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Stability of Coalition Structures and Imputations in Coalition-Games

MILAN MAREŠ

In the presented paper a model of bargaining in coalition-games is suggested. The main goal of the work is to present a simple and easy to survey method, which could be algorithmized in a sufficient degree.

INTRODUCTION

At the present time, there exist a few models of the bargaining in coalition-games with side-payments. Most of them are devoted to the problem of final distribution of the common pay-off. Such models usually investigate properties of the core (e.g. [4], [5], and many others), or introduce certain vectors of final profits, having some equilibrium properties (e.g. [2]).

More interesting from the point of view of this paper, are such models, which describe not only the final pay-offs, given in some units of transferable utility, but also the final "state of the bargaining", i.e. the final coalitions, together with the inner distributions of their profit. The most famous of them is the Aumann's and Maschler's paper [1], which gives a very illustrative model of bargaining. Author of the presented paper suggested a modification of this method in [3]. The Aumann's and Maschler's method can cause certain difficulties, namely when we want to find out the existence of some contraobjections to any objection against given configuration.

The main idea if the presented paper is that the procedure of finding of the resulting configuration may be simplified, if we find firstly the coalition structures, which may appear in the result of bargaining, and then we construct the imputations, which correspond to this coalition structures.

The aim of this paper is to describe such a procedure, which would be simple enough, which could be algorithmized, and which would not contradict the common intuitive idea about the rational result of bargaining in a conflict situation, or the classical concepts of the core.

The suggested method is described in sections 3, 4, 5, 8 and 9 of this paper. The other parts are devoted to introductory notions (parts 1 and 2), to some special situations, which may occur in the coalition-game and which enable us to simplify the procedure (sections 6, 7 and 10), and to certain correspondence between the described method and a classical concept of the game theory, namely the core (section 11).

1. COALITION-GAME

First of all we specify two notions, known from the elementary set theory.

If A is some non-empty and finite set then the family $\mathcal{M}$ of non-empty subsets of A is called a *partition* of A, iff

$$\bigcup_{M \in \mathcal{M}} M = A,$$

and for any $K, L \in \mathcal{M}$ is $K \cap L = \emptyset$.

If $\mathcal{M}$ and $\mathcal{N}$ are two partitions of the same set A, then $\mathcal{M}$ is called a *subpartition* of $\mathcal{N}$ iff for any $M \in \mathcal{M}$ there exists $N \in \mathcal{N}$ such that $M \subset N$.

The coalition-game is a conflict situation, in which its participants, players, are allowed to collaborate in admissible coalitions, and for any coalition its guaranteed final common pay-off is known. This pay-off can be arbitrarily distributed among members of the coalition.

Let I be a finite and non-empty set.

Let $\mathbf{K}$ be a non-empty class of the partitions of the set I, such that for any class $\mathbf{M} \subset \mathbf{K}$ and for any family $\mathcal{K}$ of subsets of I, where

$$\mathcal{K} \subset \bigcup_{\mathcal{M} \in \mathbf{M}} \mathcal{M}, \text{ and } \mathcal{K} \text{ is a partition of } I,$$

also $\mathcal{K} \in \mathbf{K}$.

Let, finally, v be a superadditive set function, defined on the family of subsets of I

$$\bigcup_{\mathcal{K} \in \mathbf{K}} \mathcal{K}$$

i.e.

$$(1.1) \qquad v(K) + v(L) \leqq v(K \cup L), \quad K, L, K \cup L \in \bigcup_{\mathcal{K} \in \mathbf{K}} \mathcal{K}, \quad K \cap L = \emptyset.$$

Then the triple

$$\Gamma = (I, \mathbf{K}, v)$$

is called a *coalition-game*.

Elements of the set I are called *players*, partitions of I, belonging to the class $\mathbf{K}$ are *coalition structures*, the subsets of I, belonging to the family

$$\bigcup_{\mathscr{K} \in \mathbf{K}} \mathscr{K}$$

are called *coalitions*, and the set function v is the *characteristic function* of the game Γ.

2. IMPUTATIONS AND CONFIGURATIONS

Let $\Gamma = (I, \mathbf{K}, v)$ be a coalition-game. It was already said in the previous section that the common profit $v(K)$ of any coalition K can be arbitrarily distributed among its members. The concept of imputation represents such possible distribution of pay-off.

Let n be the number of players in the set I. We define for any coalition structure $\mathscr{K} \in \mathbf{K}$ the subset $X(\mathscr{K})$ of n-dimensional Euclidian space,

$$X(\mathscr{K}) = \{\mathbf{x} = (x_i)_{i \in I} : \sum_{i \in K} x_i = v(K) \quad \text{for any} \quad K \in \mathscr{K}\} . \tag{2.1}$$

Any n-dimensional vector $\mathbf{x} \in X(\mathscr{K})$ is called an *imputation in* $\mathscr{K}$. It is obvious that $X(\mathscr{K})$ is $(n - k)$-dimensional subspace of the n-dimensional Euclidian space, where n is number of elements in I, and k is number of coalitions in $\mathscr{K}$.

If $\mathscr{K} \in \mathbf{K}$ and $\mathbf{x} \in X(\mathscr{K})$ then the pair $(\mathscr{K}, \mathbf{x})$ is called a *configuration* in the given coalition-game Γ.

3. RATIONALITY AND EFFECTIVITY

The concept of rationality is a basic one in the coalition-games theory. It is well--known from the literature, e.g. from [1], and it must not be omitted, whenever the configurations in colition-game are investigated. For our purposes, when the coalition structures and the imputations are considered separately, the rationality may be restricted to the effectivity from below, defined in this section. The effectivity from above, defined also in this section, enables us, in the final steps of our bargaining investigation, to eliminate some results of bargaining, in which the cooperation among players is not as strong as possible.

Definition 1. The configuration $(\mathscr{K}, \mathbf{x})$ is said to be *rational*, iff for all coalition structures $\mathscr{J} \in \mathbf{K}$ such that $\mathscr{J}$ is a subpartition of $\mathscr{K}$, the inequality

$$v(J) \leqq \sum_{i \in J} x_i \tag{3.1}$$

is true for all $J \in \mathscr{J}$.

Definition 2. Let $\mathscr{K} \in \mathbf{K}$ be a coalition structure. Then we say that

$\mathscr{K}$ is *effective from below*, iff there exists an imputation $\mathbf{x} \in X(\mathscr{K})$ such that $(\mathscr{K}, \mathbf{x})$ is rational;

$\mathscr{K}$ is *effective from above*, iff for any $\mathscr{L}$ effective from below and such that $\mathscr{K}$ is a subpartition of $\mathscr{L}$, the equation

$$\sum_{K \in \mathscr{K}} v(K) = \sum_{L \in \mathscr{L}} v(L) \tag{3.2}$$

is true;

$\mathscr{K}$ is *effective*, iff it is effective from below and effective from above.

The classes of configurations being effective from below, effective from above, and effective, will be benoted by symbols

$$\mathbf{K}_{\text{ef}}, \mathbf{K}^{\text{ef}} \quad \text{and} \quad \mathbf{K}_{\text{ef}}^{\text{ef}},$$

respectively.

Remark 3.1. It is obvious that $\mathbf{K}_{\text{ef}}^{\text{ef}} = \mathbf{K}_{\text{ef}} \cap \mathbf{K}^{\text{ef}}$.

Remark 3.2. Let $\mathscr{K} \in \mathbf{K}$. It is an immediate consequence of the previous Definitions that $\mathscr{K} \in \mathbf{K}_{\text{ef}}$ iff there exists $\mathbf{x} \in X(\mathscr{K})$, $\mathbf{x} = (x_i)_{i \in I}$, such that for any $\mathscr{J} \in \mathbf{K}$, where $\mathscr{J}$ is a subpartition of $\mathscr{K}$, and for all $J \in \mathscr{J}$ the following inequality holds

$$\sum_{i \in J} x_i \geqq v(J).$$

Lemma 3.1. There exists at least one effective coalition structure in any coalition-game; i.e. $\mathbf{K}_{\text{ef}}^{\text{ef}} \neq \emptyset$.

Proof. The class $\mathbf{K}_{\text{ef}}$ is always non-empty, because it contains at least the coalition structures from $\mathbf{K}$, for which no their subpartition in $\mathbf{K}$ exists. On the other hand, the finite, non-empty class $\mathbf{K}_{\text{ef}}$ contains at least one coalition structure $\mathscr{L}$, such that for any $\mathscr{K} \in \mathbf{K}_{\text{ef}}$, $\mathscr{L}$ is not a subpartition of $\mathscr{K}$. Consequently,

$$\mathscr{L} \in \mathbf{K}_{\text{ef}} \cap \mathbf{K}^{\text{ef}} = \mathbf{K}_{\text{ef}}^{\text{ef}}.$$

The following statements of this section describe some relations between a coalition structure and its subpartition, namely, if they both are effective. The obtained results will be applied in the 6th section, and, partially, also in the 10th section of this work, where they enable us to simplify the construction of the bargaining solution in some coalition-games.

