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ON CHARACTERIZATION OF DIRECTED DIVERGENCE OF TYPE β THROUGH INFORMATION EQUATION

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The directed-divergence of type β ($\beta > 0$, $\beta \neq 1$) has been characterized through an 'Information Equation' and its solution, under the homogeneity (of type β, $\beta > 0$, $\beta \neq 1$) has been obtained. Some applications of the directed-divergence of type β to Information Theory have been discussed.

1. INTRODUCTION

The information theoretic concepts as envisaged in various measures, namely Kullback's information or directed-divergence [5], Kerridge's inaccuracy [7] and Theil's information improvement [12], have found many applications in behaviourial sciences. Characterizations of these measures in arbitrary probability spaces and continuous analogs have been discussed earlier by Campbell [2], Rathie and Kannappan [8], [6], Sharma and Autar [10], Sharma and Soni [11] and Renyi [9] etc.

The object of this contribution is to characterize the directed-divergence of type β ($\beta > 0$, $\beta \neq 1$) through 'information equation' and to discuss some applications of it to information theory. Let the true probabilities of a system of events be given by the complete probability distribution:

$$P = (p_1, p_2, \ldots, p_n), \quad p_i \geqq 0, \quad \sum_{i=1}^{n} p_i = 1 .$$

Let the $Q = (q_1, q_2, \ldots, q_n)$, $q_i \geqq 0, \sum_{i=1}^{n} q_i = 1$ be the revised probability distribution. The measures of error made by the observer or the measures of information gain, estimating the discrete probability distribution Q from the probability distribution P are given by

$$I_n \begin{pmatrix} p_1, p_2, \ldots, p_n \\ q_1, q_2, \ldots, q_n \end{pmatrix} = \sum_{i=1}^{n} p_i \log (p_i q_i^{-1}) \tag{1.1}$$

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and

(1.2) $$I_n^\beta\begin{pmatrix}p_1, p_2, \ldots, p_n\\ q_1, q_2, \ldots, q_n\end{pmatrix} = (2^{\beta-1} - 1)^{-1}\left[\sum_{i=1}^{n} p_i^\beta q_i^{1-\beta} - 1\right]$$

(cf. [5], and [8]) where $\beta > 0, \beta \neq 1$.

Here we consider the 'information equation' given by

(1.3) $$I\begin{pmatrix}x, y, z\\ l, m, n\end{pmatrix} = I\begin{pmatrix}x + y, 0, z\\ l + m, 0, n\end{pmatrix} + I\begin{pmatrix}x, y, 0\\ l, m, 0\end{pmatrix}$$

in the domain $D^2 = \{(x, y, z; l, m, n); x, y, z \geqq 0, l, m, n \geqq 0, xy + yz + zx > 0, lm + mn + nl > 0\}$, a generalization of entropy equation [4] viz.

(1.4) $$H(x, y, z) = H(x + y, 0, z) + H(x, y, 0)$$
$$(x, y, z \geqq 0,\ xy + yz + zx > 0)\,.$$

The homogeneity condition considered here is defined as follows:

(1.5) $$I^\beta\begin{pmatrix}\lambda x, \lambda y, \lambda z\\ \mu l, \mu m, \mu n\end{pmatrix} = \lambda^\beta \mu^{1-\beta} I_3^\beta\begin{pmatrix}x, y, z\\ l, m, n\end{pmatrix}, \quad \lambda, \mu > 0\,, \quad \beta > 0\,, \quad \beta \neq 1\,.$$

The symmetric and homogeneous (of type β, $\beta > 0$, $\beta \neq 1$) solution of (1.3) has been given in Section 2 and its applications to information theory have been discussed in Section 3.

2. SOLUTION OF INFORMATION EQUATION AND CHARACTERIZATION OF DIRECTED-DIVERGENCE

In this section we solve the information equation (1.3) and characterize the directed-divergence of type β under the homogeneity condition (1.5). Let the measure (1.2) satisfy the following postulates:

Postulate 1. Branching property i.e.

