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A Mathematical Correction Problem

JERZY ZABCZYK

A special problem of stochastic control is considered. An approximation of the solution of the problem is given.

In this paper we shall consider the following problem. Suppose that we observe the Markov process $\{\Omega, F, P, x_t\}$

$$x_t = x - \frac{x}{|x|} t + w_t, \quad t \geqq 0, \tag{I}$$

where $x \in R^3$, $x \neq 0$ and $\{\Omega, F, P, w_t\}$ is a Brownian motion. At some stopping time T we make the correction, namely for $t > T$ we will observe a new process

$$\tilde{x}_t = y - \frac{y}{|y|}(t - T) + w_{t-T}, \text{ where } y = x_T. \tag{II}$$

The question arising in this situation is the following: what should be the stopping time T in order to maximize the probability of hitting of the ball K_α,

$$K_\alpha = \{z \in R^3; |z| \leqq \alpha\} \quad \text{where} \quad 0 < \alpha < |x|$$

i.e. in order to maximize

$$\mathrm{P}\{\text{there exists } t \geqq 0 \text{ that } |\bar{x}_t| \leqq \alpha\}$$

where

$$\bar{x}_t = \begin{cases} x_t & \text{for} \quad t \leqq T, \\ \tilde{x}_t & \text{for} \quad t > T. \end{cases}$$

Roughly speaking, equations (I) and (II) describe a situation when the deterministic trajectory in a physical system is perturbed by Gaussian white noise.

We shall show that we obtain a good approximation of the solution of the problem when we define

$$T = \inf\{t \geqq 0;\ x_t^1 \geqq 0 \text{ or } |x_t| \leqq \alpha\}$$

where $x_t = (x_t^1, x_t^2, x_t^3)$ and the starting point $x = (-r, 0, 0)$, $r > 0$.

We do not know how to calculate the exact solution of the above problem.

In the sequel we shall use the following notations:

$p(r, \alpha)$ denotes the probability of hitting of the ball K_α starting from $x = (-r, 0, 0)$ when $T = +\infty$ (without correction),

$p^{\mathrm{m}}(r, \alpha)$ denotes the maximal probability of hitting of the ball K_α (with optimal correction),

$p^{\mathrm{a}}(r, \alpha)$ the probability of hitting of the ball K_α starting from $x = (-r, 0, 0)$ when

$$T = \inf\{t \geqq 0;\ x_t^1 \geqq 0 \text{ or } |x_t| \leqq \alpha\}.$$

We shall prove

Theorem. *The following inequalities hold*

(1) $$\frac{\alpha}{r} \leqq p(r, \alpha) \leqq \frac{\alpha}{r}\, \mathrm{e}^{2\alpha},$$

(2) $$p^{\mathrm{m}}(r, \alpha) \leqq \alpha \sqrt{\frac{\pi}{2r}} \left(\mathrm{e}^{2\alpha} + \mathrm{e}^{2\alpha} \sqrt{\frac{2}{\pi r}} \right),$$

$$p^{\mathrm{a}}(r, \alpha) \geqq \alpha \sqrt{\frac{\pi}{2r}} \left[(1 + r^{-2/3})^{-3/2} \frac{2}{\sqrt{2\pi}} \int_{\alpha/r}^{3\sqrt{r}} \mathrm{e}^{-s^2/2}\, \mathrm{d}s \right]$$

and therefore

(3) $$1 \leqq \varliminf_{r \to +\infty} \frac{p(r, \alpha)}{\alpha \dfrac{1}{r}} \leqq \varlimsup_{r \to +\infty} \frac{p(r, \alpha)}{\alpha \dfrac{1}{r}} \leqq \mathrm{e}^{2\alpha},$$

$$1 \leqq \varliminf_{r \to +\infty} \frac{p^{\mathrm{a}}(r, \alpha)}{\alpha \sqrt{\dfrac{\pi}{2r}}} \leqq \varlimsup_{r \to +\infty} \frac{p^{\mathrm{m}}(r, \alpha)}{\alpha \sqrt{\dfrac{\pi}{2r}}} \leqq \mathrm{e}^{2\alpha}.$$

Proof. The potential kernel (see [1], p. 69) of the process $\{\Omega, F, P, x_t\}$

$$x_t = (-r, 0, 0) + t(1, 0, 0) + w_t, \quad t \geqq 0$$

takes the form

$$G(x, y) = \frac{1}{2\pi} \cdot \frac{1}{|x - y|}\, \mathrm{e}^{y_1 - x_1}\, \mathrm{e}^{-|x-y|}$$

where

$$x = (x_1, x_2, x_3), \quad y = (y_1, y_2, y_3).$$

But

$$2\pi\alpha\, G(x, 0) \leqq 1 \leqq 2\pi\alpha e^{2\alpha}\, G(x, 0), \quad \text{for} \quad |x| = \alpha$$

and

$$G(x, 0) = \mathrm{E}[G(x_{T'}, 0)], \quad \text{for} \quad |x| > \alpha$$

where

$$T' = \inf\{t \geqq 0 : |x_t| = \alpha\}.$$

This and the potential-theoretical characterization of the hitting probability (see [1], p. 141) give

$$2\pi\alpha\, G((-r, 0, 0), 0) \leqq p(r, \alpha) \leqq G((-r, 0, 0), 0)\, 2\pi\alpha\, e^{2\alpha}$$

and therefore (1) is true.