Lemma 3.2. Let $\mathscr{K} \in \mathbf{K}$, $\mathscr{L} \in \mathbf{K}_{\text{ef}}$, $\mathscr{K}$ is a subpartition of $\mathscr{L}$, and let

$$\sum_{K \in \mathscr{K}} v(K) = \sum_{L \in \mathscr{L}} v(L). \tag{3.3}$$

Then $\mathscr{K} \in \mathbf{K}_{\text{ef}}$ and $X(\mathscr{K}) \subset X(\mathscr{L})$.

Proof. Because $\mathscr{K}$ is a subpartition of $\mathscr{L}$, the relations (3.3) and (1.1) imply that

$$v(L) = \sum_{K \in \mathscr{K}, K \subset L} v(K), \quad \text{for all} \quad L \in \mathscr{L}.$$

It means that $X(\mathscr{K}) \subset X(\mathscr{L})$. Let $\mathscr{K} \notin \mathbf{K}_{\mathrm{ef}}$. Then, by Remark 3.2, for any $\mathbf{x} \in X(\mathscr{K})$ there exists $\mathscr{J} \in \mathbf{K}$ and $J \in \mathscr{J}$ such that $\mathscr{J}$ is a subpartition of $\mathscr{K}$ (and of $\mathscr{L}$), and

$$\sum_{i \in J} x_i < v(J).$$

Consequently, also $(\mathscr{L}, \mathbf{x})$ is not rational for any $\mathbf{x} \in X(\mathscr{K}) \subset X(\mathscr{L})$. On the other hand, (3.3) and (2.1) imply that for any $\mathbf{y} \in X(\mathscr{L}) - X(\mathscr{K})$ there exists $K \in \mathscr{K}$, such that

$$v(K) > \sum_{i \in K} y_i .$$

Because $\mathscr{K}$ is a subpartition of $\mathscr{L}$, also no configuration $(\mathscr{L}, \mathbf{y})$, where $\mathbf{y} \in X(\mathscr{L}) - X(\mathscr{K})$ is rational. It means that $\mathscr{K} \notin \mathbf{K}_{\mathrm{ef}}$ implies $\mathscr{L} \notin \mathbf{K}_{\mathrm{ef}}$, what contradicts the assumptions of Lemma.

Lemma 3.3. Let $\mathscr{J} \in \mathbf{K}^{\mathrm{ef}}$, $\mathscr{K} \in \mathbf{K}$, and let $\mathscr{J}$ be a subpartition of $\mathscr{K}$. Then $\mathscr{K} \in \mathbf{K}^{\mathrm{ef}}$.

Proof. If $\mathscr{K} \notin \mathbf{K}^{\mathrm{ef}}$ then there exists $\mathscr{M} \in \mathbf{K}_{\mathrm{ef}}$ such that $\mathscr{K}$ is a subpartition of $\mathscr{M}$ and

$$\sum_{M \in \mathscr{M}} v(M) > \sum_{K \in \mathscr{K}} v(K) \geqq \sum_{J \in \mathscr{J}} v(J), \tag{3.4}$$

where the last inequality follows from (1.1) and from the assumption that $\mathscr{J}$ is a subpartition of $\mathscr{K}$. As $\mathscr{J}$ is also a subpartition of $\mathscr{M}$, the relation (3.4) implies that $\mathscr{J} \notin \mathbf{K}^{\mathrm{ef}}$, what contradicts the assumptions of Lemma.

Corollary Let $\mathscr{J} \in \mathbf{K}^{\mathrm{ef}}$, $\mathscr{L} \in \mathbf{K}_{\mathrm{ef}}$, $\mathscr{K} \in \mathbf{K}$, and let $\mathscr{J}$ be a subpartition of $\mathscr{K}$ and $\mathscr{K}$ be a subpartition of $\mathscr{L}$. Then

$$\mathscr{J} \in \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}, \quad \mathscr{K} \in \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}, \quad \mathscr{L} \in \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}},$$

as follows from Lemmas 3.2 and 3.3.

Lemma 3.4. Let $\mathscr{K} \in \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}$, $\mathscr{L} \in \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}$ and let $\mathscr{K}$ be a subpartition of $\mathscr{L}$. Then

$$v(L) = \sum_{K \in \mathscr{K}, K \subset L} v(K), \quad \text{for all} \quad L \in \mathscr{L}, \tag{3.5}$$

$$X(\mathscr{L}) \supset X(\mathscr{K}), \tag{3.6}$$

and if for some imputation $\mathbf{x} \in X(\mathscr{L})$ the configuration $(\mathscr{L}, \mathbf{x})$ is rational, then $\mathbf{x} \in X(\mathscr{K})$ and $(\mathscr{K}, \mathbf{x})$ is rational.

Proof. The equation (3.5) is an immediate consequence of (3.2) and (1.1) and of the assumption that $\mathscr{K}$ is a subpartition of $\mathscr{L}$. Let $\mathbf{x} \in X(\mathscr{K})$. Then, by (3.5) and (2.1), also $\mathbf{x} \in X(\mathscr{L})$. Let, on the other hand, $\mathbf{x} \in X(\mathscr{L})$, and $(\mathscr{L}, \mathbf{x})$ be rational. Then, by Definition 1,

$$\sum_{i \in K} x_i \geqq v(K)$$

for all $K \in \mathscr{K}$, as $\mathscr{K}$ is a subpartition of $\mathscr{L}$. This inequality, and the equality (3.5) imply, that

$$\sum_{i \in K} x_i = v(K) \quad \text{for all} \quad K \in \mathscr{K} ,$$

and, consequently, $\mathbf{x} \in X(\mathscr{K})$. If $(\mathscr{K}, \mathbf{x})$ is not rational, then there exists $\mathscr{J} \in \mathbf{K}$, such that $\mathscr{J}$ is a subpartition of $\mathscr{K}$ and

$$v(J) < \sum_{i \in J} x_i \quad \text{for some} \quad J \in \mathscr{J} . \tag{3.7}$$

But, (3.7) contradicts the assumption of rationality of $(\mathscr{L}, \mathbf{x})$, because $\mathscr{J}$ is also a subpartition of $\mathscr{L}$. It means that $(\mathscr{K}, \mathbf{x})$ is rational.

4. SAFE COALITION STRUCTURES

This section of the presented paper includes some auxiliary relations on the class $\mathbf{K}$ of all admissible coalition structures. Our aim in this section is to define, under which assumptions a given coalition structure can be endangered by some group of other coalition structures, and when it has a possibility to stave off this endangerment.

Definition 3. Let $\mathscr{K} \in \mathbf{K}$, and let $\mathbf{M} \subset \mathbf{K}_{\text{ef}}$ be a class of coalition structures. Then we say that $\mathscr{K}$ is safety against $\mathbf{M}$ and we write $\mathscr{K}\ \sigma\ \mathbf{M}$ iff there exists $\mathbf{x} \in X(\mathscr{K})$ such that $(\mathscr{K}, \mathbf{x})$ is rational configuration, and

$$\sum_{i \in M} x_i \geqq v(M) \quad \text{for all} \quad M \in \bigcup_{\mathscr{M} \in \mathbf{M}} \mathscr{M} .$$

If $\mathscr{K}$ is not safety against some class $\mathbf{M} \subset \mathbf{K}_{\text{ef}}$, then we write $\mathscr{K}$ non σ $\mathbf{M}$.

Remark 4.1.

1. if $\mathscr{K} \in \mathbf{K}$, $\mathbf{M} \subset \mathbf{K}_{\text{ef}}$, then exactly one of the relations $\mathscr{K}\ \sigma\ \mathbf{M}$ and $\mathscr{K}$ non σ $\mathbf{M}$ is true;
2. if $\mathscr{K}' \in \mathbf{K} - \mathbf{K}_{\text{ef}}^{\text{ef}}$ then there exists $\mathscr{K} \in \mathbf{K}_{\text{ef}}^{\text{ef}}$ such that $\mathscr{K}'$ non σ $\{\mathscr{K}\}$;
3. if $\mathscr{K} \in \mathbf{K}$, $\mathbf{M} \subset \mathbf{N} \subset \mathbf{K}_{\text{ef}}$, then $\mathscr{K}\ \sigma\ \mathbf{N}$ implies $\mathscr{K}\ \sigma\ \mathbf{M}$;

4. if $(j_1, \ldots, j_n)$ is an arbitrary permutation of indices $(1, \ldots, n)$, if $\mathscr{K} \in \mathbf{K}$ and $\{\mathscr{K}_1, \ldots, \mathscr{K}_n\} \subset \mathbf{K}_{\text{ef}}$, then
   $\mathscr{K}\ \sigma\ \{\mathscr{K}_1, \ldots, \mathscr{K}_n\}$ if and only if
   $\mathscr{K}\ \sigma\ \{\mathscr{K}_{j_1}, \ldots, \mathscr{K}_{j_n}\}$;
5. if $\mathscr{K} \in \mathbf{K}$, $\mathbf{M} \subset \mathbf{K}_{\text{ef}}$, $\mathscr{J} \in \mathbf{M}$, then
   $\mathscr{K}\ \sigma\ \mathbf{M}$ if and only if $\mathscr{K}\ \sigma\ (\mathbf{M} \cup \{\mathscr{J}\})$, and
   $\mathscr{K}\ \sigma\ \mathbf{M}$ if and only if $\mathscr{K}\ \sigma\ (\mathbf{M} \cup \{\mathscr{K}\})$.