(2.1) $$I_n^\beta\begin{pmatrix}p_1, p_2, \ldots, p_n\\ q_1, q_2, \ldots, q_n\end{pmatrix} = I_{n-1}^\beta\begin{pmatrix}p_1 + p_2, p_3, \ldots, p_n\\ q_1 + q_2, q_3, \ldots, q_n\end{pmatrix} +$$
$$+ (p_1 + p_2)^\beta (q_1 + q_2)^{1-\beta} I_2^\beta\begin{pmatrix}\dfrac{p_1}{p_1 + p_2}, \dfrac{p_2}{p_1 + p_2}\\ \dfrac{q_1}{q_1 + q_2}, \dfrac{q_2}{q_1 + q_2}\end{pmatrix}$$

Postulate 2. Symmetry i.e.

(2.2) $$I_n^\beta\begin{pmatrix}p_1, p_2, \ldots, p_n\\ q_1, q_2, \ldots, q_n\end{pmatrix} = I_n^\beta\begin{pmatrix}p_{k(1)}, p_{k(2)}, \ldots, p_{k(n)}\\ q_{k(1)}, q_{k(2)}, \ldots, q_{k(n)}\end{pmatrix}$$

where $k(1), k(2), \ldots, k(n)$ is a permutation of $1, 2, \ldots, n$.

Postulate 3. Nullity i.e.

$$I_n^\beta\begin{pmatrix}1, 0, 0\\ 1, 0, 0\end{pmatrix} = 0. \tag{2.3}$$

Postulate 4. Unit i.e.

$$I_n^\beta\begin{pmatrix}1, 0, 0\\ \frac{1}{2}, \frac{1}{2}, 0\end{pmatrix} = 1. \tag{2.4}$$

Lemma 1. The function

$$I^\beta\begin{pmatrix}x, y, z\\ l, m, n\end{pmatrix} = (x + y + z)^\beta (l + m + n)^{1-\beta} I_3^\beta\begin{pmatrix}\dfrac{x}{x+y+z}, \dfrac{y}{x+y+z}, \dfrac{z}{x+y+z}\\ \dfrac{l}{l+m+n}, \dfrac{m}{l+m+n}, \dfrac{n}{l+m+n}\end{pmatrix} \tag{2.5}$$

$(x, y, z \geqq 0,\ x + y + z > 0,\ l, m, n \geqq 0, l + m + n > 0)$ $\beta \neq 1$, $\beta > 0$ satisfies the information equation (1.3).

Proof. Set $n = 3$ in Postulate 1,

$$I_3^\beta\begin{pmatrix}p_1, p_2, p_3\\ q_1, q_2, q_3\end{pmatrix} = I_2^\beta\begin{pmatrix}p_1 + p_2, p_3\\ q_1 + q_2, q_3\end{pmatrix} + \tag{2.6}$$

$$+ (p_1 + p_2)^\beta (q_1 + q_2)^{1-\beta} I_2^\beta\begin{pmatrix}\dfrac{p_1}{p_1 + p_2}, \dfrac{p_2}{p_1 + p_2}\\ \dfrac{q_1}{q_1 + q_2}, \dfrac{q_2}{q_1 + q_2}\end{pmatrix}$$

Letting $p_1 = p_2 = \frac{1}{2}$, $p_3 = 0$

$q_1 = q_2 = \frac{1}{2}$, $q_3 = 0$

and then $p_1 = p_3 = \frac{1}{2}$, $p_2 = 0$

$q_1 = q_3 = \frac{1}{2}$, $q_2 = 0$

and using Postulate 2 we get

$$I_2^\beta\begin{pmatrix}1, 0\\ 1, 0\end{pmatrix} = 0. \tag{2.7}$$

Next setting $p_3 = 0$, $q_3 = 0$ in (2.6), we get

$$I_3^\beta\begin{pmatrix}p_1, p_2, 0\\ q_1, q_2, 0\end{pmatrix} = I_2^\beta\begin{pmatrix}p_1 + p_2, 0\\ q_1 + q_2, 0\end{pmatrix} + (p_1 + p_2)^\beta (q_1 + q_2)^{1-\beta}. \tag{2.8}$$

