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Kybernetika, Vol. 22 (1986), No. 4, 320--328

Persistent URL: <http://dml.cz/dmlcz/124371>

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ERGODIC PROPERTIES OF LOCALLY STATIONARY PROCESSES

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This article deals with asymptotic properties of locally stationary processes. Necessary and sufficient conditions under whose validity a locally stationary process is ergodic are given.

The spectral theory of locally stationary processes is studied in [1]. This paper deals with locally stationary processes from the point of view of their asymptotic behaviour.

Let $x(t)$, $t \in \mathbb{R}_1$, be a locally stationary process with $E\{x(t)\} = 0$ and with covariance function

$$R(s, t) = R_1 \left(\frac{s+t}{2} \right) R_2(s-t),$$

where $R_1 \geq 0$ and R_2 is a stationary covariance function. Let us suppose that $x(t)$ is harmonizable, i.e. $x(t)$ can be expressed in the form of a stochastic integral understood in the quadratic mean, see [2],

$$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} d\xi(\lambda)$$

where $\xi(\lambda)$ is a second order stochastic process having covariance function $\gamma(\lambda, \mu)$ with bounded variation on $\mathbb{R}_1 \times \mathbb{R}_1$.

We shall need the following lemma for the proof of Theorem 1.

Lemma 1. Let $x(t)$, $t \in \mathbb{R}_1$, be a harmonizable process, let $T > 0$. Then

$$\int_{-T}^T x(t) dt = \int_{-\infty}^{+\infty} \left\{ \int_{-T}^T e^{it\lambda} dt \right\} d\xi(\lambda) \quad \text{a.s.}$$

if

$$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} d\xi(\lambda).$$

Proof. The process $x(t)$ is harmonizable, i.e. $x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} d\xi(\lambda)$ where $\int \int |d\gamma(\lambda, \mu)| < \infty$, $\gamma(\lambda, \mu) = E\{\xi(\lambda) \bar{\xi}(\mu)\}$. The stochastic integral $\int_{-\infty}^{+\infty} e^{it\lambda} d\xi(\lambda)$ is understood as a limit in the quadratic mean

$$x(t) = \text{l.i.m.} \int_{\substack{A \rightarrow -\infty \\ B \rightarrow +\infty}}^B e^{it\lambda} d\xi(\lambda).$$

At first, we prove the possibility of order change between \int_{-T}^T and \int_A^B . As $e^{it\lambda}$ is continuous and bounded $\int_{-T}^T E\left\{\left|\int_A^B e^{it\lambda} d\xi(\lambda)\right|^2\right\} dt$ is finite for every pair A, B . By the Fubini theorem then

$$\int_{-T}^T E\left\{\left|\int_A^B e^{it\lambda} d\xi(\lambda)\right|^2\right\} dt = E\left\{\int_{-T}^T \left|\int_A^B e^{it\lambda} d\xi(\lambda)\right|^2 dt\right\}$$

and this implies the existence of $\int_{-T}^T \left\{\int_A^B e^{it\lambda} d\xi(\lambda)\right\} dt$ a.s.. On the other hand, $\int_{-T}^T e^{it\lambda} dt = 2T(\sin T\lambda)/(T\lambda)$ is continuous and bounded on every interval (A, B) and hence the stochastic integral

$$\int_A^B \left\{\int_{-T}^T e^{it\lambda} dt\right\} d\xi(\lambda)$$

exists too.

By the definition of the stochastic integral in the quadratic mean

$$\int_A^B e^{it\lambda} d\xi(\lambda) = \text{l.i.m.} \sum_{\|\mathcal{D}\| \rightarrow 0} \exp(it\lambda_j) \Delta\xi(\lambda_j),$$