Lemma 4.1. Let $\mathscr{K} \in \mathbf{K}$, and let $\mathbf{O} \subset \mathbf{K}_{\text{ef}}$ be the empty class of coalition structures. Then $\mathscr{K}\ \sigma\ \mathbf{O}$ if and only if $\mathscr{K} \in \mathbf{K}_{\text{ef}}$.

Proof. By Definition 3, $\mathscr{K}\ \sigma\ \mathbf{O}$ if and only if there exists $\mathbf{x} \in X(\mathscr{K})$ such that $(\mathscr{K}, \mathbf{x})$ is rational. This is equivalent with the effectivity from below of $\mathscr{K}$.

Now, when the auxiliary relation was introduced, we can investigate the gradual mutations of coalition structures, during the bargaining process.

5. STABILITY OF COALITION STRUCTURES

In this part a method is given, which enables us to define the bargaining solution on the class of coalition structures. It enables us to find out the resulting structures. The following Definition 4 describes, which coalition structures can break a formed coalitions.

Generally, the coalition structures, which appear during the bargaining process, can be broken and substituted by some other ones. Some of them are able to reappear, sooner or later, after any such removal. This ability may be understood as certain kind of dynamic stability. The strongest possible state of the bargaining is such, in which the formed coalition structure can not be substituted by any other coalition structure. Both cases of stability, the strong one and the dynamic one, are investigated in this section, and, for some special cases of coalition structures, also in following two parts of this paper.

Definition 4. A mapping Δ from $\mathbf{K}$ into the family of subclasses of the class $\mathbf{K}_{\text{ef}}$, such that for any $\mathscr{K} \in \mathbf{K}$

$$(5.1)\quad \Delta(\mathscr{K}) = \{\mathscr{M} \in \mathbf{K}_{\text{ef}}: \textit{for some } \mathbf{M} \subset \mathbf{K}_{\text{ef}}\ \mathscr{K}\ \sigma\ \mathbf{M} \textit{ and } \mathscr{K} \text{ non } \sigma\ (\mathbf{M} \cup \{\mathscr{M}\})\}$$

is called a *domination structure* in the game Γ.

Remark 5.1. The previous Definition implies immediately that for any $\mathscr{K} \in \mathbf{K}$ is $\Delta(\mathscr{K}) \subset \mathbf{K}_{\text{ef}}$, and for any $\mathscr{K} \in \mathbf{K} - \mathbf{K}_{\text{ef}}^{\text{ef}}$ is $\Delta(\mathscr{K}) \neq \emptyset$.

Lemma 5.1. Let $\mathscr{K} \in \mathbf{K}$. Then $\Delta(\mathscr{K}) = \emptyset$ if and only if $\mathscr{K}\ \sigma\ \mathbf{K}_{\text{ef}}$.

Proof. It is an immediate consequence of Definition 4 that $\Delta(\mathscr{K}) = \emptyset$ if and only if $\mathscr{K}\ \sigma\ \mathbf{M}$ for all $\mathbf{M} \subset \mathbf{K}_{cf}$. It means that the implication $\Delta(\mathscr{K}) = \emptyset \Rightarrow \mathscr{K}\ \sigma\ \mathbf{K}_{cf}$ is true. On the other hand, if $\mathscr{K}\ \sigma\ \mathbf{K}_{cf}$, then, by Remark 4.1 (statement 3), also $\mathscr{K}\ \sigma\ \mathbf{M}$ for any $\mathbf{M} \subset \mathbf{K}_{cf}$, and $\Delta(\mathscr{K}) = \emptyset$.

Corollary. Let $\mathscr{K} \in \mathbf{K}$. Then $\Delta(\mathscr{K}) = \emptyset$ if and only if there exists $\mathbf{x} \in X(\mathscr{K})$ such that

$$\sum_{i \in K} x_i \geqq v(K) \quad \text{for all} \quad K \in \bigcup_{\mathscr{L} \in \mathbf{K}_{cf}} \mathscr{L} .$$

After this introduction of the domination structure and its elementary properties, we are going to investigate the stability of coalition structures. In the following Definition we introduce the general concept of stability, which has dynamic character.

Definition 5. Let $\mathscr{K} \in \mathbf{K}$. We say that $\mathscr{K}$ is *stable*, iff for any set $\{\mathscr{K}_1, \ldots, \mathscr{K}_n\} \subset \mathbf{K}_{cf}$ such that

$$\mathscr{K}_1 \in \Delta(\mathscr{K}) , \quad \mathscr{K}_r \in \Delta(\mathscr{K}_{r-1}) , \quad r = 2, \ldots, n ,$$

there exists a set $\{\mathscr{L}_1, \ldots, \mathscr{L}_m\} \subset \mathbf{K}_{cf}$ such that

$$\mathscr{L}_1 \in \Delta(\mathscr{K}_n) , \quad \mathscr{L}_s \in \Delta(\mathscr{L}_{s-1}) , \quad s = 2, \ldots, m , \quad \mathscr{K} \in \Delta(\mathscr{L}_m) .$$

The symbol $\mathbf{S}$ denotes the class of all stable coalition structures in the given game.

Remark 5.2. It is an immediate consequence of the previous Definition and of Remark 5.1 that

$$\mathbf{S} \subset \mathbf{K}_{cf} ,$$

$$\Delta(\mathscr{K}) = \emptyset \Rightarrow \mathscr{K} \in \mathbf{S} , \quad \text{for any} \quad \mathscr{K} \in \mathbf{K} .$$

The just defined stability of coalition structures can be interpreted as the "dynamic stability", mentioned in the introductory paragraphs of this section. It is obvious that the "strong stability", mentioned also in the introductory paragraphs, was already defined. This kind of stability is represented by the coalition structures, for which the value of domination structure is empty. By Remark 5.2, the "strong stability" is a special case of the "dynamic stability". Their further properties are described by the following statements.

Lemma 5.2. Let $\mathscr{K}, \mathscr{L} \in \mathbf{K}$, $\mathscr{K} \in \mathbf{S}$, and $\mathscr{L} \in \Delta(\mathscr{K})$. Then $\mathscr{L} \in \mathbf{S}$.

Proof. If $\mathscr{L} \notin \mathbf{S}$ then there exists a set $\{\mathscr{K}_1, \ldots, \mathscr{K}_n\} \subset \mathbf{K}_{cf}$ such that

(5.2) $$\mathscr{K}_1 \in \Delta(\mathscr{L}) , \quad \mathscr{K}_r \in \Delta(\mathscr{K}_{r-1}) , \quad r = 2, \ldots, n ,$$

and there is no set $\{\mathscr{K}_{n+1}, \ldots, \mathscr{K}_m\} \subset \mathbf{K}_{cf}$ such that

(5.3) $$\mathscr{K}_r \in \Delta(\mathscr{K}_{r-1}) , \quad r = n + 1, \ldots, m, \quad \mathscr{L} \in \Delta(\mathscr{K}_m) .$$

It means that there exists a set $\{\mathscr{L}, \mathscr{K}_1, \ldots, \mathscr{K}_n\}$ such that $\mathscr{L} \in \Delta(\mathscr{K})$ and (5.2) is true, and there is no $\{\mathscr{K}_{n+1}, \ldots, \mathscr{K}_m\}$ such that (5.3) would be true and $\mathscr{K} \in \Delta(\mathscr{K}_m)$. Hence, $\mathscr{K} \notin \mathbf{S}$, what contradicts the assumptions of Lemma.

Lemma 5.3. Let $\mathscr{K}, \mathscr{L} \in \mathbf{K}$, $\mathscr{K} \in \Delta(\mathscr{L})$, $\Delta(\mathscr{K}) = \emptyset$. Then $\mathscr{L} \notin \mathbf{S}$.

Proof. As $\Delta(\mathscr{K}) = \emptyset$, there does not exist any set $\{\mathscr{K}_1, \ldots, \mathscr{K}_n\}$ in $\mathbf{K}_{\text{ef}}$, $\mathscr{K}_1 \in \Delta(\mathscr{K})$, $\mathscr{K}_r \in \Delta(\mathscr{K}_{r-1})$, $r = 2, \ldots, n$, $\mathscr{L} \in \Delta(\mathscr{K}_n)$.

Lemma 5.4. Let $\mathscr{J}, \mathscr{K} \in \mathbf{K}_{\text{ef}}$, and let

$$\sum_{K \in \mathscr{K}} v(K) > \sum_{J \in \mathscr{J}} v(J)\,. \tag{5.4}$$

Then $\mathscr{K} \in \Delta(\mathscr{J})$.

