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General Coalition-Games

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The main purpose of this paper is to suggest a general model of coalition-games including, as its special cases, the most important types of such games. It concerns the main types of coalition-games investigated in the well-known literature, as well as the games which may, and probably shall, appear in the future as consequences of new applications of coalition-games theory. This paper presents such a general coalition-game model, shows its connections to some more special games, and also suggests a generalization of classical strong solutions of coalition-games, due for the presented game model.

0. INTRODUCTION

There exist a few models of coalition-games, investigated in many papers. Let us mention the most important of them: the games with side-payments and the games without them. These games are used, and slightly modified, for the purposes of some classical applications, and also in economical or psychological investigations, etc. But, the range of applications will probably increase. It means that the number of different modifications of the basic game models will increase, too. As some of the new non-standard applications will surely need only some of the properties assumed for the basic coalition-game model, the problem of essentiality of those properties and assumptions will appear. It concerns, especially, the assumption of superadditivity of the coalition pay-offs. This assumption is very natural in case of the usual applications of coalition-games to the cooperative strategical behaviour, but it is problematic if the game model is used for modeling some other problems of groups or clusters forming.

The main reason of the presented paper, as well as of the author's papers [5] and [9], is to suggest a game model including only the essential and necessary properties of the considered game. It will be advantageous if those assumptions reflect the most natural and generally valid properties of situations modelled by the coalition-games theory. This paper introduces also some basic properties of the suggested game model, and the concept of solution of such game is presented here. One of the goals of the following paragraphs is to show that the coalition-game model, suggested here, really covers a wide class of more special game concepts which were investigated in the well known literature, or which may appear in the future as a consequence of applications of the game theory into new areas.

Last but not least, I should like to express here my thanks to RNDr Albert Perez, DrSc, from the Institute of Information Theory and Automation of Czechoslovak Academy of Sciences, whose interesting ideas about qualitatively new possibilities of the coalition-games applications into new fields, especially into the cluster analysis, turned my attention to the problems formulated in this paper and in paper [9].

1. GENERAL COALITION-GAME

The concept of general coalition-game, introduced here, is investigated and discussed in the following sections of this paper and in papers [5] and [9].

In the whole paper we denote by R the set of all real numbers and by N the set of all positive integers. Symbol $\{x_1, \ldots, x_n\}$ denotes the set containing exactly the elements $x_1, \ldots, x_n$, where $n \in N$; symbol $\{x\colon x \textit{ has the property } \boldsymbol{P}\}$ denotes the set of exactly all elements x having the property $\boldsymbol{P}$.

Let us consider a non-empty and finite set I, and denote by $\mathscr{I}$ the class of all non-empty subsets of I, i.e.

$$\mathscr{I} = 2^I - \{\emptyset\}\,.$$

Let us consider a mapping $\boldsymbol{V}$ from the class 2^I into the family of subsets of R^I,

$$\boldsymbol{V} : 2^I \to 2^{R^I}\,,$$

such that for every $K \in 2^I$

(1.1) $\boldsymbol{V}(K)$ *is closed;*

(1.2) *if* $\mathbf{x} = (x_i)_{i\in I} \in \boldsymbol{V}(K)$, $\mathbf{y} = (y_i)_{i\in I} \in R^I$, *and* $x_i \geqq y_i$ *for all* $i \in K$, *then* $\mathbf{y} \in \boldsymbol{V}(K)$;

(1.3) $\boldsymbol{V}(K) \neq \emptyset\,,\ \boldsymbol{V}(K) = R^I \Leftrightarrow K = \emptyset$.

Then the pair

$$\Gamma = (I, \boldsymbol{V})$$

will be called a *general coalition-game*, the set I will be the set of *players*, and $\boldsymbol{V}$ will be the *general characteristic function* of the game Γ.

The concept of general coalition-game, in some degree similar to the one defined above, was introduced already in [8] and also in some other works. However, it contained the superadditivity assumption of characteristic function, and it was, consequently, less general than the one considered here. There are also some other, more sophisticated, differences between the game model presented here and that one given in [8]. The superadditivity assumption, especially, can be easily formulated even for the general characteristic function V, and it would be natural in many applications of the suggested game model. But, it is not necessary, and also not desirable, in some other situations in which the general coalition-game may be applied. It is, why the superadditivity of the general characteristic function is not assumed here.

It is useful to introduce some other notions describing the elements of the general coalition-game and the concepts derived from them. The sets belonging to the class $\mathscr{I}$ are called *coalitions*, and any partition of I into disjoint coalitions is called a *coalition structure*. Every vector $\mathbf{x} \in R^I$ is called an *imputation*. If there exists a coalition structure $\mathscr{K}$ such that

$$\mathbf{x} \in \bigcap_{K \in \mathscr{K}} V(K)$$

then $\mathbf{x}$ is an *admissible imputation* in the given game Γ.