where \mathcal{D} is a partition of (A, B) . Now

$$\begin{aligned} & E\left\{\left|\int_{-T}^T \left\{\int_A^B e^{it\lambda} d\xi(\lambda)\right\} dt - \int_{-T}^T \left\{\sum_j \exp(it\lambda_j) \Delta\xi(\lambda_j)\right\} dt\right|^2\right\} \leq \\ & \leq E\left\{\left|\int_{-T}^T \left|\int_A^B e^{it\lambda} d\xi(\lambda) - \sum_j \exp(it\lambda_j) \Delta\xi(\lambda_j)\right| dt\right|^2\right\} = \\ & = \int_{-T}^T E\left\{\left|\int_A^B e^{it\lambda} d\xi(\lambda) - \sum_j \exp(it\lambda_j) \Delta\xi(\lambda_j)\right|^2\right\} dt \rightarrow 0 \quad \text{as } \|\mathcal{D}\| \rightarrow 0 \end{aligned}$$

because

$$E\left\{\left|\int_A^B e^{it\lambda} d\xi(\lambda) - \sum_j \exp(it\lambda_j) \Delta\xi(\lambda_j)\right|^2\right\} \rightarrow 0 \quad \text{as } \|\mathcal{D}\| \rightarrow 0$$

for every $t \in [-T, T]$ and

$$\left\{E\left\{\left|\int_A^B e^{it\lambda} d\xi(\lambda) - \sum_j \exp(it\lambda_j) \Delta\xi(\lambda_j)\right|^2\right\}\right\}_{\mathcal{D}}$$

is bounded in t on $[-T, T]$. But

$$\int_{-T}^T \sum_j \exp(it\lambda_j) \Delta\xi(\lambda_j) dt = \sum_j \left\{\int_{-T}^T \exp(it\lambda_j) dt\right\} \Delta\xi(\lambda_j) \approx$$

$$= \sum_j 2T \frac{\sin(T\lambda_j)}{T\lambda_j} \Delta\xi(\lambda_j) \rightarrow \int_A^B \left\{ \int_{-T}^{+T} e^{it\lambda} dt \right\} d\xi(\lambda) \quad \text{as } \|\mathcal{D}\| \rightarrow 0$$

in the quadratic mean. As the limit in the quadratic mean is defined unambiguously a.s. we proved

$$\int_{-T}^{+T} \left\{ \int_A^B e^{it\lambda} d\xi(\lambda) \right\} dt = \int_A^B \left\{ \int_{-T}^{+T} e^{it\lambda} dt \right\} d\xi(\lambda) \quad \text{a.s.}$$

In a similar way as before we can prove the existence of the integrals

$$\int_{-\infty}^{+\infty} \left\{ \int_{-T}^{+T} e^{it\lambda} dt \right\} d\xi(\lambda), \quad \int_{-T}^{+T} \left\{ \int_{-\infty}^{+\infty} e^{it\lambda} d\xi(\lambda) \right\} dt.$$

As

$$\int_{-\infty}^{+\infty} \left\{ \int_{-T}^{+T} e^{it\lambda} dt \right\} d\xi(\lambda) = \text{l.i.m.}_{\substack{A \rightarrow -\infty \\ B \rightarrow +\infty}} \int_A^B \left\{ \int_{-T}^{+T} e^{it\lambda} dt \right\} d\xi(\lambda)$$

the previous results give immediately

$$\int_{-\infty}^{+\infty} \left\{ \int_{-T}^{+T} e^{it\lambda} dt \right\} d\xi(\lambda) = \int_{-T}^{+T} \left\{ \int_{-\infty}^{+\infty} e^{it\lambda} d\xi(\lambda) \right\} dt \quad \text{a.s.} \quad \square$$

Theorem 1. Let $x(t)$, $t \in \mathbb{R}_1$, be a harmonizable random process, $E\{x(t)\} = 0$, then as $T \rightarrow \infty$

$$\frac{1}{2T} \int_{-T}^{+T} x(t) dt \rightarrow \xi(\langle 0 \rangle) \quad \text{in the quadratic mean}$$

where $\xi(\langle 0 \rangle)$ is the jump at 0 of $\xi(\cdot)$.

