\ \frac{4}{9}\Delta_i & \text{for } j = 2\,, \\ \frac{5}{18}\Delta_i & \text{for } j = 3\,, \end{cases} \qquad y_{ij} = \begin{cases} s_i - \frac{1}{2}\sqrt{(\frac{3}{5})}\,\Delta_i & \text{for } j = 1\,, \\ s_i & \text{for } j = 2\,, \\ s_i + \frac{1}{2}\sqrt{(\frac{3}{5})}\,\Delta_i & \text{for } j = 3\,. \end{cases}$$

Instead of (6.1) we have

$$f(x) - \lambda \sum_{i=1}^{n} \sum_{j=1}^{3} w_{ij} \, q(x \mid y_{ij}) f(y_{ij}) = 0\,. \tag{6.2}$$

Inserting y_{ij} for x we obtain the homogeneous system of linear equations, where $f(y_{ij})$ are unknown and λ is a parameter. This leads to the problem of finding the eigenvalues and the eigenvectors of a matrix. We look for the solution which corresponds to such a value of λ, which is near to 1. In our numerical investigations really a value of λ was very near to 1 (e.g. 0·997), whereas the others were substantially smaller (below 0·5), and the identification of the proper λ from the set of all eigenvalues was easy.

According to (6.2) an (approximate) formula for the stationary density f was found in the form

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{3} k_{ij}\, q(x \mid y_{ij}), \tag{6.3}$$

where

$$k_{ij} = \lambda w_{ij} f(y_{ij}), \quad \sum_{i=1}^{n} \sum_{j=1}^{3} k_{ij} = 1,$$

and

$$q(x \mid y_{ij}) \text{ is the density of } \begin{cases} N(\alpha y_{ij}, \sigma^2) & \text{if } y_{ij} < r, \\ N(\beta y_{ij}, \sigma^2) & \text{if } y_{ij} > r. \end{cases} \tag{6.4}$$

The moments corresponding to $f(x)$ can be easily found, because

$$\int_{-\infty}^{\infty} x^k f(x)\, \mathrm{d}x = \sum_{i=1}^{n} \sum_{j=1}^{3} k_{ij} \int_{-\infty}^{\infty} x^k\, q(x \mid y_{ij})\, \mathrm{d}x$$

and $\int x^k\, q(x \mid y_{ij})\, \mathrm{d}x$ is a known moment of a normal distribution (6.4). Thus we get the expectation μ, variance v^2, α_3 and α_4.

This method also allows to calculate the correlation coefficient ϱ of the variables $X = X_s$ and $Y = Y_{s-1}$ in the stationary case. Let Y have the density f (6.3). Then the joint density $h(x, y)$ of X and Y is

$$h(x, y) = \begin{cases} (2\pi)^{-1/2} \exp\{-\frac{1}{2}(x - \alpha y)^2\} f(y) & \text{if } y < r, \\ (2\pi)^{-1/2} \exp\{-\frac{1}{2}(x - \beta y)^2\} f(y) & \text{if } y \geqq r. \end{cases}$$

We have

$$\mathsf{E}XY = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, h(x, y)\, \mathrm{d}x\, \mathrm{d}y =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{3} k_{ij} \int_{-\infty}^{r} y\, q(y \mid y_{ij}) \int_{-\infty}^{\infty} x(2\pi)^{-1/2} \exp\{-\tfrac{1}{2}(x - \alpha y)^2\}\, \mathrm{d}x\, \mathrm{d}y +$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{3} k_{ij} \int_{r}^{\infty} y\, q(y \mid y_{ij}) \int_{-\infty}^{\infty} x(2\pi)^{-1/2} \exp\{-\tfrac{1}{2}(x - \beta y)^2\}\, \mathrm{d}x\, \mathrm{d}y =$$

$$= \alpha \sum_{i=1}^{n} \sum_{j=1}^{3} k_{ij} \int_{-\infty}^{r} y^2\, q(y \mid y_{ij})\, \mathrm{d}y + \beta \sum_{i=1}^{n} \sum_{j=1}^{3} k_{ij} \int_{r}^{\infty} y^2\, q(y \mid y_{ij})\, \mathrm{d}y.$$