The assumption of the complete cooperation expressed by the fact that the class $\mathscr{I}$ contains all non-empty subsets of I (it means that all non-empty coalitions are considered to be possible and admissible) is seemingly strong and also innatural for some applications. In fact, this assumption is a formal one only. It enables us to simplify the formal notation and the formulation of some results. The results introduced here or their very closed analogies keep valid even if some reasonable form of incomplete cooperation is considered, e.g. if we suppose that $\mathscr{I}$ is such a class of subsets of I that for every $K \in \mathscr{I}$ there exist $L_1, \ldots, L_n \in \mathscr{I}$, $L_i \cap K = \emptyset$, $L_i \cap L_j = \emptyset$, for $i \neq j$, $i, j = 1, \ldots, n$, such that $K \cup L_1 \cup \ldots \cup L_n = I$, i.e. the set $\{K, L_1, \ldots, L_n\}$ forms a coalition structure.

On the other hand, the assumption of closedness of the general characteristic function V is essential for the further steps of the presented work. But, it is acceptable from the point of view of the existing coalition-games models, and we may suppose it to be acceptable also for the expected future applications of the presented general coalition-game model.

The same is true even in case of assumptions (1.2) and (1.3). They are essential for the results defived in this work, and they will be probably important even for other investigations of the suggested game model. However, they are natural, and it is not so easy to find an application of the coalition-games theory, for which that assumptions would not be acceptable.

The general coalition-game introduced here has numerous special cases into which it may be modified. In the following four sections we briefly mention some of them which seem to be most important and most interesting ones.

Let us consider the finite and non-empty set of players I, and the set of coalitions $\mathcal{I}$. Let us suppose that every coalition $K \in \mathcal{I}$ has a non-empty and finite set A_K of pure strategies and the corresponding set S_K of mixed strategies to its disposal. Every mixed strategy is a probability distribution on the set of pure strategies. It is usual to suppose that $A_{K \cup L} \supset A_K \times A_L$ for $K, L \in \mathcal{I}$, $K \cap L = \emptyset$. This assumption represents a superadditivity property for strategies. Its interpretation is obvious and well-known from the literature.

Let us suppose, further, that there exist elementary utility functions e_i, $i \in I$, which prescribe to every coalition structure $\mathcal{K}$ and to every vector of pure strategies

$$\mathbf{a} = (a_K)_{K \in \mathcal{K}} \in \mathop{\times}_{K \in \mathcal{K}} A_K = A_{\mathcal{K}}$$

a real-valued vector

$$(e_i(\mathcal{K}, \mathbf{a}))_{i \in I} .$$

These elementary utility functions can be extended in very natural way to the mixed utility functions u_i, $i \in I$, defined on the class of pairs consisting of coalition structures and of corresponding vectors of mixed strategies. Then the mixed utility functions u_i, $i \in I$, prescribe to every coalition structure $\mathcal{K}$ and to every vector of mixed strategies

$$\mathbf{s} = (s_K)_{K \in \mathcal{K}} \in \mathop{\times}_{K \in \mathcal{K}} S_K$$

the utilities

(2.1) $$u_i(\mathcal{K}, \mathbf{s}) = \sum e_i(\mathcal{K}, \mathbf{a}) \prod_{K \in \mathcal{K}} s_K(a_K) ,$$

where the sum is taken over $\mathbf{a} = (a_K)_{K \in \mathcal{K}} \in A_{\mathcal{K}}$.

Then the coalition-game with mixed utilities is defined as the triple

$$\Gamma_1 = (I, \{A_K\}_{K \in \mathcal{I}}, \{u_i\}_{i \in I}) .$$

We may suppose that the utilities are not transitive, it means that there is no possibility of side-payments in the given game Γ_1. (The opposite case of games with transitive utility is discussed in Sections 4 and 5). We may suppose, too, that the players in Γ_1 are allowed not to exploit all the obtained utility. It means that if a coalition structure $\mathcal{K}$ and a vector of mixed strategies $\mathbf{s}$ prescribe to a player $i \in I$ some utility $u_i(\mathcal{K}, \mathbf{s})$ then they prescribe to him also all utility values $u_i^* \leqq u_i(\mathcal{K}, \mathbf{s})$. For the coalition-game with mixed utilities the following statement can be easily proved.

Theorem 1. The game with mixed utilities Γ_1 is a special case of the general coalition-game $\Gamma = (I, V)$, where for every $K \in \mathcal{I}$ is

(2.2) $V(K) = \{\mathbf{x} = (x_i)_{i \in I}$: *there exists a coalition structure* $\mathscr{K}$ *and a vector of mixed strategies* $\mathbf{s} = (s_K)_{K \in \mathscr{K}}$ *such that for all* $i \in I$ *is* $x_i \leqq u_i(\mathscr{K}, \mathbf{s})\}$.

Moreover, the sets $V(K)$ are convex closed sets for all coalitions $K \in \mathscr{I}$.