To obtain a lower bound for $p^a(r, \alpha)$ let us denote

$$S = \inf\{t \geqq 0;\, x_t^1 \geqq 0\}.$$

Then (see [3])

$$\mathrm{P}(S \in \mathrm{d}t) = \frac{r}{\sqrt{(2\pi t^3)}}\, e^{-(r-t)^2/2t}\, \mathrm{d}t, \quad t > 0.$$

Hence

$$\mathrm{P}(x_S \in \mathrm{d}y) = g_r(y)\, \mathrm{d}y, \quad y = (y_1, y_2)$$

where

$$g_r(y) = \int_0^{+\infty} \frac{1}{[\sqrt{(2\pi t)}]^2}\, e^{-|y|^2/2t} \cdot \frac{r}{\sqrt{(2\pi t^3)}}\, e^{-(r-t)^2/2t}\, \mathrm{d}t =$$

$$= \left[\frac{r}{2\pi}\, e^{-[\sqrt{(r^2+|y|^2)}-r]}\right] \cdot \frac{1}{\sqrt{(r^2 + |y|^2)^3}} \cdot [1 + \sqrt{(r^2 + |y|^2)}].$$

Let us assume that

$$T = \inf\{t \geqq 0;\, x_t^1 \geqq 0 \text{ or } |x_t| \leqq \alpha\}$$

and

$$h(x) = \begin{cases} 1 & \text{for} \quad |x| \leqq \alpha, \\ \dfrac{\alpha}{|x|} & \text{for} \quad |x| > \alpha. \end{cases}$$

Since $h(x_T) \geqq h(x_S)$ we have

$$p^a(r, \alpha) \geqq \mathrm{E}(h(x_T)) \geqq \mathrm{E}(h(x_S)).$$

But

$$\mathrm{E}(h(x_S)) = \alpha \int_{|y|>\alpha} g_r(y) \frac{1}{|y|}\, \mathrm{d}y + \int_{|y|\leqq\alpha} g_r(y)\, \mathrm{d}y \geqq$$

$$\geqq \alpha \int_{|y|>\alpha} g_r(y) \frac{1}{|y|}\, \mathrm{d}y\,.$$

Using the polar coordinates we obtain

$$\mathrm{E}(h(x_S)) \geqq \alpha \int_{\alpha}^{+\infty} \frac{r}{\sqrt{(r^2+s^2)^3}} \left[1 + \sqrt{(r^2+s^2)}\right] \mathrm{e}^{-s^2/[s+\sqrt{(s^2+r^2)}]}\, \mathrm{d}s \geqq$$

$$\geqq \alpha \int_{\alpha}^{+\infty} \frac{r\sqrt{(2\pi r)}}{\sqrt{(r^2+s^2)^3}} \left[1 + \sqrt{(r^2+s^2)}\right] \frac{1}{\sqrt{(2\pi r)}}\, \mathrm{e}^{-s^2/2r}\, \mathrm{d}s\,.$$

Since

$$\frac{r\sqrt{(2\pi r)}}{\sqrt{(r^2+s^2)^3}} \left[1 + \sqrt{(r^2+s^2)}\right] \geqq \frac{r^2\sqrt{(2\pi r)}}{\sqrt{(r^2+s^2)^3}}$$

we have for any $\beta > 0$

$$\mathrm{E}(h(x_S)) \geqq \left[\frac{1}{\sqrt{(2\pi r)}} \int_{\alpha}^{\beta\sqrt{r}} \mathrm{e}^{-s^2/2r}\, \mathrm{d}s\right] \frac{\alpha r^2 \sqrt{(2\pi r)}}{\sqrt{(r^2+\beta^2 r)^3}}\,,$$

$$\text{(III)} \qquad \mathrm{E}(h(x_S)) \geqq \left[\frac{2}{\sqrt{(2\pi)}} \int_{\alpha/\sqrt{r}}^{\beta} \mathrm{e}^{-s^2/2}\, \mathrm{d}s\right] \alpha \sqrt{\frac{\pi}{2r}}\, \frac{r^3}{\sqrt{(r^2+\beta^2 r)^3}}\,.$$

At last, applying (III) for $\beta = \sqrt[3]{r}$ we finally obtain

$$p^{a}(r,\alpha) \geqq \left[\frac{2}{\sqrt{(2\pi)}} \int_{\alpha/\sqrt{r}}^{\sqrt[3]{r}} \mathrm{e}^{-s^2/2}\, \mathrm{d}s\right] \frac{1}{\sqrt{\left(1+\frac{1}{r^{2/3}}\right)^3}} \cdot \alpha \sqrt{\frac{\pi}{2r}}\,.$$