We insert from (6.4) and use Lemma 2.3. Finally we get

$$\mathsf{E}XY = \alpha \sum_{i} \sum_{\substack{j \\ y_{ij} < r}} k_{ij}[(1 + \alpha^2 y_{ij}^2)\, \Phi(r - \alpha y_{ij}) - (2\pi)^{-1/2} (r + \alpha y_{ij}) \exp\{-\tfrac{1}{2}(r - \alpha y_{ij})^2\}] +$$

$$+ \alpha \sum_{i} \sum_{\substack{j \\ y_{ij} \geqq r}} k_{ij}[(1 + \beta^2 y_{ij}^2)\, \Phi(r - \beta y_{ij}) - (2\pi)^{-1/2} (r + \beta y_{ij}) \exp\{-\tfrac{1}{2}(r - \beta y_{ij})^2\}] +$$

$$+ \beta \sum_{i} \sum_{\substack{j \\ y_{ij} < r}} k_{ij}[(1 + \alpha^2 y_{ij}^2)\, \Phi(\alpha y_{ij} - r) + (2\pi)^{-1/2} (r + \alpha y_{ij}) \exp\{-\tfrac{1}{2}(r - \alpha y_{ij})^2\}] +$$

$$+ \beta \sum_{i} \sum_{\substack{j \\ y_{ij} \geqq r}} k_{ij}[(1 + \beta^2 y_{ij}^2)\, \Phi(\beta y_{ij} - r) + (2\pi)^{-1/2} (r + \beta y_{ij}) \exp\{-\tfrac{1}{2}(r - \beta y_{ij})^2\}] .$$

Now, the correlation coefficient ϱ is calculated by

$$\varrho = (\mathsf{E}XY - \mu^2)/v^2 .$$

7. SOME NUMERICAL RESULTS

The exact solution based on explicit formulas from Section 4 is available only for $r = 0$ and $\alpha = -\beta$. However, in these cases it enables to compare the accuracy of other methods.

The Markov approximation, the results of which are given in Tables 1, 2 and 3, was obtained from a division of the real line into 10 intervals. The bounds of these interval (except for $-\infty$ and ∞) are introduced in the row A of the Table 6.

For the numerical solution, which was described in Section 6, the following points a_i were chosen:

$$-8, \; -5, \; -2, \; -1, \; 0, \; 1, \; 2, \; 6.$$

Again, the results are contained in Tables 1, 2 and 3.

For each combination (r, α, β) ten simulations were produced. Each simulation had the length $N = 500$. The results are given in the tables in the columns with the heading "simulations". The first number is the average of the corresponding sample statistics calculated from 10 realizations. The second number (in the parenthesis) represents the sample deviation of the type

$$[\tfrac{1}{9} \sum_{i=1}^{10} (x_i - \bar{x})^2]^{1/2} ,$$

where x_i are the sample statistics.

The accuracy of the Markov approximation surely depends on the number of states and on their location. Some information about it can be obtained from the Tables 4 and 5. The investigated cases are denoted by the letters $A - G$ and the bounds of the used intervals are in Table 6. To make the comparisons easier, the Table 4 contains also column with the exact results and the Table 5 a column with the numerical solution (because in the latter case the exact result is not known).

The correlation functions and the spectral densities for $r = 0$, $\alpha = 0.9$ and $\beta = -0.9$ are drawn in Fig. 1 and Fig. 2, respectively. The solid line is the average from 10 simulations. The spectral densities in the simulations were estimated by Parzen formula. The dashed line represents the result from the Markov approximation when there are 10 states corresponding to the scheme A introduced in Table 6. The dotted line belongs to the Markov approximation with 20 states and their scheme is E.

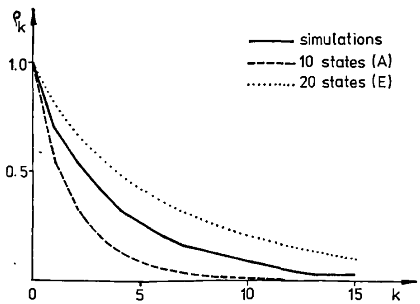

Fig. 1. Estimates of the correlation function for $r = 0$, $\alpha = 0.9$, $\beta = -0.9$.

Table 1.

		r = 0			
		Exact	Markov approximation	Numerical solution	Simulations
$\alpha = 0.5$, $\beta = -0.5$	μ	−0·461	−0·488	−0·461	−0·428 (0.048)
	v^2	1·121	1·192	1·121	1·112 (0.074)
	α_3	−0·035	0·017	−0·035	−0·029 (0·129)
	α_4	3·010	2·874	3·010	2·902 (0·177)
	ϱ	0·173	0·174	0·173	0·153 (0·060)
$\alpha = 0.8$, $\beta = -0.8$	μ	−1·064	−1·033	−1·063	−1·002 (0·088)
	v^2	1·646	1·440	1·646	1·624 (0·157)
	α_3	−0·245	0·165	−0·246	−0·228 (0·160)
	α_4	3·134	2·528	3·136	3·065 (0·232)
	ϱ	0·522	0·449	0·522	0·496 (0·058)
$\alpha = 0.9$, $\beta = -0.9$	μ	−1·647	−1·332	−1·639	−1·533 (0·134)
	v^2	2·549	1·514	2·513	2·447 (0·331)
	α_3	−0·472	0·347	−0·441	−0·460 (0·220)
	α_4	3·321	2·451	3·220	3·194 (0·371)
	ϱ	0·724	0·548	0·723	0·701 (0·047)

Table 2.