Proof. We have to prove the properties of the general characteristic function V, introduced on Section 1. Property (1.1) follows from the inequalities in (2.2), as well as property (1.2). This property is interpreted for the described type of games by the consideration introduced above the theorem. Also property (1.3) follows immediately from the relation (2.2). The convexity of the sets $V(K)$, $K \in \mathscr{I}$, follows from (2.1), as the set of possible values of the utility function $u_i(\mathscr{K}, \mathbf{s})$ is, for a given coalition structure $\mathscr{K}$ and player $i \in I$, a convex combination of the values of elementary utility functions $e_i(\mathscr{K}, \mathbf{a})$, where $\mathbf{a} \in A_{\mathscr{K}}$.

3. MARKET GAMES

The previous section of this paper was devoted to the games, in which no possibility of the utility transition was admitted. The only possibility of certain utility distribution among members of the same coalition is the choice of a proper mixed strategy. Then the utility is distributed among the members of coalition in statistical sense. They may hope to obtain, after a sufficiently large number of repetitions of the same game, coalition structure and strategies, the total profit proportional to the bargained rate of utilities.

In this section, we shall mention another type of coalition-games, in which a specific possibility of utility transmission exists. There does not still exist any common and lineary useful representant of utility, but the utility may be transfered by means of some individually useful representants of it. It means that the transfered utility may change (decrease or increase) its value during the transmission process. We call such games the market games, and we interpret and describe them in the following way.

We suppose that there exists a set of players I, which is finite and non-empty, with the class of coalitions $\mathscr{I}$. We suppose, further, that after the realization of the strategical part of the game, or after another arbitrary process, every coalition $K \in \mathscr{I}$ obtains its guaranted prize, which is represented by some quantity of goods α_K. These goods are distributed among members of the coalition K into amounts α_i, $i \in K$, such that

$$\alpha_K = \sum_{i \in K} \alpha_i \,. \tag{3.1}$$

There exist utility functions u_i, $i \in I$, defined on the set of all possible quantities of goods, which represent the utility of any possible amount of goods for individual players. They prescribe a real number $u_i(\alpha_i)$ to every quantity of goods and each player.

It is useful to note that there usually exist more kinds of goods in the given market game. If we denote their number by $m \in N$, then the amounts of goods α_K and α_i are real-valued vectors

$$\alpha_K = (\alpha_K^j)_{j=1,\dots,m}\,, \quad \alpha_i = (\alpha_i^j)_{j=1,\dots,m}\,,$$

with components representing the amounts of all considered kinds of goods. Then the utility functions map R^m into R, $u_i : R^m \to R$, $i \in I$, and the condition (3.1) can be substituted by the system of equalities

$$\alpha_K^j = \sum_{i \in K} \alpha_i^j \quad \text{for all} \quad j = 1, \dots, m\,. \tag{3.2}$$

It is obvious that the real possibilities of distribution of goods, it means also of distribution of utility, are very various.

We shall denote by $\mathbb{A}(K)$, $K \in \mathscr{I}$, the sets of all possible distributions of the quantity of goods α_K among players in K, e.g.

$$\mathbb{A}(K) = \{(\alpha_i)_{i \in K} : \alpha_i = (\alpha_i^j)_{j=1,\dots,m} \in R^m, \sum_{i \in K} \alpha_i^j = \alpha_K^j, j = 1, \dots, m\}\,, \tag{3.3}$$
$$m \in N,\ m \geqq 1\,.$$

As the amounts of goods α_i are m-dimensional real-valued vectors $(\alpha_j^i)_{j=1,\dots,m} \in R^m$, the classes $\mathbb{A}(K)$, $K \in \mathscr{I}$, are subsets of the space $(R^m)^K$. It follows from (3.1) and (3.2) immediately that the sets $\mathbb{A}(K)$ defined by (3.3) are closed and convex subsets of $(R^m)^K$, as they are Cartesian products of hyperplanes of R^K spaces.

After introducing all necessary concepts, we may define the market game. It is the quadruple

$$\Gamma_2 = (I, \{\alpha_K\}_{K \in \mathscr{I}}, \{\mathbb{A}(K)\}_{K \in \mathscr{I}}, \{u_i\}_{i \in I})\,.$$

It is evident that the condition (3.1) and its specification (3.2) are not the only conditions which may be used for the definition of the sets of possible distributions of goods $\mathbb{A}(K)$. Condition (3.2) is usually completed by some additional conditions, usually limiting the minimal values of α_i^j, e.g. $\alpha_i^j \geqq 0$, $i \in K$, $K \in \mathscr{I}$, $j = 1, \dots, m$, etc. It is not difficult to verify that the validity of following two theorems is preserved even for other forms of the sets $\mathbb{A}(K)$, $K \in \mathscr{I}$, which are closed, respectively convex, subsets of $(R^m)^K$.