Proof. We suppose that $x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} d\xi(\lambda)$ with $E\{d\xi(\lambda) \overline{d\xi(\mu)}\} = d\gamma(\lambda, \mu)$. Then by use of the previous Lemma 1

$$\frac{1}{2T} \int_{-T}^{+T} x(t) dt = \frac{1}{2T} \int_{-T}^{+T} \left[\int_{-\infty}^{+\infty} e^{it\lambda} d\xi(\lambda) \right] dt = \frac{1}{2T} \int_{-\infty}^{+\infty} \frac{e^{iT\lambda} - e^{-iT\lambda}}{i\lambda} d\xi(\lambda).$$

Now

$$E \left\{ \left| \frac{1}{2T} \int_{-\infty}^{+\infty} \frac{e^{iT\lambda} - e^{-iT\lambda}}{i\lambda} d\xi(\lambda) \right|^2 \right\} = \frac{1}{4T^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{(e^{iT\lambda} - e^{-iT\lambda})(e^{-iT\mu} - e^{iT\mu})}{(i\lambda)(-i\mu)} d\gamma(\lambda, \mu)$$

according to Lemma 1 in [1].

Let $\varepsilon > 0$ and let us consider a partition of $\mathbb{R}_1 \times \mathbb{R}_1$ in the following form

$$\begin{aligned} \mathbb{R}_1 \times \mathbb{R}_1 &= [-\varepsilon, +\varepsilon] \times [-\varepsilon, +\varepsilon] \cup [-\varepsilon, +\varepsilon] \times \mathbb{R}_1 \cup \mathbb{R}_1 \times \\ &\times [-\varepsilon, +\varepsilon] \cup \{(\lambda, \mu) \in \mathbb{R}_1 \times \mathbb{R}_1 : |\lambda| > \varepsilon, |\mu| > \varepsilon\}. \end{aligned}$$

Using this partition we obtain

$$\left| \frac{1}{4T^2} \int_{|\lambda| > \varepsilon} \int_{|\mu| > \varepsilon} \frac{(e^{iT\lambda} - e^{-iT\lambda})(e^{iT\mu} - e^{-iT\mu})}{(+i\lambda)(-i\mu)} d\gamma(\lambda, \mu) \right| \leq$$

$$\leq \frac{1}{4T^2} \int_{|\lambda| > \varepsilon} \int_{|\mu| > \varepsilon} \frac{4}{\varepsilon^2} |\mathrm{d}\gamma(\lambda, \mu)| \rightarrow 0 \quad \text{as } T \rightarrow \infty$$

because $\gamma(\lambda, \mu)$ is of bounded variation on $\mathbb{R}_1 \times \mathbb{R}_1$. Further, as $\varepsilon \downarrow 0$

$$(1) \quad \frac{1}{4} \int_{-\varepsilon}^{+\varepsilon} \int_{-\varepsilon}^{+\varepsilon} \frac{(e^{iT\lambda} - e^{-iT\lambda})(e^{-iT\mu} - e^{iT\mu})}{(+iT\lambda)(-iT\mu)} \mathrm{d}\gamma(\lambda, \mu) \rightarrow \mathrm{d}\gamma(0, 0) = \\ = \lim_{\varepsilon \downarrow 0} [\gamma(\varepsilon, \varepsilon) - \gamma(\varepsilon, -\varepsilon) - \gamma(-\varepsilon, \varepsilon) + \gamma(-\varepsilon, -\varepsilon)] \quad \text{uniformly in } T.$$

Surely, we can write (let us put $f(T, \lambda, \mu) = \frac{(e^{iT\lambda} - e^{-iT\lambda})(e^{-iT\mu} - e^{iT\mu})}{(-4iT\lambda iT\mu)}$ for simplicity)

$$\int_{-\varepsilon}^{+\varepsilon} \int_{-\varepsilon}^{+\varepsilon} f(T, \lambda, \mu) \mathrm{d}\gamma(\lambda, \mu) - \mathrm{d}\gamma(0, 0) = \\ = \int_{-1}^{+1} \int_{-1}^{+1} (\psi_\varepsilon(\lambda, \mu) f(T, \lambda, \mu) - \delta_{(0,0)}(\lambda, \mu)) \mathrm{d}\gamma(\lambda, \mu)$$