		$r = 1$		
		Markov approximation	Numerical solution	Simulations
$\alpha = 0{\cdot}5$, $\beta = -0{\cdot}5$	μ	$-0{\cdot}364$	$-0{\cdot}343$	$-0{\cdot}307$ $(0{\cdot}053)$
	v^2	$1{\cdot}297$	$1{\cdot}217$	$1{\cdot}216$ $(0{\cdot}087)$
	α_3	$0{\cdot}023$	$-0{\cdot}023$	$-0{\cdot}037$ $(0{\cdot}138)$
	α_4	$2{\cdot}851$	$2{\cdot}988$	$2{\cdot}882$ $(0{\cdot}176)$
	ϱ	$0{\cdot}217$	$0{\cdot}219$	$0{\cdot}200$ $(0{\cdot}056)$
$\alpha = 0{\cdot}9$, $\beta = -0{\cdot}9$	μ	$-1{\cdot}219$	$-1{\cdot}540$	$-1{\cdot}447$ $(0{\cdot}149)$
	v^2	$1{\cdot}801$	$2{\cdot}829$	$2{\cdot}784$ $(0{\cdot}305)$
	α_3	$0{\cdot}425$	$-0{\cdot}308$	$-0{\cdot}300$ $(0{\cdot}190)$
	α_4	$2{\cdot}517$	$3{\cdot}084$	$3{\cdot}058$ $(0{\cdot}355)$
	ϱ	$0{\cdot}562$	$0{\cdot}722$	$0{\cdot}699$ $(0{\cdot}039)$

Table 3.

		$r = 2$		
		Markov approximation	Numerical solution	Simulations
$\alpha = 0{\cdot}5$, $\beta = -0{\cdot}5$	μ	$-0{\cdot}261$	$-0{\cdot}145$	$-0{\cdot}099$ $(0{\cdot}060)$
	v^2	$1{\cdot}362$	$1{\cdot}314$	$1{\cdot}302$ $(0{\cdot}071)$
	α_3	$0{\cdot}009$	$-0{\cdot}035$	$-0{\cdot}047$ $(0{\cdot}076)$
	α_4	$2{\cdot}832$	$2{\cdot}980$	$2{\cdot}915$ $(0{\cdot}153)$
	ϱ	$0{\cdot}278$	$0{\cdot}356$	$0{\cdot}342$ $(0{\cdot}051)$
$\alpha = 0{\cdot}9$, $\beta = -0{\cdot}9$	μ	$-1{\cdot}115$	$-1{\cdot}326$	$-1{\cdot}267$ $(0{\cdot}286)$
	v^2	$2{\cdot}045$	$3{\cdot}437$	$3{\cdot}399$ $(0{\cdot}506)$
	α_3	$0{\cdot}440$	$-0{\cdot}179$	$-0{\cdot}153$ $(0{\cdot}187)$
	α_4	$2{\cdot}480$	$2{\cdot}878$	$2{\cdot}817$ $(0{\cdot}240)$
	ϱ	$0{\cdot}590$	$0{\cdot}747$	$0{\cdot}720$ $(0{\cdot}030)$

The numerical experience contained in Tables 1–6 and in Fig. 1–2 leads to the following conclusions.

a. The numerical solution is very near to the exact values. Its quality decreases a little, when $|\alpha|, |\beta| \to 1$.

b. Markov approximations give only rough estimates of the considered characteristics. The precision is not better even if the number of the states grows. We must remark, however, that this may be a consequence of the inaccuracies arising in the process of solving great number of linear equations with an ill-conditioned matrix. It would be interesting to analyze this problem in detail. In any case, the Figures 1 and 2 clearly show that the estimates of the correlation function and of the spectral density are rather bad, whereas the approximations of μ and v^2 can be of some use.

Table 4.