Theorem 2. Let all sets $\mathbb{A}(K)$, $K \in I$, be closed subsets of $(R^m)^K$, and let the utility functions u_i, $i \in I$, be continuous on R^m. Then the market game Γ_2 is a special case of the general coalition-game $\Gamma = (I, V)$, where for every coalition $K \in \mathscr{I}$ is

$$V(K) = \{\mathbf{x} = (x_i)_{i \in I}: \textit{there exists } (\alpha_i)_{i \in K} \in \mathbb{A}(K) \tag{3.4}$$
$$\textit{such that for all } i \in K \textit{ is } x_i \leqq u_i(\alpha_i)\}\,.$$

Proof. If $\mathbb{A}(K)$ are closed sets and u_i are continuous functions, then the inequalities in (3.4) imply also the closedness of the sets $V(K)$. Moreover, the same inequalities in (3.4) and the finiteness of the vectors α_K imply the validity of (1.2) and (1.3) for the sets $V(K)$.

Theorem 3. If all sets $\mathbb{A}(K)$, $K \in \mathscr{I}$, are closed and convex in $(R^m)^K$, and if all utility functions u_i, $i \in I$, are continuous and concave on R^m, i.e.

$$u_i(\lambda\alpha_i + (1 - \lambda)\beta_i) \geqq \lambda u_i(\alpha_i) + (1 - \lambda) u_i(\beta_i)$$

for

$$0 \leqq \lambda \leqq 1\,, \quad \alpha_i\,, \quad \beta_i \in R^m\,,$$

then the sets $V(K)$ defined by (3.4) are convex subsets of R^I.

Proof. Let us choose $\mathbf{x} = (x_i)_{i\in I} \in V(K)$, $\mathbf{y} = (y_i)_{i\in I} \in V(K)$ and $(\alpha_i)_{i\in K} \in \mathbb{A}(K)$, $(\beta_i)_{i\in K} \in \mathbb{A}(K)$ such that for all $i \in K$ is $u_i(\alpha_i) \geqq x_i$ and $u_i(\beta_i) \geqq y_i$. If $0 \leqq \lambda \leqq 1$, then for all $i \in K$ is

$$\lambda x_i + (1 - \lambda) y_i \leqq \lambda u_i(\alpha_i) + (1 - \lambda) u_i(\beta_i) \leqq u_i(\lambda\alpha_i + (1 - \lambda)\beta_i)\,.$$

Moreover, the convexity of $\mathbb{A}(K)$ implies that

$$(\lambda\alpha_i + (1 - \lambda)\beta_i)_{i\in K} \in \mathbb{A}(K)\,,$$

and consequently,

$$(\lambda x_i + (1 - \lambda) y_i)_{i\in I} \in V(K)\,,$$

where the property (1.2) was used for $i \notin K$.

Conditions of closedness and convexity of $\mathbb{A}(K)$, $K \in \mathscr{I}$, as well as the condition of continuity of u_i, $i \in I$, are natural and usually accepted in economical models. It means that they are no exceptional constrains of the practical applicability of the results proved above to economical situations. Even the concavity of utility functions is usually assumed in economical models and may be, without any serious loss of generality, assumed in the market game model, if it is necessary.

4. COALITION-GAMES WITH SIDE-PAYMENTS

In this section we suppose that the utility obtained by players is completely transferable among them. It means that there exists a linear representant of utility, common for all players. Such games are well known in the literature. We suppose that there exists a set of players I with the class of coalitions $\mathscr{I}$, and that there exists a mapping v from the class $\mathscr{I}$ into R, usually called the characteristic function of the coalition-game with side-payments. It is usually supposed that the characteristic function v is superadditive, i.e.

$$v(K \cup L) \geqq v(K) + v(L), \quad \text{for} \quad K, L \in \mathscr{I}, K \cap L = \emptyset .$$

Then the coalition-game with side-payments is the pair

$$\Gamma_3 = (I, v) .$$

The values $v(K)$, $K \in \mathscr{I}$, of the characteristic function v, represent the total utility guaranted for the whole coalition K. The players in K are allowed to distribute this common utility $v(K)$ in arbitrary way. During this distribution, the total sum of utilities in the coalition K does not change, and it is equal to $v(K)$. If we denote by $(u_i)_{i \in K}$, $u_i \in R$, the vector of the final utilities obtained by players in K, then

$$\sum_{i \in K} u_i = v(K) , \tag{4.1}$$

and all real-valued vectors fulfilling (4.1) represent the admissible distributions of utility in K. Even in this case, we may accept the assumption that players are not forced to use all the obtained utility. It means that also all real-valued vectors $(u'_i)_{i \in K}$ such that

$$\sum_{i \in K} u'_i \leqq v(K) \tag{4.2}$$

are admissible distributions of utility among players in the coalition K.