if $0 < \varepsilon < 1$, $\psi_\varepsilon(\lambda, \mu) = 1$ for $(\lambda, \mu) \in [-\varepsilon, +\varepsilon] \times [-\varepsilon, +\varepsilon]$, $\psi_\varepsilon(\lambda, \mu) = 0$ otherwise, $\delta_{(0,0)}(0, 0) = 1$ and $\delta_{(0,0)}(\lambda, \mu) = 0$ otherwise. As $\{\psi_\varepsilon(\lambda, \mu) f(T, \lambda, \mu) - \delta_{(0,0)}(\lambda, \mu)\}_{0 < \varepsilon < 1}$ is a nonincreasing net tending to 0, hence tending uniformly in $[-1, +1] \times [-1, +1]$, if $\varepsilon \downarrow 0$, then (1) holds. In a similar way, $\psi_\varepsilon(\lambda, \mu) - \delta_{(0,0)}(\lambda, \mu) \rightarrow 0$ uniformly as $\varepsilon \downarrow 0$ and the following inequality

$$|\psi_\varepsilon(\lambda, \mu) f(T, \lambda, \mu) - \delta_{(0,0)}(\lambda, \mu)| \leq \psi_\varepsilon(\lambda, \mu) - \delta_{(0,0)}(\lambda, \mu)$$

holds for every (λ, μ) because $|f(T, \lambda, \mu)| \leq 1$ and $f(T, 0, 0) = 1$. These facts prove the uniform convergence of (1) with respect to T .

Similarly, we can estimate

$$\left| \frac{1}{4T^2} \int_{-\varepsilon}^{+\varepsilon} \int_{|\mu| > \varepsilon} \frac{(e^{iT\lambda} - e^{-iT\lambda})(e^{-iT\mu} - e^{iT\mu})}{(i\lambda)(-i\mu)} \mathrm{d}\gamma(\lambda, \mu) \right| \leq \\ \leq \frac{1}{4T^2} \int_{-\varepsilon}^{+\varepsilon} \int_{|\mu| > \varepsilon} |\mathrm{d}\gamma(\lambda, \mu)| \rightarrow 0 \quad \text{as } T \rightarrow \infty \text{ for every } \varepsilon > 0.$$

Analogously,

$$\frac{1}{4T^2} \int_{|\lambda| > \varepsilon} \int_{-\varepsilon}^{+\varepsilon} \frac{(e^{iT\lambda} - e^{-iT\lambda})(e^{-iT\mu} - e^{iT\mu})}{(i\lambda)(-i\mu)} \mathrm{d}\gamma(\lambda, \mu) \rightarrow 0 \quad \text{as } T \rightarrow \infty$$

for every $\varepsilon > 0$. Summarizing these results we obtain that

$$\mathbb{E} \left| \frac{1}{2T} \int_{-T}^{+T} x(t) \mathrm{d}t - \zeta(\langle 0 \rangle) \right|^2 \rightarrow 0, \quad \text{as } T \rightarrow \infty$$

where $\zeta(\langle 0 \rangle)$ is the jump of the random process $\zeta(\cdot)$ at 0. \square

If $x(t)$ has a spectral density function, i.e. $\mathrm{d}\gamma(\lambda, \mu) = f(\lambda, \mu) \mathrm{d}\lambda \mathrm{d}\mu$, then $\zeta(\langle 0 \rangle) =$

= 0 and $x(t)$ is ergodic because

$$\frac{1}{2T} \int_{-T}^{+T} x(t) dt \rightarrow E\{x(0)\} = 0 \quad \text{as } T \rightarrow \infty.$$

In case the existence of spectral density is not supposed, we must calculate

$$\begin{aligned} E\{\xi(+\varepsilon) - \xi(-\varepsilon)\}^2 &= \\ &= \gamma(+\varepsilon, +\varepsilon) - \gamma(-\varepsilon, +\varepsilon) - \gamma(+\varepsilon, -\varepsilon) + \gamma(-\varepsilon, -\varepsilon) = \int_{-\varepsilon}^{+\varepsilon} \int_{-\varepsilon}^{+\varepsilon} dd\gamma(\lambda, \mu). \end{aligned}$$