$$. I_2^\beta\begin{pmatrix}\dfrac{p_1}{p_1 + p_2}, \dfrac{p_2}{p_1 + p_2}\\ \dfrac{q_1}{q_1 + q_2}, \dfrac{q_2}{q_1 + q_2}\end{pmatrix}$$

which in accordance with $p_1 + p_2 = 1 = q_1 + q_2$ and (2,7) yields

$$(2.9) \qquad I_3^\beta\begin{pmatrix} p_1, p_2, 0 \\ q_1, q_2, 0 \end{pmatrix} = I_2^\beta\begin{pmatrix} p_1, p_2 \\ q_1, q_2 \end{pmatrix}.$$

Therefore (2.6), on using (2.9) takes the form

$$(2.10) \quad I_3^\beta\begin{pmatrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \end{pmatrix} = I_3^\beta\begin{pmatrix} p_1 + p_2, 0, p_3 \\ q_1 + q_2, 0, q_3 \end{pmatrix} + (p_1 + p_2)^\beta (q_1 + q_2)^{1-\beta} \cdot$$

$$\cdot\, I_3^\beta\begin{pmatrix} \dfrac{p_1}{p_1 + p_2}, \dfrac{p_2}{p_1 + p_2}, 0 \\ \dfrac{q_1}{q_1 + q_2}, \dfrac{q_2}{q_1 + q_2}, 0 \end{pmatrix}.$$

Next setting

$$p_1 = \frac{x}{x + y + z}, \quad p_2 = \frac{y}{x + y + z}, \quad p_3 = \frac{z}{x + y + z}$$

and

$$q_1 = \frac{l}{l + m + n}, \quad q_2 = \frac{m}{l + m + n}, \quad q_3 = \frac{n}{l + m + n}$$

in (2.10), we get on simplification

(2.11)

$$I_3^\beta\begin{pmatrix} \dfrac{x}{x + y + z}, \dfrac{y}{x + y + z}, \dfrac{z}{x + y + z} \\ \dfrac{l}{l + m + n}, \dfrac{m}{l + m + n}, \dfrac{n}{l + m + n} \end{pmatrix} = \frac{1}{(x + y + z)^\beta (l + m + n)^{1-\beta}} \cdot$$

$$\cdot \left[(x + y + z)^\beta (l + m + n)^{1-\beta} I_3^\beta\begin{pmatrix} \dfrac{x + y}{x + y + z}, 0, \dfrac{z}{x + y + z} \\ \dfrac{l + m}{l + m + n}, 0, \dfrac{n}{l + m + n} \end{pmatrix} + \right.$$

$$\left. + (x + y)^\beta (l + m)^{1-\beta} I_3^\beta\begin{pmatrix} \dfrac{x}{x + y}, \dfrac{y}{x + y}, 0 \\ \dfrac{l}{l + m}, \dfrac{m}{l + m}, 0 \end{pmatrix} \right].$$

Finally using the functional relation (2.5), we mark that $I^\beta\begin{pmatrix} x, y, z \\ l, m, n \end{pmatrix}$ satisfies (1.3) and this proves Lemma 1. □

Next we prove the main theorem which relaxes the regularity condition.

Theorem 1. The symmetric and homogeneous solution of type β, of (1.3) satisfying Postulate 3 and Postulate 4 is given by

(2.12)

$$I^{\beta}\begin{pmatrix} x, y, z \\ l, m, n \end{pmatrix} = A_{\beta}[x^{\beta}l^{1-\beta} + y^{\beta}m^{1-\beta} + z^{\beta}n^{1-\beta} - (x+y+z)^{\beta}(l+m+n)^{1-\beta}]$$

where $A_{\beta} = (2^{\beta-1} - 1)^{-1}$.