It can be easily seen that the coalition-game introduced in this way is a special case of the general coalition-game. In fact, the need to generalize the concept of coalition-game with side-payments motivated the creation of the general coalition-game. The relation between general coalition-game and the game with side-payments is more precisely formulated in the following statement.

Theorem 4. The coalition-game with side-payments Γ_3 is a special case of the general coalition-game $\Gamma = (I, V)$, where for every $K \in \mathscr{I}$ is

$$V(K) = \{\mathbf{x} = (x_i)_{i \in I}: \sum_{i \in K} x_i \leqq v(K)\} . \tag{4.3}$$

The sets $V(K)$ are half-spaces of R^I. The superadditivity of the mapping v implies that for all $K, L \in \mathscr{I}$, $K \cap L = \emptyset$, is

$$V(K \cup L) \supset V(K) \cap V(L) . \tag{4.4}$$

Proof. The first statement of this theorem is obvious. The sets $V(K)$ are closed half-spaces of R^I, as follows from (4.3). The same formula implies the conditions (1.2) and (1.3), immediately. If the mapping v is superadditive, i.e.

$$v(K \cup L) \geqq v(K) + v(L), \quad K, L \in \mathscr{I}, \quad K \cap L = \emptyset ,$$

then, for an arbitrary $\mathbf{x} \in V(K) \cap V(L)$, is

$$\sum_{i \in K} x_i \leqq v(K) \quad \text{and} \quad \sum_{i \in L} x_i \leqq v(L),$$

as follows from (4.3). As $K \cap L = \emptyset$, the above inequalities imply that

$$\sum_{i \in K \cup L} x_i \leqq v(K) + v(L) \leqq v(K \cup L),$$

and, hence $\mathbf{x} \in V(K \cup L)$.

5. COALITION-GAMES WITH RESTRICTED SIDE-PAYMENTS

The coalition-games mentioned in this section represent a special modification of the coalition-games with side-payments introduced in the previous one. We assume that the coalition-game with side-payments $\Gamma_3 = (I, \mathbf{v})$ with the set of players I and characteristic function $\mathbf{v}$ is given. We do not suppose, here, that for any coalition $K \in \mathscr{I}$ all distributions $(u_i)_{i \in K}$, fulfilling (4.1), of the total utility $v(K)$ are admissible. We suppose that for any coalition $K \in \mathscr{I}$ a set $T(K) \subset R^K$ is given, such that all vectors $(u_i)_{i \in K} \in T(K)$ fulfil (4.1), and $T(K)$ contains exactly all admissible distributions of the total utility $v(K)$ of the coalition K. As usually, even in this case we suppose that players need not accept or utilize all the utility which they obtain. It means that if $(u_i)_{i \in K}$ belongs to $T(K)$ and $(u_i')_{i \in K} \in R^I$ is such that $u_i' \leqq u_i$ for all $i \in K$, then $(u_i')_{i \in K}$ represents also an admissible distribution of utility among members of the coalition K. Then the coalition-game with restricted side-payments is the triple

$$\Gamma_4 = (I, \mathbf{v}, \{T(K)\}_{K \in \mathscr{I}}).$$

The extremal example of such games is the coalition-game with restricted side-payments, where the distribution of utility among members of coalitions is strictly given. In such game the sets $T(K)$ are one-element sets for all coalitions $K \in \mathscr{I}$.

It is more than obvious that the coalition-game with restricted side-payments Γ_4 is a special modification of the general coalition-game $\Gamma = (I, V)$ with

$$V(K) = \{\mathbf{x} = (x_i)_{i \in I}: \textit{there exists } \mathbf{y} = (y_i)_{i \in K} \in T(K) \textit{ such that } x_i \leqq y_i \textit{ for all } i \in K\}, \quad K \in \mathscr{I}, \tag{5.1}$$

if and only if the sets $T(K)$ are closed. Then the closedness of $V(K)$, as well as the properties (1.2) and (1.3), follow from (5.1) immediately. Moreover, if the sets $T(K)$ are convex, then also $V(K)$ are convex.

The restrictions, stronger or weaker, put on the sets of admissible distributions of utility among members of coalitions in coalition-games with side-payments, represent a frequent element of the practical applications of those games. Then the superadditivity of the characteristic function v often looses much of its sense, and the consequence of superadditivity of the characteristic function v for the general characteristic function V, proved in the second part of Theorem 4, is not generally true.

6. STRONGLY STABLE COALITION STRUCTURES

In this section, we define and investigate the concept of the bargaining solution in the class of the coalition structures of a general coalition-game. More special cases of that game were mentioned in the previous sections, and much more about some of them we can find in literature. There are also more kinds of solutions defined in the literature for the particular special types of coalition-games. We can divide them, at least roughly, into two groups. The former one of them contains the "ideal" solutions fulfilling the demands of all players and coalitions, like the core of the coalition-game with side-payments. The necessity to fulfil the exactly given complex of demands causes certain uniformity of these solutions. They differ in their formal description, as they are adapted for differently defined games. However, the essential properties, expressed by that formal description, are remarcably similar.