At this moment we use the results of [1]; under assumption of local stationarity the following relation

$$\int_{-\varepsilon}^{+\varepsilon} \int_{-\varepsilon}^{+\varepsilon} dd\gamma(\lambda, \mu) = \iint_{A(\varepsilon)} dF_1(u) dF_2(v)$$

holds, where $A(\varepsilon) = \{(u, v) \in \mathbb{R}_1 \times \mathbb{R}_1; -\varepsilon < u + (v/2) < +\varepsilon, -\varepsilon < u - (v/2) < +\varepsilon\}$.
Now

$$\begin{aligned} \iint_{A(\varepsilon)} dF_1(u) dF_2(v) &= \int_0^\varepsilon \int_{-2\varepsilon+2u}^{2\varepsilon-2u} dF_1(u) dF_2(v) + \int_{-\varepsilon}^0 \int_{-2\varepsilon-2u}^{2u+2\varepsilon} dF_1(u) dF_2(v) = \\ &= \int_0^\varepsilon [F_2(2\varepsilon - 2u) - F_2(-(2\varepsilon - 2u))] dF_1(u) + \\ &+ \int_{-\varepsilon}^0 [F_2(2u + 2\varepsilon) - F_2(-(2u + 2\varepsilon))] dF_1(u). \end{aligned}$$

This equality yields immediately the following conclusion.

Theorem 2. A locally stationary harmonizable random process is ergodic if and only if at least one of its spectral measures is continuous at 0.

A very simple sufficient condition ensuring ergodicity of a second order random process is that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R(s, t) ds dt < \infty,$$

because then

$$E \left| \frac{1}{2T} \int_{-T}^{+T} x(t) dt \right|^2 = \frac{1}{4T^2} \int_{-T}^{+T} \int_{-T}^{+T} R(s, t) ds dt \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

In the case of locally stationary processes this condition can be weakened. Thanks to the decomposition $R(s, t) = R_1((s+t)/2) R_2(s-t)$ we can write

$$\begin{aligned} \int_{-T}^{+T} \int_{-T}^{+T} R(s, t) ds dt &= \int_{-T}^{+T} \int_{-T}^{+T} R_1\left(\frac{s+t}{2}\right) R_2(s-t) ds dt = \\ &= \int_0^T \int_{2u-2T}^{2T-2u} R_1(u) R_2(v) du dv + \int_{-T}^0 \int_{-2T-2u}^{2T+2u} R_1(u) R_2(v) du dv. \end{aligned}$$

As R_2 is a stationary covariance function, it is $|R_2(v)| \leq R_2(0)$, then

$$\begin{aligned} \frac{1}{4T^2} \int_{-T}^{+T} \int_{-T}^{+T} R(s, t) \, ds \, dt &\leq \frac{R_2(0)}{T^2} \left[\int_0^T (T-u) R_1(u) \, du + \int_{-T}^0 (T+u) R_1(u) \, du \right] = \\ &= \frac{R_2(0)}{T} \left[\int_{-T}^{+T} \left(1 - \frac{|u|}{T}\right) R_1(u) \, du \right]. \end{aligned}$$

This inequality gives, of course, that the integrability of R_1 is a sufficient condition for ergodicity of the locally stationary process $x(t)$.

Now, let us study a possible estimate of covariance function of a locally stationary process. As we need the fourth moments, let us consider a Gaussian case. The basic property of covariance function of a locally stationary process gives that

$$E\{X_{u+v/2} \bar{X}_{u-v/2}\} = R_1(u) R_2(v) \quad \text{for every pair } u, v.$$

It is reasonable to study the asymptotic behaviour of integrals

$$\int_{-T}^{+T} X_{u+v/2} \bar{X}_{u-v/2} \, du, \quad \int_{-T}^{+T} X_{u+v/2} \bar{X}_{u-v/2} \, dv,$$

when these integrals are considered in the quadratic mean sense. It holds according to Lemma 1 in [1]

$$E \left\{ \int_{-T}^{+T} X_{u+v/2} \bar{X}_{u-v/2} \, du \right\} = R_2(v) \int_{-T}^{+T} R_1(u) \, du$$

and similarly

$$E \left\{ \int_{-T}^{+T} X_{u+v/2} \bar{X}_{u-v/2} \, dv \right\} = R_1(u) \int_{-T}^{+T} R_2(v) \, dv.$$