Proof. By homogeneity, we have

$$I^{\beta}\begin{pmatrix} x, 0, 0 \\ l, 0, 0 \end{pmatrix} = x^{\beta}l^{1-\beta} I_3^{\beta}\begin{pmatrix} 1, 0, 0 \\ 1, 0, 0 \end{pmatrix} = 0 \tag{2.13}$$

$$x, l > 0, \quad \beta \neq 1, \quad \beta > 0.$$

Define a function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ such that

$$f(x; l) = I^{\beta}\begin{pmatrix} 1-x, x, 0 \\ 1-l, l, 0 \end{pmatrix} = \tag{2.14}$$

$$= \lambda^{\beta}\mu^{1-\beta} I^{\beta}\begin{pmatrix} \frac{1-x}{\lambda}, \frac{x}{\lambda}, 0 \\ \frac{1-l}{\mu}, \frac{l}{\mu}, 0 \end{pmatrix} = \lambda^{\beta}\mu^{1-\beta} I^{\beta}\begin{pmatrix} \frac{1-x}{\lambda}, 0, \frac{x}{\lambda} \\ \frac{1-l}{\mu}, 0, \frac{l}{\mu} \end{pmatrix}$$

$$x \in [0, 1], \quad l = [0, 1], \quad \lambda, \mu > 0.$$

With this substitution on the right hand side of (1.3) and then using symmetry, we get

$$(x+y+z)^{\beta}(l+m+n)^{1-\beta} f\left(\frac{z}{x+y+z}; \frac{n}{l+m+n}\right) +$$

$$+ (x+y)^{\beta}(l+m)^{1-\beta} f\left(\frac{y}{x+y}; \frac{m}{l+m}\right) =$$

$$= (x+y+z)^{\beta}(l+m+n)^{1-\beta} f\left(\frac{y}{x+y+z}; \frac{m}{l+m+n}\right) +$$

$$+ (x+z)^{\beta}(l+n)^{1-\beta} f\left(\frac{z}{x+z}; \frac{n}{l+n}\right)$$

or

(2.15)

$$f\left(\frac{z}{x+y+z}; \frac{n}{l+m+n}\right) + \left(\frac{x+y}{x+y+z}\right)^{\beta}\left(\frac{l+m}{l+m+n}\right)^{1-\beta} f\left(\frac{y}{x+y}; \frac{m}{l+m}\right) =$$

$$= f\left(\frac{y}{x+y+z}; \frac{m}{l+m+n}\right) + \left(\frac{x+z}{x+y+z}\right)^{\beta}\left(\frac{l+n}{l+m+n}\right)^{1-\beta} f\left(\frac{z}{x+z}; \frac{n}{l+n}\right)$$

Putting

$$a = \frac{z}{x + y + z}, \quad b = \frac{y}{x + y + z}, \quad \alpha = \frac{n}{l + m + n}, \quad \theta = \frac{m}{l + m + n}$$

(2.15) becomes

$$f(a; \alpha) + (1 - a)^\beta (1 - \alpha)^{1-\beta} f\left(\frac{b}{1 - a}; \frac{\theta}{1 - \alpha}\right) = \tag{2.16}$$

$$= f(b; \theta) + (1 - b)^\beta (1 - \theta)^{1-\beta} f\left(\frac{a}{1 - b}; \frac{\alpha}{1 - \theta}\right)$$

which is a functional equation which has the following solution

$$f(a; \alpha) = A_\beta[a^\beta \alpha^{1-\beta} + (1 - a)^\beta (1 - \alpha)^{1-\beta} - 1], \quad \beta > 0, \quad \beta \neq 1 \tag{2.17}$$