Proof. For any $\mathbf{x} \in X(\mathscr{J})$ there exists $K \in \mathscr{K}$ such that

$$v(K) > \sum_{i \in K} x_i.$$

Consequently, $\mathscr{J}$ non $\sigma \{\mathscr{K}\}$, and by Lemma 4.1, $\mathscr{J}\ \sigma\ \mathbf{O}$, where $\mathbf{O}$ is empty class of coalition structures. It means that $\mathscr{K} \in \Delta(\mathscr{J})$.

Corollary. If $\mathscr{J}, \mathscr{K} \in \mathbf{K}_{\text{ef}}$, $\Delta(\mathscr{K}) = \emptyset$, and (5.4) is true, then $\mathscr{J} \notin \mathbf{S}$, as follows from Lemmas 5.3 and 5.4.

Lemma 5.5. Let $\mathscr{K} \in \mathbf{K}$, $\Delta(\mathscr{K}) = \emptyset$. Then

$$\sum_{K \in \mathscr{K}} v(K) = \max \{ \sum_{J \in \mathscr{J}} v(J) \colon \mathscr{J} \in \mathbf{K}_{\text{ef}} \} = \max \{ \sum_{J \in \mathscr{J}} v(J) \colon \mathscr{J} \in \mathbf{K}_{\text{ef}}^{\text{ef}} \}\,.$$

Proof. By the previous Lemma 5.4, $\Delta(\mathscr{K}) = \emptyset$ implies that for all $\mathscr{J} \in \mathbf{K}_{\text{ef}}$ the relation (5.4) is true. It is an immediate consequence of Definition 2 that

$$\max \{ \sum_{J \in \mathscr{J}} v(J) \colon \mathscr{J} \in \mathbf{K}_{\text{ef}}^{\text{ef}} \} = \max \{ \sum_{J \in \mathscr{J}} v(J) \colon \mathscr{J} \in \mathbf{K}_{\text{ef}} \}\,.$$

Corollary. If $\Delta(\mathscr{M}) = \emptyset$ for some $\mathscr{M} \in \mathbf{K}$ then for any $\mathscr{K} \in \mathbf{S}$ is

$$\sum_{K \in \mathscr{K}} v(K) = \max \{ \sum_{J \in \mathscr{J}} v(J) : \mathscr{J} \in \mathbf{K}_{\text{ef}} \}\,,$$

as follows from Lemmas 5.3, 5.4 and 5.5.

Theorem 5.1. There exists at least one effective and stable coalition structure in any coalition-game; i.e.

$$\mathbf{S} \cap \mathbf{K}_{\text{ef}}^{\text{ef}} \neq \emptyset\,.$$

Proof. Let us introduce a binary relation δ on the class $\mathbf{K}$ in the following way: For any $\mathscr{K}, \mathscr{K}' \in \mathbf{K}$ we write

$$\mathscr{K} \,\delta\, \mathscr{K}'$$

iff there exists a set $\{\mathscr{K}_1, \ldots, \mathscr{K}_n\} \subset \mathbf{K}_{\mathrm{ef}}$ such that

$$\mathscr{K}_1 \in \Delta(\mathscr{K}'), \quad \mathscr{K}_r \in \Delta(\mathscr{K}_{r-1}), \quad r = 2, \ldots, n, \quad \mathscr{K} \in \Delta(\mathscr{K}_n)$$

and there is no set $\{\mathscr{L}_1, \ldots, \mathscr{L}_m\} \subset \mathbf{K}_{\mathrm{ef}}$ such that

$$\mathscr{L}_1 \in \Delta(\mathscr{K}), \quad \mathscr{L}_s \in \Delta(\mathscr{L}_{s-1}), \quad s = 2, \ldots, m, \quad \mathscr{K}' \in \Delta(\mathscr{L}_m).$$

Such relation δ is a partial ordering on the finite set $\mathbf{K}$, and it is antisymetrical, antireflexive, and transitive. Consequently, there exists at least one maximal element in $\mathbf{K}$, according to the partial ordering δ, and any coalition structure is maximal according to δ, if and only if it is stable, i.e.

$$\mathbf{S} = \{\mathscr{M} \in \mathbf{K} : \mathscr{M} \textit{ is maximal element in } \mathbf{K} \textit{ according to } \delta\} \subset \mathbf{K}_{\mathrm{ef}},$$

Let us choose $\mathscr{K} \in \mathbf{S}$. If $\mathscr{M}$ is effective, then the desired stable and effective coalition structure is found. If $\mathscr{M} \in \mathbf{K}_{\mathrm{ef}} - \mathbf{K}^{\mathrm{ef}}$ then there exists, by Lemma 5.4 and by Definition 2, at least one

$$\mathscr{N} \in \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}} \cap \Delta(\mathscr{M}).$$

By Lemma 5.1, also $\mathscr{N} \in \mathbf{S}$, and in this case, $\mathscr{N}$ is the desired coalition structure.

Theorem 5.2. If there exists at least one coalition structure $\mathscr{K} \in \mathbf{K}$ with empty value of domination structure Δ, then the domination structure is empty for all, and exactly all, stable coalition structures; in symbols, if $\Delta(\mathscr{K}) = \emptyset$ for some $\mathscr{K} \in \mathbf{K}$, then

$$\mathbf{S} = \{\mathscr{M} \in \mathbf{K} : \Delta(\mathscr{M}) = \emptyset\}.$$

Proof. Let $\Delta(\mathscr{K}) = \emptyset$. Then, by Lemma 5.1, $\mathscr{K} \,\sigma\, \mathbf{K}_{\mathrm{ef}}$. It means that there exists $\mathbf{x} \in X(\mathscr{K})$ such that

$$\sum_{i \in J} x_i \geqq v(J) \quad \text{for all} \quad J \in \{M : M \in \mathscr{M}, \mathscr{M} \in \mathbf{K}_{\mathrm{ef}}\}. \tag{5.5}$$

If $\mathscr{M} \in \mathbf{S}$ then, by Lemmas 5.4 and 5.5,

$$\sum_{M \in \mathscr{M}} v(M) = \sum_{K \in \mathscr{K}} v(K) = \sum_{i \in I} x_i,$$

as $\mathscr{K} \notin \Delta(\mathscr{M})$, by Lemma 5.3. Moreover, (5.5) implies that

$$\sum_{i \in M} x_i \geqq v(M) \quad \text{for all} \quad M \in \mathscr{M}, \mathscr{M} \in \mathbf{S},$$

so that

$$\sum_{i \in M} x_i = v(M) \quad \text{for all} \quad M \in \mathcal{M},\ \mathcal{M} \in \mathbf{S}\,.$$

Consequently, $\mathbf{x} \in X(\mathcal{M})$, and, because of (5.5), $\Delta(\mathcal{M}) = \emptyset$ for all $\mathcal{M} \in \mathbf{S}$, by Lemma 5.3 and its Corollary. On the other hand, if $\Delta(\mathcal{L}) = \emptyset$ for some $\mathcal{L} \in \mathbf{K}$ then $\mathcal{L} \in \mathbf{S}$ (c.f. Remark 5.2).

The just proved Theorem 5.2 provokes a question, whether the class $\mathbf{S}$ is different from $\mathbf{K}_{ef}$ (or $\mathbf{K}_{ef}^{ef}$), when $\Delta(\mathcal{K}) \neq \emptyset$ for all $\mathcal{K} \in \mathbf{K}$. If it were so, then it would have no sense to define $\mathbf{S}$ as a special notion, and we could operate with the classes $\mathbf{K}_{ef}$ and $\{\mathcal{K} \in \mathbf{K} : \Delta(\mathcal{K}) = \emptyset\}$ only. The following Example shows that $\mathbf{S}$ is generally different from $\mathbf{K}_{ef}$ even from $\mathbf{K}_{ef}^{ef}$.