Let us prove that the sequence

$$\left\{ \int_{-T}^{+T} X_{u+v/2} \bar{X}_{u-v/2} \, dv \right\}, \quad T > 0,$$

is fundamental in the quadratic mean. If $T \geq S$, then

$$\begin{aligned} E \left| \int_{-T}^{+T} X_{u+v/2} \bar{X}_{u-v/2} \, dv - \int_{-S}^{+S} X_{u+v/2} \bar{X}_{u-v/2} \, dv \right|^2 &= E \left| \int_{-T}^{-S} + \int_S^{+T} \right|^2 = \\ &= 4E \left| \operatorname{Re} \int_S^T X_{u+v/2} \bar{X}_{u-v/2} \, dv \right|^2 \end{aligned}$$

because $S \leq v \leq T$ if and only if $-T \leq -v \leq -S$ and $X_{u+v/2} \bar{X}_{u-v/2} = \bar{X}_{u-v/2} X_{u+v/2}$.

For the sake of simplicity we shall consider, further, a real case. Then

$$\begin{aligned} E \left| \int_S^T X_{u+v/2} X_{u-v/2} \, dv \right|^2 &= \int_S^T \int_S^T E \{ X_{u+v/2} X_{u-v/2} X_{u-z/2} X_{u+z/2} \} \, dv \, dz = \\ &= \int_S^T \int_S^T R_1^2(u) R_2(v) R_2(z) \, dv \, dz + \end{aligned}$$

$$\begin{aligned}
& + \int_S^T \int_S^T R_1\left(u + \frac{v+z}{4}\right) R_2\left(\frac{v-z}{2}\right) R_1\left(u - \frac{v+z}{4}\right) R_2\left(\frac{z-v}{2}\right) dv dz + \\
& + \int_S^T \int_S^T R_1\left(u + \frac{v-z}{4}\right) R_2\left(\frac{v+z}{2}\right) R_1\left(u + \frac{z-v}{4}\right) R_2\left(-\frac{v+z}{2}\right) dv dz = \\
= & R_1^2(u) \left(\int_S^T R_2(z) dz\right)^2 + 4 \iint_{\substack{[S \leq 2x+y \leq T] \\ [S \leq 2x-y \leq T]}} R_1(u+x) R_2(y) R_1(u-x) R_2(-y) dx dy + \\
& + 4 \iint_{\substack{[S \leq 2x+y \leq T] \\ [S \leq y-2x \leq T]}} R_1(u+x) R_2(y) R_1(u-x) R_2(-y) dx dy = \\
= & R_1^2(u) \left(\int_S^T R_2(z) dz\right)^2 + \int_{S/2}^{(S+T)/4} \int_{S-2x}^{2x-S} R_1(u+x) R_2^2(y) R_1(u-x) dx dy + \\
& + 4 \int_{(S+T)/4}^{T/2} \int_{2x-T}^{T-2x} R_1(u+x) R_2^2(y) R_1(u-x) dx dy + \\
& + 4 \int_{S/2}^{(T+S)/4} \int_{(S-y)/2}^{(y-S)/2} R_1(u+x) R_2^2(y) R_1(u-x) dx dy + \\
& + 4 \int_{(S+T)/4}^{T/2} \int_{(y-T)/2}^{(T-y)/2} R_1(u+x) R_1(u-x) R_2^2(y) dx dy.
\end{aligned}$$

Now, using $R_2(y) = R_2(-y)$ and $R_1(u + (-x)) R_1(u - (-x)) = R_1(u + x) \cdot R_1(u - x)$ one can write

$$\begin{aligned}
& \int_{(T+S)/4}^{T/2} \int_{S-2x}^{2x-S} R_1(u+x) R_1(u-x) R_2^2(y) dx dy = \\
= & 2 \int_{(T+S)/4}^{T/2} R_1(u+x) R_1(u-x) \int_0^{2x-S} R_2^2(y) dy dx.
\end{aligned}$$

(Similarly for the other integrals.) These results yield the following Theorem 3.