The latter group of solutions is much more varied. It contains the solutions which have not all ideal properties, but which are acceptable for the most powerful players and coalitions, and which exist even in such games, in which the ideal solution is impossible. This group is considerably wide, it contains such solutions like von Neumann - Morgenstern solution or Aumann - Maschler bergaining set for the coalition-games with side-payments, and a lot of other solutions advantageous for different purposes and applications. In this section and in the following one, we suggest an "ideal" solution of the general coalition-game. It means, a generalization of solutions included into the first group is presented here. Such solutions are called, here, strong or strongly stable solutions. Certain attempt to suggest also a solution of the second, non-ideal, type is presented in author's paper [9], even if such solution can not be a generalization of all so various solutions of that kind.

The solutions defined for particular coalition-games in the literature are usually intent on the definition of the resulting pay-offs or utilities obtained by players. It is true, namely, for the "ideal" solutions. However, not only the final pay-offs are important. In some cases, we are interested in the coalitions or coalition structures which enable the desired pay-offs. This opinion was mentioned in author's paper [3], and, likewise, some problems connected with the applications of the coalition-games theory to some new areas show that the finding of the resultant coalition structures is an important element of the coalition-game solution. In case of the strongly stable solutions of the classical types of coalition-games, like the coalition-game with side-payments or the coalition-game with strategies and mixed utilities, the form of the resulting coalition structures is not so important. It is caused by the superadditivity assumption which means that the strongly stable pay-offs, if they exist, are always achievable for the all-players coalition I. As no form of superadditivity is assumed for the general coalition-game, we are intersted here in both, the resulting coalition structures and the resulting imputations realized by these coalition structures.

In this section, we suggest the definition of strongly stable coalition structures and derive some of their properties. First of all, we introduce the following auxiliary symbol.

If $\mathscr{M} \subset \mathscr{I}$ is a set of coalitions then we denote by $\boldsymbol{V}(\mathscr{M})$ the set

$$\boldsymbol{V}(\mathscr{M}) = \bigcap_{K \in \mathscr{M}} \boldsymbol{V}(K) . \tag{6.1}$$

In this way, we have practically extended the range of the general characteristic function $\boldsymbol{V}$ from the class of all coalitions (or from the class of all one-element subclasses of $\mathscr{I}$) to the family of all non-empty subclasses of $\mathscr{I}$. As we suppose that $\boldsymbol{V}(\emptyset) = R^I$, according to (1.3), then $\boldsymbol{V}$ is defined for all subclasses of $\mathscr{I}$. This form of the definition of the general characteristic function $\boldsymbol{V}$ was used in [5]. The definition of the general characteristic function $\boldsymbol{V}$ used in this paper is more useful for the purposes of this work, and it is also better for the illustration of mutual connections between general coalition-game and its special cases. Even if both definitions of $\boldsymbol{V}$, given in [5] and in this paper, are formally different, the results obtained in [5] are still valid for the general characteristic function $\boldsymbol{V}$ defined by conditions (1.1), (1.2) and (1.3) in Section 1, especially if the auxiliary concept (6.1) is used. It means that we may refer and use the results derived in [5] for the purposes of this work.

Definition 1. If $K \in \mathscr{I}$ is a coalition and $\boldsymbol{V}(K)$ is the value of general characteristic function then the set

$$\boldsymbol{V}^*(K) = \{\mathbf{x} = (x_i)_{i \in I} \in R^I : \textit{for all } \mathbf{y} = (y_i)_{i \in I} \in \boldsymbol{V}(K) \\ \textit{either } x_i > y_i \textit{ for some } i \in K, \text{ or } y_i = x_i \\ \textit{for all } i \in K\}$$

is called the *superoptimum* of the coalition K.

For the superoptimum function $\boldsymbol{V}^*$, the following statements are true.

Remark 1. It follows from Definition 1 immediately that for every coalition $K \in \mathscr{I}$ and every $\mathbf{x} \in \boldsymbol{V}^*(K)$, $\mathbf{y} \in \boldsymbol{R}^I$, such that $x_i \leqq y_i$ for all $i \in K$, is also $\mathbf{y} \in \boldsymbol{V}^*(K)$.

Remark 2. It follows from Definition 1 and from (1.1) that for every coalition $K \in \mathscr{I}$ is $\boldsymbol{V}(K) \cap \boldsymbol{V}^*(K) \neq \emptyset$.

Remark 3. It follows from Definition 1 that for every coalition $K \in \mathscr{I}$ is $\boldsymbol{V}(K) \cup \cup \boldsymbol{V}^*(K) = R^I$. This statement was also proved in [5] as Lemma 1.

Lemma 1. If $K \in \mathscr{I}$ is a coalition and $\mathbf{x}$ is an imputation such that $\mathbf{x} \in V(K) - V^*(K)$ then there exists an imputation $\mathbf{y} \in V(K) \cap V^*(K)$ such that $x_i \leqq y_i$ for all $i \in I$.