Example. Let $\Gamma = (I, \mathbf{K}, \mathbf{v})$ be a coalition-game. Let $I = \{1, 2, 3, 4\}$, $\mathbf{K}$ containes all a priori possible partitions of the set I, and $\mathbf{v}$ be the following one:

$$\mathbf{v}(\{i\}) = \mathbf{v}(\{i, j\}) = 0 \quad \text{for all} \quad i, j \in I,\ i \neq j,$$

$$\mathbf{v}(\{i, j, k\}) = \mathbf{v}(I) = 1 \quad \text{for all} \quad i, j, k \in I,\ i \neq j \neq k \neq i\,.$$

We denote the coalition structures

$$\begin{aligned}
&\mathcal{K}_1 = \{\{1\}, \{2\}, \{3\}, \{4\}\} && \mathcal{K}_9 = \{\{3\}, \{1, 2\}, \{4\}\}\\
&\mathcal{K}_2 = \{\{1\}, \{2, 3\}, \{4\}\} && \mathcal{K}_{10} = \{\{3\}, \{1, 2, 4\}\}\\
&\mathcal{K}_3 = \{\{1\}, \{2\}, \{3, 4\}\} && \mathcal{K}_{11} = \{\{4\}, \{1, 2, 3\}\}\\
&\mathcal{K}_4 = \{\{1\}, \{3\}, \{2, 4\}\} && \mathcal{K}_{12} = \{\{1, 2\}, \{3, 4\}\}\\
&\mathcal{K}_5 = \{\{1\}, \{2, 3, 4\}\} && \mathcal{K}_{13} = \{\{1, 3\}, \{2, 4\}\}\\
&\mathcal{K}_6 = \{\{2\}, \{1, 3\}, \{4\}\} && \mathcal{K}_{14} = \{\{1, 4\}, \{2, 3\}\}\\
&\mathcal{K}_7 = \{\{2\}, \{1, 4\}, \{3\}\} && \mathcal{K}_{15} = \{\{1, 2, 3, 4\}\}\\
&\mathcal{K}_8 = \{\{2\}, \{1, 3, 4\}\} &&
\end{aligned}$$

Then it can be simply verified, that

$$\begin{aligned}
\mathbf{K}_{ef} &= \mathbf{K} - \{\mathcal{K}_{15}\}\,,\\
\mathbf{K}_{ef}^{ef} &= \{\mathcal{K}_5, \mathcal{K}_8, \mathcal{K}_{10}, \mathcal{K}_{11}, \mathcal{K}_{12}, \mathcal{K}_{13}, \mathcal{K}_{14}\}\,,\\
\Delta(\mathcal{K}_5) &= \{\mathcal{K}_8, \mathcal{K}_{10}, \mathcal{K}_{11}\}\,,\\
\Delta(\mathcal{K}_8) &= \{\mathcal{K}_5, \mathcal{K}_{10}, \mathcal{K}_{11}\}\,,\\
\Delta(\mathcal{K}_{10}) &= \{\mathcal{K}_5, \mathcal{K}_8, \mathcal{K}_{11}\}\,,\\
\Delta(\mathcal{K}_{11}) &= \{\mathcal{K}_5, \mathcal{K}_8, \mathcal{K}_{10}\}\,,\\
\Delta(\mathcal{K}_{12}) &= \Delta(\mathcal{K}_{13}) = \Delta(\mathcal{K}_{14}) = \{\mathcal{K}_5, \mathcal{K}_8, \mathcal{K}_{10}, \mathcal{K}_{11}\}\,.
\end{aligned}$$

Consequently, $\Delta(\mathcal{K}) \neq \emptyset$ for all $\mathcal{K} \in \mathbf{K}$, and

$$\mathbf{S} = \{\mathcal{K}_5, \mathcal{K}_8, \mathcal{K}_{10}, \mathcal{K}_{11}\} \neq \mathbf{K}_{ef} \neq \mathbf{K}_{ef}^{ef} \neq \mathbf{S}\,.$$

It is possible to introduce some further properties of the class $\mathbf{S}$, namely some relations between $\mathbf{S}$ and $\mathbf{K}_{\mathrm{ef}} - \mathbf{S}$.

Theorem 5.3. Let $\mathbf{S}$ be the class of stable coalition structures, let $\mathbf{K}_{\mathrm{ef}} \neq \mathbf{S}$, and let

$$\alpha = \min \{ \sum_{K \in \mathscr{K}} v(K) : \mathscr{K} \in \mathbf{S} \} , \quad \beta = \max \{ \sum_{K \in \mathscr{K}} v(K) : \mathscr{K} \in \mathbf{K}_{\mathrm{ef}} - \mathbf{S} \} .$$

Then

$$\alpha \geqq \beta .$$

Proof. Lemma 5.4 implies that for $\mathscr{K} \in \mathbf{S}$ and $\mathscr{L} \in \mathbf{K}_{\mathrm{ef}}$, for which

$$\sum_{L \in \mathscr{L}} v(L) > \sum_{K \in \mathscr{K}} v(K) ,$$

also $\mathscr{L} \in \mathbf{S}$. It means that

$$\beta \leqq \sum_{K \in \mathscr{K}} v(K)$$

for all $\mathscr{K} \in \mathbf{S}$ and, consequently, $\alpha \geqq \beta$.

Theorem 5.4. Let $\mathscr{I}$ be such a partition of the set I, that for any $\mathscr{K} \in \mathbf{K}$, $\mathscr{I}$ is a subpartition of $\mathscr{K}$. If $\mathscr{I}$ is admissible coalition structure, $\mathscr{I} \in \mathbf{K}$, then

$$\Delta(\mathscr{I}) = \emptyset \Leftrightarrow \mathscr{I} \in \mathbf{S} \Leftrightarrow \mathscr{I} \in \mathbf{K}^{\mathrm{ef}} ,$$

and if $\mathscr{K} \in \mathbf{K}$ is such that

$$\sum_{J \in \mathscr{I}} v(J) = \sum_{K \in \mathscr{K}} v(K)$$

then $\mathscr{K}$ is stable if an only if $\Delta(\mathscr{K}) = \emptyset$, and $\Delta(\mathscr{K}) = \emptyset$ if and only if $\mathscr{I} \in \mathbf{S}$.

Proof. If $\mathscr{I} \in \mathbf{K}^{\mathrm{ef}}$ then for all $\mathscr{K} \in \mathbf{K}_{\mathrm{ef}}$ is

$$\sum_{J \in \mathscr{I}} v(J) = \sum_{K \in \mathscr{K}} v(K) , \tag{5.6}$$

and for any $\mathbf{x} \in X(\mathscr{I})$ is

$$\sum_{i \in K} x_i = v(K) \quad \text{for all} \quad K \in \{ M : M \in \mathscr{M}, \mathscr{M} \in \mathbf{K}_{\mathrm{ef}} \} .$$

It means that $\mathscr{I} \sigma \mathbf{K}_{\mathrm{ef}}$, and by Lemma 5.1, $\Delta(\mathscr{I}) = \emptyset$. On the other hand, let $\mathscr{I}$ is not effective from above. Then, by Remark 5.1, there exists $\mathscr{M} \in \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}$ such that $\mathscr{M} \in \Delta(\mathscr{I})$. If $\mathscr{L} \in \mathbf{K}_{\mathrm{ef}}$ and $\mathbf{M} \subset \mathbf{K}_{\mathrm{ef}}$ then the relation $\mathscr{L} \sigma \mathbf{M}$ implies $\mathscr{L} \sigma (\mathbf{M} \cup \{\mathscr{I}\})$, as for any $\mathbf{x} \in X(\mathscr{L})$ such that $(\mathscr{L}, \mathbf{x})$ is rational·

$$\sum_{i \in J} x_i \geqq v(J) \quad \text{for all} \quad J \in \mathscr{I} ,$$

when $\mathscr{I}$ is a subpartition of $\mathscr{K}$. Consequently, $\mathscr{I} \notin \Delta(\mathscr{L})$ for any $\mathscr{L} \in \mathbf{K}_{\mathrm{ef}}$. This implies that $\mathscr{I} \notin \mathbf{S}$, if $\mathscr{I} \notin \mathbf{K}^{\mathrm{ef}}$. The implication $\Delta(\mathscr{I}) = \emptyset \Rightarrow \mathscr{I} \in \mathbf{S}$ follows from Definition 5, or from Remark 5.2, immediately. We have already proved that

$\mathscr{I} \in \mathbf{S} \Rightarrow \mathscr{I} \in \mathbf{K}^{\text{ef}} \Rightarrow \Delta(\mathscr{I}) = \emptyset \Rightarrow \mathscr{I} \in \mathbf{S}$, what finishes the proof of the first statement of Theorem. Now, let $\mathscr{K} \in \mathbf{K}$ and let (5.6) be true. Then $X(\mathscr{I}) \subset X(\mathscr{K})$. If $\mathscr{I} \in \mathbf{S}$ then $\Delta(\mathscr{I}) = \emptyset$, and $\mathscr{I}\ \sigma\ \mathbf{K}_{\text{ef}}$, by Lemma 5.1. It means that the $\mathbf{x} \in X(\mathscr{I})$, which fulfil the requirements of Definition 3 for the relation $\mathscr{I}\ \sigma\ \mathbf{K}_{\text{ef}}$, belong also to the set $X(\mathscr{K})$, and they are sufficient even for the validity conditions of the relation $\mathscr{K}\ \sigma\ \mathbf{K}_{\text{ef}}$. Because $\mathscr{K}\ \sigma\ \mathbf{K}_{\text{ef}}$ implies that $\Delta(\mathscr{K}) = \emptyset$, we have proved

(5.7) $$\mathscr{I} \in \mathbf{S} \Rightarrow \Delta(\mathscr{K}) = \emptyset \Rightarrow \mathscr{K} \in \mathbf{S}\,.$$

On the other hand, if $\mathscr{I} \notin \mathbf{S}$ then, by the first statement of this Theorem, $\mathscr{I} \notin \mathbf{K}^{\text{ef}}$; there exists $\mathscr{L} \in \mathbf{K}_{\text{ef}}$ such that

$$\sum_{L \in \mathscr{L}} v(L) > \sum_{J \in \mathscr{I}} v(J) = \sum_{J \in \mathscr{K}} v(K)\,,$$

and, by Lemma 5.4, $\mathscr{L} \in \Delta(\mathscr{K})$, $\Delta(\mathscr{K}) \neq \emptyset$. But, it is possible to prove, analogously to the method, used in the first part of this proof, that for any $\mathscr{M} \in \mathbf{K}_{\text{ef}}$, $\mathscr{K} \notin \Delta(\mathscr{M})$, so that $\mathscr{K} \notin \mathbf{S}$. Consequently,

(5.8) $$\mathscr{I} \notin \mathbf{S} \Rightarrow \Delta(\mathscr{K}) \neq \emptyset \Rightarrow \mathscr{K} \notin \mathbf{S}\,.$$

The implications (5.7) and (5.8) imply the validity of the latter statement of the proved Theorem.