Theorem 3. Let $\int_{-\infty}^{+\infty} R_2(y) dy < \infty$, $\int_{-\infty}^{+\infty} R_1^2(x) dx < \infty$, then there exists the limit in the quadratic mean

$$\text{l.i.m.}_{T \rightarrow \infty} \int_{-T}^{+T} X_{u+v/2} X_{u-v/2} dv = \int_{-\infty}^{+\infty} X_{u+v/2} X_{u-v/2} dv$$

and

$$\mathbb{E} \left\{ \int_{-\infty}^{+\infty} X_{u+v/2} X_{u-v/2} dv \right\} = R_1(u) \int_{-\infty}^{+\infty} R_2(v) dv$$

for every $u \in \mathbb{R}_1$.

Proof. We have proved that

$$\mathbb{E} \left| \int_{-T}^{+T} X_{u+v/2} X_{u-v/2} dv - \int_{-S}^{+S} X_{u+v/2} X_{u-v/2} dv \right|^2 = 4R_1^2(u) \left(\int_S^T R_2(v) dv \right)^2 +$$

$$\begin{aligned}
& + 32 \int_{(T+S)/4}^{T/2} R_1(u+v) R_2(u-v) \int_0^{2x-T} R_2^2(y) dy dx + \\
& + 32 \int_{S/2}^{S+T} R_1(u+x) R_1(u-x) \int_0^{2x-S} R_2^2(y) dy + \\
& + 32 \int_{S/2}^{(T+S)/4} R_2^2(y) \int_0^{(y-S)/2} R_1(u+x) R_1(u-x) dx dy + \\
& + 32 \int_{(T+S)/4}^{T/2} \int_0^{(T-y)/2} R_1(u+x) R_1(u-x) dx dy \leq \\
& \leq R_1^2(u) \left(\int_S^T R_2(y) dy \right)^2 + 32 \int_0^\infty R_2^2(y) dy \int_{S/2}^{T/2} R_1(u+x) R_1(u-x) dx + \\
& + 32 \int_0^\infty R_1(u+x) R_1(u-x) dx \int_{S/2}^{T/2} R_2^2(y) dy .
\end{aligned}$$

If $S \nearrow \infty$, $T \nearrow \infty$, then $E \left| \int_S^T - \int_{S'}^{T'} \right|^2 \rightarrow 0$, so the completeness of the L_2 -space ensures the existence of a limit in the quadratic mean

$$\int_{-\infty}^{+\infty} X_{u+v/2} X_{u-v/2} dv .$$

The equality

$$E \left\{ \int_{-\infty}^{+\infty} X_{u+v/2} X_{u-v/2} dv \right\} = R_1(u) \int_{-\infty}^{+\infty} R_2(v) dv$$

holds because convergence in the quadratic mean implies convergence in the mean. \square

The quite analogous conclusion we can make in the case of

$$\text{l.i.m.}_{T \rightarrow \infty} \int_{-T}^{+T} X_{u+v/2} X_{u-v/2} du = \int_{-\infty}^{+\infty} X_{u+v/2} X_{u-v/2} du$$

with

$$E \left\{ \int_{-\infty}^{+\infty} X_{u+v/2} X_{u-v/2} du \right\} = R_2(v) \int_{-\infty}^{+\infty} R_1(u) du .$$

In other words, Theorem 3 states that there exists an unbiased estimator of $R_1(u)$, resp. $R_2(v)$, with finite variance because

$$E \left\{ \frac{\int_{-\infty}^{+\infty} X_{u+v/2} X_{u-v/2} dv}{\int_{-\infty}^{+\infty} R_2(v) dv} \right\} = R_1(u),$$

resp.

$$E \left\{ \frac{\int_{-\infty}^{+\infty} X_{u+v/2} X_{u-v/2} du}{\int_{-\infty}^{+\infty} R_1(u) du} \right\} = R_2(v).$$

(Received May 31, 1985.)

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