To obtain an upper bound for $p^{m}(r,\alpha)$ remark that

$$p^{m}(r,\alpha) = \sup_{T} \mathrm{E}(p(|x_T|,\alpha))$$

where T is any stopping time connected with the Markov process $\{\Omega, F, P, x_t\}$. The function

$$k(x) = \sup_{T} \mathrm{E}\left[p(|x + T(1,0,0) + w_T|, \alpha)\right], \quad x \in R^3\,,$$

is the least excessive majorant (see [2]) of $p(|x|,\alpha)$, and $k(-r,0,0) = p^{m}(r,\alpha)$. Therefore it is sufficient to find excessive function f for the process $\{\Omega, F, P, x_t\}$ for which

$$f(x) \geqq \alpha e^{2\alpha} \frac{1}{|x|}, \quad x \in R^3 .$$

Since

$$\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) + \frac{\partial}{\partial x_1}\right]\frac{1}{\varrho} = -2\pi\delta_0 - \frac{x_1}{\varrho^3}$$

where $\varrho(x) = |x|$ and δ_0 is the Dirac measure $(\delta_0\{0\} = 1)$, it follows that

$$\alpha e^{2\alpha} \frac{1}{|x|} = 2\pi\alpha\, e^{2\alpha}\, G(x, 0) + \alpha e^{2\alpha} \int_{R^3} G(x, y) \frac{y_1}{|y|^3}\, dy =$$

$$= 2\pi\alpha\, e^{2\alpha}\, G(x, 0) + \alpha e^{2\alpha} \int_{y_1 \geqq 0} G(x, y) \frac{y_1}{|y|^3}\, dy +$$

$$+ \alpha e^{2\alpha} \int_{y_1 < 0} G(x, y) \frac{y_1}{|y|^3}\, dy .$$

Consequently we can assume that

$$f(x) = 2\pi\alpha\, e^{2\alpha}\, G(x, 0) + \alpha e^{2\alpha} \int_{y_1 \geqq 0} G(x, y) \frac{y_1}{|y|^3}\, dy .$$

Thus for $x = (-r, 0, 0)$

$$p^m(r, \alpha) \leqq \alpha e^{2\alpha} \frac{1}{r} + \alpha e^{2\alpha} \frac{1}{2\pi} \int_{y_1 \geqq 0} \frac{1}{|x - y|}\, e^{y_1 - x_1}\, e^{-|x-y|} \frac{y_1}{|y|^3}\, dy .$$

After the change of variables $(|y - (-r, 0, 0)| = R)$ we obtain:

$$\frac{1}{2\pi} \int_{y_1 \geqq 0} \frac{y_1}{|x - y|\, |y|^3} \,.\, e^{y_1 - x_1}\, e^{-|x-y|}\, dy =$$

$$= \frac{1}{2\pi} \int_r^{+\infty} 2\pi R \left[\int_0^{R-r} \frac{y_1}{R}\, e^{y_1 + r}\, e^{-R} \,.\, (R^2 - r^2 - 2ry_1)^{-3/2}\, dy_1\right] dR =$$

$$= \int_0^{+\infty} \left[\int_0^s y_1 e^{y_1 - s} (s^2 + 2r(s - y_1))^{-3/2}\, dy_1\right] ds =$$

$$= \int_0^{+\infty} \left[\int_0^s e^{-t} (s - t) (s^2 + 2rt)^{-3/2}\, dt\right] ds \leqq$$

$$\leqq \int_0^{+\infty} \left[\int_0^s e^{-t} s(s^2 + 2rt)^{-3/2}\, dt\right] ds = \int_0^{+\infty} e^{-t} (t^2 + 2rt)^{-1/2}\, dt \leqq$$

$$\leqq \frac{1}{\sqrt{2r}} \int_0^{+\infty} e^{-t} \frac{1}{\sqrt{t}} \sqrt{\frac{2r}{t + 2r}}\, dt \leqq \frac{1}{\sqrt{2r}} \int_0^{+\infty} e^{-t} \frac{1}{\sqrt{t}}\, dt = \sqrt{\frac{\pi}{2r}}$$

because

$$\int_0^{+\infty} e^{-t} \frac{1}{\sqrt{t}} \, dt = \sqrt{\pi} .$$

Finally, we conclude that

$$p^m(r, \alpha) \leqq \alpha e^{2\alpha} \frac{1}{r} + e^{2\alpha} \alpha \sqrt{\frac{\pi}{2r}}$$

and this completes the proof.

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VÝTAH

Jeden matematický problém opravy odchýlené trajektorie

JERZY ZABCZYK

V práci se uvažuje následující problém: Deterministická trajektorie v prostoru, směřující konstantní rychlostí do počátku, je zatížena chybou typu Brownova pohybu. Odchýlenou trajektorii je možno jedenkrát opravit tak, že opět směřuje do počátku. Za kritérium vhodnosti korekce je zvolena pravděpodobnost zasažení koule o poloměru α se středem v počátku. Optimální strategie není známa. Autor doporučuje provádět korekci při průchodu trajektorie rovinou, vedenou počátkem a kolmou na (deterministický) směr jejího pohybu. Odvozuje nerovnosti, umožňující porovnat tuto strategii s optimální.

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