In this section, we have introduced the class $\mathbf{S}$, which contains all stable coalition structures, and which has properties, described in the previous statements.

When we consider all possible transitions from one coalition structure to another one, during the bargaining process, we see that they can be distributed into two groups.

One of them includes such situations, in which some coalitions in the original coalition structure were broken. These situations have, generally, rahter antagonistic character – they can come true against the interestes and wishes of some groups of players. They represent the real conflict in the bargaining process.

The other group of transitions includes the situations, in which only some groups of small coalitions were unified into larger coalitions. Formally, some coalition structure from $\mathbf{K}_{\text{ef}} - \mathbf{K}_{\text{ef}}^{\text{ef}}$ was substituted by another one from $\mathbf{K}_{\text{ef}}^{\text{ef}}$. These transitions may be realized without lost of profit for any player, and with growth of profit for at least some of them. We may expect that they will be realized as soon as possible, and that the coalition structures, in which such transitions are possible, will not occur among the results of rational bargaining.

It enables us to introduce the Bargaining Solution on the class of all coalition structures.

The *Bargaining Solution on the class of all coalition structures* is the class of all stable and effective coalition structures in the given coalition-game. i.e. the class

(5.9) $$\mathbf{S} \cap \mathbf{K}_{\text{ef}}^{\text{ef}}\,.$$

We have proved in this section that the Bargaining Solution on the class of all coalition structures is always non-empty and that it is equal to the class

$$\{\mathscr{K} \in \mathbf{K} : \Delta(\mathscr{K}) = \emptyset\},$$

if and only if that class is non-empty.

Moreover, this Bargaining Solution on the class of all coalition structures is satisfactory also from the point of view of demands, which may be intuitevely putted on the properties of expected result of bargaining in coalition-games.

6. EFFECTIVE SUBPARTITIONS

This section is devoted to a special relation between coalition structures, namely to the relation between a coalition structure and its subpartition, if they both are effective.

In the whole following section we suppose that

(6.1) $\mathscr{K} \in \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}, \quad \mathscr{L} \in \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}$, *and* $\mathscr{K}$ *is a subpartition of* $\mathscr{L}$.

This situation was already mentioned in Lemma 3.4, and the relation between a coalition structure and its subpartition, without the effectivity assumption, was investigated also in Lemmas 3.2 and 3.3 If such a pair of coalition structures appears in some coalition-game, the procedure of finding out the Bargaining Solution on the class of all coalition structures can be simplified, namely if some of them belongs to the Solution.

Lemma 6.1. Let (6.1) be fulfilled for some $\mathscr{K}$, $\mathscr{L} \in \mathbf{K}$, and let $\mathbf{M} \subset \mathbf{K}_{\mathrm{ef}}$, $\mathscr{M} \in \mathbf{K}$. **Then**

$$\mathscr{M}\ \sigma\ (\mathbf{M} \cup \{\mathscr{K}\}) \Rightarrow \mathscr{M}\ \sigma\ (\mathbf{M} \cup \{\mathscr{K}, \mathscr{L}\}),$$

$$\mathscr{L}\ \sigma\ \mathbf{M} \Rightarrow \mathscr{K}\ \sigma\ \mathbf{M},$$

$$\mathscr{K}\ \sigma\ (\mathbf{M} \cup \{\mathscr{L}\}) \Leftrightarrow \mathscr{K}\ \sigma\ \mathbf{M}, \quad \text{and}$$

$$\mathscr{L}\ \sigma\ (\mathbf{M} \cup \{\mathscr{K}\}) \Leftrightarrow \mathscr{L}\ \sigma\ \mathbf{M}.$$

Proof. Let $\mathscr{M} \in \mathbf{K}$ and $\mathscr{M}\ \sigma\ (\mathbf{M} \cup \{\mathscr{K}\})$. Then, by Lemma 4.1, $\mathscr{M} \in \mathbf{K}_{\mathrm{ef}}$, and there exists $\mathbf{x} \in X(\mathscr{M})$ such that $(\mathscr{M}, \mathbf{x})$ is rational, and

$$\sum_{i \in J} x_i \geqq v(J) \quad \text{for all} \quad J \in \mathscr{K} \cup \Big(\bigcup_{\mathscr{J} \in \mathbf{M}} \mathscr{J}\Big). \tag{6.2}$$

Hence,

$$\sum_{i \in K} x_i \geqq v(K) \quad \text{for all} \quad K \in \mathscr{K}. \tag{6.3}$$

By Lemma 3.4 and by (6.3),

$$\sum_{i \in L} x_i \geqq \sum_{K \in \mathscr{K}, K \subset L} v(K) = v(L) \quad \text{for all} \quad L \in \mathscr{L}\,.$$

Consequently, (6.2) is true also for all

$$J \in \mathscr{K} \cup \mathscr{L} \cup \left(\bigcup_{\mathscr{J} \in \mathbf{M}} \mathscr{J} \right),$$

and $\mathscr{M}\ \sigma\left(\mathbf{M} \cup \{\mathscr{K}, \mathscr{L}\}\right)$.

The second statement, $\mathscr{L}\ \sigma\ \mathbf{M} \Rightarrow \mathscr{K}\ \sigma\ \mathbf{M}$, is an immediate consequence of Definition 3, assumption (6.1), and of the second part of Lemma 3.4.

Let $\mathscr{K}\ \sigma\ \mathbf{M}$. Then there exists $\mathbf{x} \in X(\mathscr{K})$, for which the requirements of Definition 3 are fulfilled. By Lemma 3.4, also

$$\sum_{i \in L} x_i = \sum_{K \in \mathscr{K}, K \subset L} v(K) = v(L) \quad \text{for any} \quad L \in \mathscr{L}\,,$$

and $\mathbf{x}$ is also the imputation, for which the conditions of the relation $\mathscr{K}\ \sigma\left(\mathbf{M} \cup \{\mathscr{L}\}\right)$ are secured. The implication

$$\mathscr{K}\ \sigma\ \mathbf{M} \Rightarrow \mathscr{K}\ \sigma\left(\mathbf{M} \cup \{\mathscr{L}\}\right)$$

is proved and the opposite implication follows from Remark 4.1 (part 3). Let $\mathscr{L}\ \sigma\ \mathbf{M}$, and let $\mathbf{y} \in X(\mathscr{L})$ be the imputation, for which the requirements of Definition 3 for the relation $\mathscr{L}\ \sigma\ \mathbf{M}$ are fulfilled. Because of the rationality of $(\mathscr{L}, \mathbf{y})$ and of the validity of (6.1),

$$\sum_{i \in K} y_i \geqq v(K) \quad \text{for all} \quad K \in \mathscr{K}\,,$$

it means that

$$\mathscr{L}\ \sigma\ \mathbf{M} \Rightarrow \mathscr{L}\ \sigma\left(\mathbf{M} \cup \{\mathscr{K}\}\right) \Rightarrow \mathscr{L}\ \sigma\ \mathbf{M}\,,$$

where the latter implication follows from Remark 4.1.

Corollary. The last two statements of the previous Lemma imply that for any $\mathscr{K}$ and $\mathscr{L}$ satisfying (6.1) is

$$\mathscr{K} \notin \Delta(\mathscr{L}) \quad \text{and} \quad \mathscr{L} \notin \Delta(\mathscr{K})\,.$$

Lemma 6.2. Let $\mathscr{K}, \mathscr{L} \in \mathbf{K}$ fulfil (6.1), let $\mathbf{M} \subset \mathbf{K}_{ef}$, and let

$$\mathbf{J}(\mathscr{L}) = \{\mathscr{J} \in \mathbf{K}_{ef} : \mathscr{J} \textit{ is a subpartition of } \mathscr{L}\}\,.$$

Then $\mathscr{L}\ \sigma\,\mathbf{J}(\mathscr{L})$, and $\mathscr{L}\ \sigma\ \mathbf{M} \Leftrightarrow \mathscr{L}\ \sigma\left(\mathbf{J}(\mathscr{L}) \cup \mathbf{M}\right) \Leftrightarrow \mathscr{K}\ \sigma\left(\mathbf{J}(\mathscr{L}) \cup \mathbf{M}\right)$.