Proof. If $\mathbf{x} \notin V^*(K)$ then there exists $\mathbf{y} \in V(K)$ such that $x_i \leqq y_i$ for all $i \in K$ and, according to (1.2), the imputation $\mathbf{y}$ may be chosen so that also $y_i \geqq x_i$ for all $i \in I - K$. The closedness assumption (1.1) implies that the imputation $\mathbf{y}$ may be chosen in such a way that $\mathbf{y}$ is a boundary point of the set $V(K)$. Hence, there is no imputation $\mathbf{z} \in V(K)$ such that $z_i \geqq y_i$ for all $i \in K$. It means that $\mathbf{y}$ may be chosen so that $\mathbf{y} \in V(K) \cap V^*(K)$.

Lemma 2. If $K, L \in \mathscr{I}$ are coalitions such that $K \subset L$ and

$$V(K) = [V(L) \cap R^K] \times R^{I-K}$$

then also

$$V^*(K) = [V^*(L) \cap R^K] \times R^{I-K}.$$

Proof.

$$\begin{aligned} V^*(K) &= \{\mathbf{x} = (x_i)_{i\in I} \in R^I : \forall\, \mathbf{y} = (y_i)_{i\in I} \in V(K) \\ &\quad \textit{either}\ \exists\,(i \in K,\ y_i < x_i),\ \textit{or}\ \forall\,(i \in K,\ y_i = x_i)\} = \\ &= \{\mathbf{x} = (x_i)_{i\in I} \in R^I : \forall\, \mathbf{y} = (y_i)_{i\in I} \in [V(L) \cap R^K] \times R^{I-K} \\ &\quad \textit{either}\ \exists\,(i \in K,\ y_i < x_i)\ \textit{or}\ \forall\,(i \in K,\ y_i = x_i)\} = \\ &= [V^*(L) \cap R^K] \times R^{I-K}. \end{aligned}$$

Definition 2. If $\mathscr{K}$ is a coalition structure then we say that $\mathscr{K}$ is *strongly stable* iff there exists an imputation $\mathbf{x} \in V(\mathscr{K})$ such that $\mathbf{x}$ belongs to the superoptimum of all coalitions in $\mathscr{I}$.

Some fundamental properties of strongly stable coalition structures were derived in [5] already. Here, we repeat the most important of them and introduce a few more. The following lemma was proved in [5] as Lemma 2.

Lemma 3. If $\mathscr{K}$ is a coalition structure and if there exists $L \in \mathscr{I}$ such that $V(\mathscr{K}) \subset V(L) - V^*(L)$ then $\mathscr{K}$ is not strongly stable.

Corollary. If a coalition structure $\mathscr{K}$ is strongly stable then $V(\mathscr{K}) \cap V(L) \cap V^*(L) \neq \emptyset$ for all $L \in \mathscr{I}$, as follows from Lemma 3 and from (1.2).

Definition 3. We say that a coalition $L \in \mathscr{I}$ is effective iff there exists an imputation $\mathbf{x} \in V(L)$ such that $\mathbf{x} \in V^*(J)$ for all $J \in \mathscr{I}$, $J \subset L$. If $\mathscr{L}$ is a coalition structure then we say that $\mathscr{L}$ is effective from below iff all coalitions in $\mathscr{L}$ are effective.

Remark 4. As every coalition $K \in \mathscr{I}$ is a subset of itself, it follows from Definition 3 immediately that K is effective iff there exists $\mathbf{x} \in V(K) \cap V^*(K)$ such that $\mathbf{x} \in V^*(J)$ for all $J \subset K$, $J \in \mathscr{I}$.

Lemma 4. An arbitrary coalition structure $\mathscr{L}$ is effective from below iff there exist $\mathbf{x} \in V(\mathscr{L})$ such that $\mathbf{x} \in V^*(J)$ for all $J \in \mathscr{I}$ such that $J \subset L$ for some $L \in \mathscr{L}$.

The previous lemma was proved in [5] as Lemma 3. Its statement follows immediately from Definition 3.

Remark 5. The coalition structure $\{I\}$ containing exactly one coalition of all players is effective from below iff the coalition I is effective, as follows from Definition 3 immediately.

Remark 6. All one-player coalitions are effective, as follows from Remark 2. It means that each player $i \in I$ belongs to some effective coalition and that the coalition structure containing exactly all one-player coalitions is always effective from below.

Lemma 5. If $L \in \mathscr{I}$ is not effective then there exists a set of coalitions

$$\mathscr{I}^*(L) = \{J\colon J \in \mathscr{I},\ J \subset L,\ J \text{ is } \textit{effective}\}$$

such that

$$V^*(L) \supset \bigcap_{J \in \mathscr{I}^*(L)} V^*(J)\,.$$

The previous lemma was introduced in [5] as Lemma 4.