(cf. [8]) under the boundary conditions

$$f(1; 1) = f(0; 0) \tag{2.18}$$

and

$$f(1, \tfrac{1}{2}) = f(0; \tfrac{1}{2}) = 1. \tag{2.19}$$

From (2.18) we have

$$A_\beta = (2^{1-\beta} - 1)^{-1}. \tag{2.20}$$

Next (2.14) and (2.17) gives

$$I^\beta \begin{pmatrix} 1 - a, a, 0 \\ 1 - \alpha, \alpha, 0 \end{pmatrix} = A_\beta[a^\beta \alpha^{1-\beta} + (1 - a)^\beta (1 - \alpha)^{1-\beta} - 1]. \tag{2.21}$$

Returning to the substitution,

$$a = \frac{z}{x + y + z}, \quad b = \frac{y}{x + y + z}, \quad \alpha = \frac{n}{l + m + n}, \quad \theta = \frac{m}{l + m + n}$$

(2.21) takes the form

$$I^\beta \begin{pmatrix} \dfrac{x + y}{x + y + z}, & \dfrac{z}{x + y + z}, & 0 \\ \dfrac{l + m}{l + m + n}, & \dfrac{n}{l + m + n}, & 0 \end{pmatrix} =$$

$$= A_\beta\left[\left(\frac{z}{x + y + z}\right)^\beta \left(\frac{n}{l + m + n}\right)^{1-\beta} + \left(\frac{x + y}{x + y + z}\right)^\beta \left(\frac{l + m}{l + m + n}\right)^{1-\beta} - 1\right]$$

or

$$I^\beta \begin{pmatrix} x + y, z, 0 \\ l + m, n, 0 \end{pmatrix} = A_\beta[z^\beta n^{1-\beta} + (x + y)^\beta (l + m)^{1-\beta} - \tag{2.22}$$

$$- (x + y + z)^\beta (l + m + n)^{1-\beta}].$$

Also (2.22) and (2.2) gives

$$(2.23)\qquad I^{\beta}\begin{pmatrix} x, & y, & 0 \\ l, & m, & 0 \end{pmatrix} = A_{\beta}[x^{\beta}l^{1-\beta} + y^{\beta}m^{1-\beta} - (x + y)^{\beta}(l + m)^{1-\beta}] .$$

Finally (2.22), (2.23), (2.2) and (2.5) with (1.3) give the required result i.e.

$$I^{\beta}\begin{pmatrix} x, & y, & z \\ l, & m, & n \end{pmatrix} = A_{\beta}[x^{\beta}l^{1-\beta} + y^{\beta}m^{1-\beta} + z^{\beta}n^{1-\beta} - (x + y + z)^{\beta}(l + m + n)^{1-\beta}]$$

where $A_{\beta} = (2^{\beta-1} - 1)^{-1}$.

This completes the proof of the Theorem 1. □

3. APPLICATIONS TO INFORMATION THEORY

Theorem 2. Let $P = (p_1, p_2, \ldots, p_n) \in \Delta_n$ and $Q = (q_1, q_2, \ldots q_n) \in \Delta_n$ be two complete probability distributions; their directed divergence of type β satisfying Postulates 1 – 5 is given by

$$(3.1)\qquad I_n^{\beta}\begin{pmatrix} p_1, p_2, \ldots, p_n \\ q_1, q_2, \ldots, q_n \end{pmatrix} = (2^{\beta-1} - 1)^{-1}\left[\sum_{i=1}^{n} p_i^{\beta}q_i^{1-\beta} - 1\right]$$

where $\beta \neq 1, \beta > 0$.

Proof. For probability distributions $(p_1, p_2, p_3) \in \Delta_3$ and $(q_1, q_2, q_3) \in \Delta_3$, $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$, we have from (2.12)

$$(3.2)\qquad I_3^{\beta}\begin{pmatrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \end{pmatrix} = A_{\beta}[p_1^{\beta}q_1^{1-\beta} + p_2^{\beta}q_2^{1-\beta} + p_3^{\beta}q_3^{1-\beta} - 1] .$$

Also from (2.9), when $p_1 + p_2 = 1$, $q_1 + q_2 = 1$, we have

$$(3.3)\qquad I_2^{\beta}\begin{pmatrix} p_1, p_2 \\ q_1, q_2 \end{pmatrix} = A_{\beta}[p_1^{\beta}q_1^{1-\beta} + p_2^{\beta}q_2^{1-\beta} - 1] .$$