Proof. According to Remark 3.2, $\mathscr{L} \in \mathbf{K}_{ef}$ iff there exitsts $\mathbf{x} \in X(\mathscr{L})$ such that

$$\sum_{i \in J} x_i \geqq v(J) \quad \text{for all} \quad \mathscr{J} \in \mathbf{J}(\mathscr{L})\,, \quad J \in \mathscr{J}\,.$$

Consequently, $\mathscr{L} \in \mathbf{K}_{\text{ef}}$ implies $\mathscr{L}\ \sigma\ \mathbf{J}(\mathscr{L})$. Let $\mathscr{L}\ \sigma\ \mathbf{M}$. Then there exists $\mathbf{x} \in X(\mathscr{L})$, such that $(\mathscr{L}, \mathbf{x})$ is rational, and

$$(6.4) \qquad \sum_{i \in M} x_i \geqq v(M) \quad \text{for all} \quad M \in \mathscr{M},\ \mathscr{M} \in \mathbf{M}\,.$$

It is equivalent with the statement that (6.4) is true for all $M \in \mathbf{M} \cup \mathbf{J}(\mathscr{L})$, and that

$$(6.5) \qquad \mathscr{L}\ \sigma\ (\mathbf{M} \cup \mathbf{J}(\mathscr{L}))\,.$$

If (6.5) is true then, by Lemma 6.1, also

$$(6.6) \qquad \mathscr{K}\ \sigma\ (\mathbf{M} \cup \mathbf{J}(\mathscr{L}))\,.$$

The relation (6.6) means that there exists $\mathbf{x} \in X(\mathscr{K})$ such that $(\mathscr{K}, \mathbf{x})$ is rational, and (6.4) is true for all $M \in \mathbf{M} \cup \mathbf{J}(\mathscr{L})$. Then, according to Lemma 3.4, is $\mathbf{x} \in X(\mathscr{L})$ and $(\mathscr{L}, \mathbf{x})$ is also rational. It means that $\mathscr{L}\ \sigma\ \mathbf{M}$. Summing up the proved implications, we obtain

$$\mathscr{L}\ \sigma\ \mathbf{M} \Rightarrow \mathscr{L}\ \sigma\ (\mathbf{M} \cup \mathbf{J}(\mathscr{L})) \Rightarrow \mathscr{K}\ \sigma\ (\mathbf{M} \cup \mathbf{J}(\mathscr{L})) \Rightarrow \mathscr{L}\ \sigma\ \mathbf{M}\,,$$

and the statement of Lemma is proved.

Corollary. Let (6.1) be true for some $\mathscr{K}$, $\mathscr{L} \in \mathbf{K}$, let $\mathbf{M} \subset \mathbf{K}_{\text{ef}}$, and let $\mathbf{J}(\mathscr{L}) = \{\mathscr{J} \in \mathbf{K}_{\text{ef}} : \mathscr{J}$ *is a subpartition of* $\mathscr{L}\}$. If $\mathscr{K}\ \sigma\ \mathbf{M}$ and $\mathscr{L}$ non $\sigma\ \mathbf{M}$ then there exists a class $\mathbf{J} \subset \mathbf{J}(\mathscr{L})$, $\mathbf{J} \neq \emptyset$, such that $\mathscr{K}$ non $\sigma\ (\mathbf{M} \cap \mathbf{J})$.

Lemma 6.3. If for $\mathscr{K}$, $\mathscr{L} \in \mathbf{K}$ (6.1) is true then $\Delta(\mathscr{K}) \supset \Delta(\mathscr{L})$.

Proof. Let $\mathscr{M} \in \Delta(\mathscr{L})$. It means that there exists $\mathbf{M} \subset \mathbf{K}_{\text{ef}}$, and

$$(6.7) \qquad \mathscr{L}\ \sigma\ \mathbf{M} \quad \text{and} \quad \mathscr{L}\ \text{non}\ \sigma\ (\mathbf{M} \cup \{\mathscr{M}\})\,.$$

By Lemma 6.1, also $\mathscr{K}\ \sigma\ \mathbf{M}$. If $\mathscr{K}$ non $\sigma\ (\mathbf{M} \cup \{\mathscr{M}\})$ then $\mathscr{M} \in \Delta(\mathscr{K})$. If $\mathscr{K}\ \sigma\ (\mathbf{M} \cup \{\mathscr{M}\})$ and (6.7) is true then, by Lemma 6.2 and its Corollary, there exists $\mathbf{J} \subset \mathbf{K}_{\text{ef}}$, $\mathbf{J} \neq \emptyset$, such that

$$(6.8) \qquad \mathscr{K}\ \text{non}\ \sigma\ (\mathbf{M} \cup \mathbf{J} \cup \{\mathscr{M}\})\,.$$

It is possible to choose $\mathbf{J}$ in such a way that $\mathbf{J} = \mathbf{J}(\mathscr{L})$. Then

$$\mathscr{K}\ \sigma\ (\mathbf{M} \cup \mathbf{J})\,,$$

and (6.8) is true. It means that $\mathscr{M} \in \Delta(\mathscr{K})$, and $\Delta(\mathscr{L}) \subset \Delta(\mathscr{K})$.

Lemma 6.4. Let (6.1) be true for some $\mathscr{K}$, $\mathscr{L} \in \mathbf{K}$, and let $\mathscr{M} \in \mathbf{K}$. Then $\mathscr{L} \in \Delta(\mathscr{M})$ implies $\mathscr{K} \in \Delta(\mathscr{M})$.

Proof. Let $\mathscr{L} \in \Delta(\mathscr{M})$, and let $\mathbf{M} \subset \mathbf{K}_{ef}$ be such that $\mathscr{M}\ \sigma\ \mathbf{M}$ and $\mathscr{M}$ non $\sigma\ (\mathbf{M} \cup \cup \{\mathscr{L}\})$. Then $\mathscr{M}$ non $\sigma\ (\mathbf{M} \cup \{\mathscr{K}\})$, because

$$\mathscr{M}\ \sigma\ (\mathbf{M} \cup \{\mathscr{K}\}) \Rightarrow \mathscr{M}\ \sigma\ (\mathbf{M} \cup \{\mathscr{K}\} \cup \{\mathscr{L}\}) \Rightarrow \mathscr{M}\ \sigma\ (\mathbf{M} \cup \{\mathscr{L}\}),$$

according to Lemma 6.1 and Remark 4.1. Consequently, $\mathscr{K} \in \Delta(\mathscr{M})$.

Theorem 6.1. Let $\mathscr{K}, \mathscr{L} \in \mathbf{K}_{ef}^{ef}$ be effective coalition structures, and let $\mathscr{K}$ be a subpartition of $\mathscr{L}$. Then $\mathscr{K}$ is stable if $\mathscr{L}$ is stable, and the value of domination structure for $\mathscr{K}$ is empty if and only if the value of domination structure for $\mathscr{L}$ is empty; in symbols,

$$\mathscr{L} \in \mathbf{S} \Rightarrow \mathscr{K} \in \mathbf{S},$$

$$\Delta(\mathscr{L}) = \emptyset \Leftrightarrow \Delta(\mathscr{K}) = \emptyset.$$

Proof. Before proving the statement of Theorem, we prove the following auxiliary property of non-rational configurations.

Let $\mathbf{x} \in X(\mathscr{L})$, and let $(\mathscr{L}, \mathbf{x})$ be non-rational. Then there exists $\mathscr{M} \in \mathbf{K}_{ef}$ and $M \in \mathscr{M}$ such that $\mathscr{M}$ is a subpartition of $\mathscr{L}$ and

$$\sum_{i \in M} x_i < v(M).$$

By Definition 1, we can choose $\mathscr{J} \in \mathbf{K}$ and $J \in \mathscr{J}$ such that $\mathscr{J}$ is a subpartition of $\mathscr{L}$, and

$$\sum_{i \in J} x_i < v(J).$$

If $\mathscr{J}$ was chosen in such a way that $\mathscr{J} \in \mathbf{K}_{ef}$ then the problem is solved. Let $\mathscr{J} \in \mathbf{K} - \mathbf{K}_{ef}$. Then we can construct $\mathscr{J}^* \in \mathbf{K}$ such that $J \in \mathscr{J}^*$, $\mathscr{J}^*$ is a subpartition of $\mathscr{J}$, and for any $\mathscr{N} \in \mathbf{K}$, where $J \in \mathscr{N}$, $\mathscr{N}$ is not a subpartition of $\mathscr{J}^*$. If $\mathscr{J}^* \notin \mathbf{K}_{ef}$ then we choose $\mathbf{y} \in X(\mathscr{J}^*)$ such that $y_i > x_i$ for all $i \in J$. As $\mathscr{J}^* \notin \mathbf{K}_{ef}$, $(\mathscr{J}^*, \mathbf{y})$ is not rational, and we can repeat the described procrdure. It means that there exists $\mathscr{N} \in \mathbf{K}$. and $N \in \mathscr{N}$ such that $\mathscr{N}$ is a subpartition of $\mathscr{J}^*$ and

$$v(N) > \sum_{i \in N} y_i,$$

where N must be a subset of J. If $\mathscr{N}$ was chosen in such a way that $\mathscr{N} \in \mathbf{K}_{ef}$, then the problem is solved. If $\mathscr{N} \in \mathbf{K} - \mathbf{K}_{ef}$, we repeat the described procedure as long as we necessarily reach, after finite number of steps, a coalition structure $\mathscr{M} \in \mathbf{K}$ and a coalition $M \in \mathscr{M}$ such that $M \subset N \subset J$, $\mathscr{M}$ is a subpartition of $\mathscr{N}$ (and of $\mathscr{J}$, too),

$$v(M) > \sum_{i \in M} y_i > \sum_{i \in M} x_i,$$

and $\mathscr{M}$ is effective from below, $\mathscr{M} \in \mathbf{K}_{ef}$.