$r = 0, \alpha = 0.9, \beta = -0.9$								
	Exact	Markov approximation						
		10 states (A)	10 states (B)	10 states (C)	10 states (D)	20 states (E)	26 states (F)	26 states (G)
μ	−1·647	−1·332	−1·878	−1·859	−1·898	−1·962	−1·761	−1·707
v^2	2·549	1·514	3·210	3·212	3·229	3·739	2·759	2·740
α_3	−0·472	0·347	−0·512	−0·550	−0·515	−0·814	−0·466	−0·533
α_4	3·321	2·451	3·145	3·055	3·021	3·832	3·324	3·503
ϱ	0·724	0·548	0·753	0·765	0·765	0·801	0·736	0·736

Table 5.

$r = 1, \alpha = 0.8, \beta = -0.6$						
	Numerical solution	Markov approximation				
		10 states (A)	10 states (B)	10 states (C)	10 states (D)	20 states (E)
μ	−0·871	−0·843	−0·954	−0·913	−0·938	−0·933
v^2	1·853	1·676	2·239	2·104	2·154	2·091
α_3	−0·176	0·182	−0·156	−0·363	−0·347	−0·382
α_4	3·062	2·535	3·623	3·326	3·255	3·544
ϱ	0·558	0·498	0·546	0·580	0·587	0·591

c. Although a simulation of the length $N = 500$ is usually considered as long enough for AR processes of the first and second order, in case of a threshold model this may be true only for small values $|\alpha|$ and $|\beta|$. Tables 1 – 3 clearly show that the

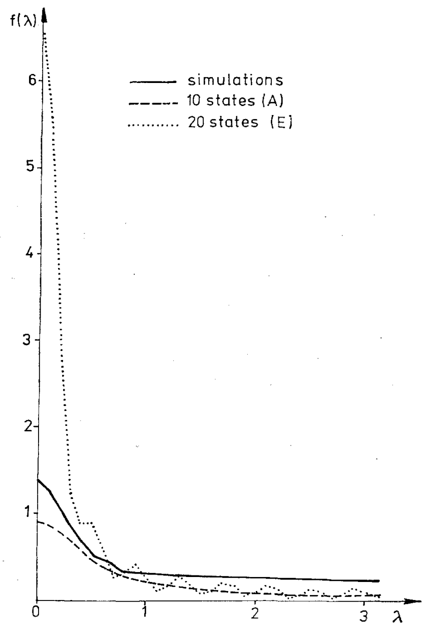

Fig. 2. Estimates of the spectral density for $r = 0$, $\alpha = 0{\cdot}9$, $\beta = -0{\cdot}9$.

Table 6.

The bounds of intervals used in Tables 4 and 5.

A	− 2·5	− 1·5	−1·0	−0·5	0·0	0·5	1·0	1·5	2·5	
B	− 6·0	− 4·0	−3·0	−2·0	−1·0	0·0	1·0	2·0	5·0	
C	− 6·0	− 4·0	−3·0	−2·2	−1·6	−1·0	0·0	1·0	2·0	
D	− 6·0	− 4·0	−2·8	−2·1	−1·4	−0·7	0·0	1·0	2·0	
E	− 9·0	− 6·0	−4·0	−3·0	−2·5	−2·0	−1·5	−1·0	−0·5	0·0
	0·5	1·0	1·5	2·0	2·5	3·0	4·0	6·0	9·0	
F	−13·0	−11·0	−9·0	−8·0	−7·0	−6·0	−5·0	−4·0	−3·0	−2·0
	− 1·7	− 1·4	−1·1	−0·8	−0·5	−0·2	0·0	0·5	1·0	1·5
	2·0	3·0	4·0	5·0	7·0					
G	−12·0	− 9·0	−8·0	−7·0	−6·0	−5·0	−4·0	−3·6	−3·2	−2·8
	− 2·4	− 2·0	−1·7	−1·4	−1·1	−0·8	−0·5	−0·2	0·0	0·5
	1·0	1·5	2·0	3·0	5·0					

variability of characteristics from the simulations is too large for $\alpha = 0{\cdot}9$, $\beta = -0{\cdot}9$. Although the results of simulations introduced in our tables are close to exact values, we must not forget, that they are averages; each of them needed, in fact, some $10 \times \times 500$ simulated variables!

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Doc. RNDr. Jiří Anděl, DrSc., doc. RNDr. Ivan Netuka, CSc., RNDr. Karel Zvára, CSc., Univerzita Karlova, matematicko-fyzikální fakulta (Charles University — Faculty of Mathematics and Physics), Sokolovská 83, 186 00 Praha 8. Czechoslovakia.