Lemma 6. If a coalition $L \in \mathscr{I}$ is effective then there always exists a coalition structure $\mathscr{L}$, effective from below, such that $L \in \mathscr{L}$.

Proof. The statement of this lemma follows from Remark 6 immediately. If L is an effective coalition then we can construct a coalition structure $\mathscr{L}$ containing L and exactly all one-player coalitions $\{i\}$ such that $i \in I - L$. Then $\mathscr{L}$ is effective from below.

The following two theorems were proved in [5] as Theorem 1 and Theorem 2. They follow from the previous lemmas.

Theorem 5. If $\mathbf{x} \in R^I$ is an imputation belonging to the superoptimum of all effective coalitions then it belongs to the superoptimum of all coalitions.

Theorem 6. A coalition structure $\mathscr{K}$ is strongly stable if and only if there exists an imputation $\mathbf{x} \in V(\mathscr{K})$ such that $\mathbf{x}$ belongs to the superoptimum $V^*(J)$ of all effective coalitions $J \in \mathscr{I}$.

After having introduced the concept of the strongly stable coalition structures, we may define and investigate the strong stability of imputations. As it was said in the introduction of the previous section, the solutions presented, for special types of coalition-games, in the literature concern usually the imputations only. It means that more connections between the strongly stable imputations defined below and some special strong solutions from the literature can be found. For the core of the coalition-game with side-payments, the connection is derived at the end of this section.

Definition 4. We say that an imputation $\mathbf{x} \in R^I$ is *strongly stable* iff it is admissible and it belongs to the superoptimum of all coalitions.

Remark 7. The imputation $\mathbf{x} \in R^I$ is strongly stable iff there exists a coalition structure $\mathscr{K}$ such that

$$\mathbf{x} \in V(\mathscr{K}) \cap \Big(\bigcap_{L \in \mathscr{I}} V^*(L) \Big),$$

as follows from the previous definition immediately.

The following two theorems were proved in [5] as Theorem 3 and Theorem 4. They describe interesting properties of strongly stable imputations, and follow from lemmas and theorems given in the previous section, as well as from Definition 4.

Theorem 7. There exists a strongly stable imputation in a given general coalition-game Γ if and only if there is at least one strongly stable coalition structure in Γ.

Theorem 8. An imputation $\mathbf{x} \in V(\mathscr{K})$, where $\mathscr{K}$ is a coalition structure, is strongly stable if and only if $\mathbf{x} \in V^*(J)$ for all effective coalitions $J \in \mathscr{I}$.

The strong mutual connection existing between strongly stable imputations and the core of the coalition-game with side payments is formulated and proved by the next theorem. If $\Gamma_3 = (I, \boldsymbol{v})$ is a coalition-game with side-payments and with super-additive characteristic function $\boldsymbol{v}$, which was discussed in Section 4, then the core of Γ_3 is the set of imputations

$$\mathbb{C} = \{\mathbf{x} = (x_i)_{i \in I} \in R^I : \sum_{i \in I} x_i \leqq \boldsymbol{v}(I), \textit{ and for all } K \in \mathscr{I} \textit{ is } \sum_{i \in K} x_i \geqq \boldsymbol{v}(K)\} ; \tag{7.1}$$

then the following statement holds.

Theorem 9. Let us suppose that the considered general coalition-game Γ is a coalition game with side-payments and with the superadditive characteristic function $\boldsymbol{v}$, i.e. that for all coalitions $K \in \mathcal{I}$ there exists a real number $v(K)$ such that

$$V(K) = \{\mathbf{x} = (x_i)_{i \in I}: \sum_{i \in K} x_i \leqq v(K)\} . \tag{7.2}$$

Then an imputation $\mathbf{x} \in R^I$ is strongly stable if and only if $\mathbf{x} \in V(I)$ and for all coalitions $K \in \mathcal{I}$ is

$$\sum_{i \in K} x_i \geqq v(K) .$$

Proof. The superadditivity of the characteristic function v and relation (7.2) imply that any imputation $\mathbf{x}$ is admissible iff the inequality

$$\sum_{i \in I} x_i \leqq v(I)$$

holds. Moreover, if (7.2) is fulfilled for all $K \in \mathcal{I}$ then there exists, for every $\mathbf{y} \in R^I$ such that

$$\sum_{i \in K} y_i < v(K) ,$$

an imputation $\mathbf{x} \in V(K)$ such that $x_i > y_i$ for all $i \in K$. It means that for any $K \in \mathcal{I}$ it is

$$V^*(K) = \{\mathbf{y} \in R^I: \sum_{i \in K} y_i \geqq v(K)\} ,$$

and the statement is proved.

Corollary. The previous theorem implies immediately that the core of a coalition-game with side-payments is identical with the class of exactly all strongly stable imputations in the sense of Definition 4.

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