Applying the mathematical induction, we get the required result i.e. (3.1). Hence the theorem. □

Theorem 3. Let $P = (p_1, p_2, \ldots, p_m)$, $\sum_{i=1}^{m} p_i = 1$, $Q = (q_1, q_2, \ldots, q_n)$, $\sum_{j=1}^{n} q_j = 1$ and $Q_i = (q_{i1}, q_{i2}, \ldots, q_{in})$, $\sum_{j=1}^{n} q_{ij} = 1$, $i = 1, 2, \ldots, m$ be the probability distributions, then we have

$$(3.4)\qquad I_n^{\beta}\begin{pmatrix} \sum_{i=1}^{m} p_i q_{i1}, & \sum_{i=1}^{m} p_i q_{i2}, & \ldots, & \sum_{i=1}^{m} p_i q_{in} \\ q_1, & q_2, & \ldots\ldots, & q_n \end{pmatrix} \leqq \sum_{i=1}^{m} p_i I_n^{\beta}\begin{pmatrix} q_{i1}, q_{i2}, \ldots, q_{in} \\ q_1, q_2, \ldots, q_n \end{pmatrix}$$

Proof.

$$I_n^\beta\begin{pmatrix}\sum_{i=1}^m p_iq_{i1},\ldots,\sum_{i=1}^m p_iq_{in}\\ q_1,\ldots\ldots,q_n\end{pmatrix}=(2^{\beta-1}-1)^{-1}\left[\sum_{j=1}^n\left(\sum_{i=1}^m p_iq_{ij}\right)^\beta q_j^{1-\beta}-1\right].$$

Refer, [3] (p. 532)

$$(3.5)\qquad \left(\sum_{i=1}^m p_iq_{ij}\right)^\beta \geqq \sum_{i=1}^m p_iq_{ij}^\beta,\quad \text{for}\quad \beta<1$$

$$\leqq \sum_{i=1}^m p_iq_{ij}^\beta,\quad \text{for}\quad \beta>1\,.$$

Multiplying by $q_j^{1-\beta}$ and summing over all j's, we have

$$\sum_{j=1}^n\left(\sum_{i=1}^m p_iq_{ij}\right)^\beta q_j^{1-\beta}\gtrless\sum_{j=1}^n\left(\sum_{i=1}^m p_iq_{ij}^\beta\right)q_j^{1-\beta}$$

according as $\beta\gtrless 1$. However since $(2^{\beta-1}-1)\gtrless 0$ according as $\beta\gtrless 1$, we have when $\beta\neq 1$

$$(3.6)\qquad (2^{\beta-1}-1)^{-1}\left[\sum_{j=1}^n\left(\sum_{i=1}^m p_iq_{ij}\right)^\beta q_j^{1-\beta}-1\right]\leqq$$

$$\leqq(2^{\beta-1}-1)^{-1}\left[\sum_{i=1}^m\sum_{j=1}^n p_iq_{ij}^\beta q_j^{1-\beta}-1\right].$$

Thus

$$I_n^\beta\begin{pmatrix}\sum_{i=1}^m p_iq_{i1},\ldots,\sum_{i=1}^m p_iq_{in}\\ q_1,\ldots\ldots,q_n\end{pmatrix}\leqq(2^{\beta-1}-1)^{-1}\sum_{i=1}^m p_i\left[\sum_{j=1}^n q_{ij}^\beta q_j^{1-\beta}-1\right]=$$

$$=\sum_{i=1}^m p_i I_n^\beta\begin{pmatrix}q_{i1},q_{i2},\ldots,q_{in}\\ q_1,q_2,\ldots,q_n\end{pmatrix}.$$

This completes the proof of the theorem. □

We conclude that the characterization of directed divergence under homogeneity condition have applications in Mathematical Economics, Production Theory and Utility Theory.

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