The auxiliary statement is proved, and we may start the proof of the Theorem. Let $\mathscr{L} \in \mathbf{S}$ and let $\Delta(\mathscr{L}) \neq \emptyset$. Then there exists a set $\{\mathscr{M}_1, \dots, \mathscr{M}_n\} \subset \mathbf{K}_{\mathrm{ef}}$ such that

$$\mathscr{M}_1 \in \Delta(\mathscr{L}), \quad \mathscr{M}_r \in \Delta(\mathscr{M}_{r-1}), \quad r = 2, \dots, n.$$

By Lemma 6.3, also $\mathscr{M}_1 \in \Delta(\mathscr{K})$. Because $\mathscr{L} \in \mathbf{S}$, there exists a set $\{\mathscr{M}_{n+1}, \dots, \mathscr{M}_p\} \subset \subset \mathbf{K}_{\mathrm{ef}}$, such that

$$\mathscr{M}_r \in \Delta(\mathscr{M}_{r-1}), \quad r = n+1, \dots, p, \quad \mathscr{L} \in \Delta(\mathscr{M}_p).$$

By Lemma 6.4 also $\mathscr{K} \in \Delta(\mathscr{M}_p)$. Let us consider an arbitrary set $\{\mathscr{N}_1, \dots, \mathscr{N}_m\} \subset \mathbf{K}_{\mathrm{ef}}$ such that

$$\mathscr{N}_1 \in \Delta(\mathscr{K}), \quad \mathscr{N}_s \in \Delta(\mathscr{N}_{s-1}), \quad s = 2, \dots, m. \tag{6.9}$$

Then there may be constructed a set

$$\mathbf{L} = \{\mathscr{L}, \mathscr{M}_1, \dots, \mathscr{M}_n, \mathscr{M}_{n+1}, \dots, \mathscr{M}_p, \mathscr{K}, \mathscr{N}_1, \dots, \mathscr{N}_m\}.$$

Because $\mathscr{L} \in \mathbf{S}$, the existence of the set $\mathbf{L}$ implies that there exists a set $\{\mathscr{N}_{m+1}, \dots \dots, \mathscr{N}_q\} \subset \mathbf{K}_{\mathrm{ef}}$, such that

$$\mathscr{N}_s \in \Delta(\mathscr{N}_{s-1}), \quad s = m+1, \dots, q, \quad \mathscr{L} \in \Delta(\mathscr{N}_q).$$

By Lemma 6.4, also $\mathscr{K} \in \Delta(\mathscr{N}_q)$. Because of the general validity of the previous consideration for all sets $\{\mathscr{N}_1, \dots, \mathscr{N}_m\} \subset \mathbf{K}_{\mathrm{ef}}$ with the property (6.9), the coalition structure $\mathscr{K}$ is stable. This proved the first statement of the Theorem, for $\mathscr{L} \in \mathbf{S}$, $\Delta(\mathscr{L}) \neq \emptyset$.

Let $\Delta(\mathscr{L}) = \emptyset$. Then $\mathscr{L}\ \sigma\ \mathbf{K}_{\mathrm{ef}}$, by Lemma 5.3, and $\mathscr{K}\ \sigma\ \mathbf{K}_{\mathrm{ef}}$, as follows from Lemma 6.1. Hence, also $\Delta(\mathscr{K}) = \emptyset$, $\mathscr{K} \in \mathbf{S}$.

Let $\Delta(\mathscr{K}) = \emptyset$. Then $\mathscr{K}\ \sigma\ \mathbf{K}_{\mathrm{ef}}$. It means that there exists $\mathbf{x} \in X(\mathscr{K})$ such that $(\mathscr{K}, \mathbf{x})$ is rational, and

$$\sum_{i \in J} x_i \geqq v(J) \quad \text{for all} \quad J \in \{M : M \in \mathscr{M}, \mathscr{M} \in \mathbf{K}_{\mathrm{ef}}\}. \tag{6.10}$$

By Lemma 3.4, $\mathbf{x} \in X(\mathscr{L})$, and (6.10) is going on. It means that the relation $\mathscr{L}\ \sigma\ \mathbf{K}_{\mathrm{ef}}$ is true, if $(\mathscr{L}, \mathbf{x})$ is rational. Let us suppose that $(\mathscr{L}, \mathbf{x})$ is non-rational. Then, by the auxiliary statement, proved in the first part of this proof, there exists $\mathscr{M} \in \mathbf{K}_{\mathrm{ef}}$ and $M \in \mathscr{M}$ such that

$$\sum_{i \in M} x_i < v(M).$$

It is in contradiction with (6.10). Hence $(\mathscr{L}, \mathbf{x})$ is rational, and $\mathscr{L}\ \sigma\ \mathbf{K}_{\mathrm{ef}}$. Consequently, by Lemma 5.3, $\Delta(\mathscr{L}) = \emptyset$.

Besides the just proved Theorem 6.1, we may use another methods for the verification, whether $\mathscr{K}$ and $\mathscr{L}$, satisfying (6.1) are stable or not. The following Remarks represent two equivalent formulations of one such method.

Remark 6.1. Let $\mathscr{K}, \mathscr{L} \in \mathbf{K}$ satisfy (6.1). If there exists $\mathscr{M} \in \mathbf{S}$ such that $\mathscr{L} \in \Delta(\mathscr{M})$, then also $\mathscr{L} \in \mathbf{S}$ and $\mathscr{K} \in \mathbf{S}$. If there exists $\mathscr{N} \in \mathbf{K} - \mathbf{S}$ such that $\mathscr{N} \in \Delta(\mathscr{L})$ then also $\mathscr{L} \notin \mathbf{S}$ and $\mathscr{K} \notin \mathbf{S}$, as follows from Lemma 5.2.

Remark 6.2. Let (6.1) be true for some $\mathscr{K}, \mathscr{L} \in \mathbf{K}$. Let

$$\alpha = \min \{ \sum_{M \in \mathscr{M}} v(M) : \mathscr{M} \in \mathbf{S} \} ,$$

$$\beta = \max \{ \sum_{M \in \mathscr{M}} v(M) : \mathscr{M} \in \mathbf{K}_{\text{ef}} - \mathbf{S} \} ,$$

$$\mu = \sum_{K \in \mathscr{K}} v(K) = \sum_{L \in \mathscr{L}} v(L) \text{ (cf. Lemma 3.4).}$$

Then, by Theorem 5.3, $\mu > \alpha$ implies $\mathscr{L} \in \mathbf{S}$ and $\mathscr{K} \in \mathbf{S}$, and $\mu < \beta$ implies $\mathscr{L} \notin \mathbf{S}$ and $\mathscr{K} \notin \mathbf{S}$.

7. STRICTLY BOUNDED COOPERATION

The bargaining model, and especially the Bargaining Solution on the class of all coalition structures, are very simple for a special kind of coalition-games.

Let $\Gamma = (I, \mathbf{K}, v)$ be a coalition-game. We say that Γ is a *game with strictly bounded cooperation*, iff for any pair of coalition structures $\mathscr{K}, \mathscr{L} \in \mathbf{K}$, $\mathscr{K}$ is a subpartition of $\mathscr{L}$ or $\mathscr{L}$ is a subpartition of $\mathscr{K}$.

For such games Theorem 5.4 can be always applied, and, moreover, the following simple results can be obtained.

Lemma 7.1. Let $\mathscr{K} \in \mathbf{K}_{\text{ef}}$ be such that for any $\mathscr{J} \in \mathbf{K}_{\text{ef}}$ is $\mathscr{J}$ a subpartition of $\mathscr{K}$. Then $\Delta(\mathscr{K}) = \emptyset$.

Proof. It follows from Definition 2 that $\mathscr{K} \in \mathbf{K}_{\text{ef}}^{\text{ef}}$. The condition $\mathscr{K} \in \mathbf{K}_{\text{ef}}$ means, by Remark 3.2, that there exists $\mathbf{x} \in X(\mathscr{K})$ such that for any $\mathscr{J} \in \mathbf{K}$ where $\mathscr{J}$ is a subpartition of $\mathscr{K}$, and for any $J \in \mathscr{J}$, is

$$v(J) \leqq \sum_{i \in J} x_i .$$

It means, by assumptions, that $\mathscr{K} \; \sigma \; \mathbf{K}_{\text{ef}}$, and, by Lemma 5.3, $\Delta(\mathscr{K}) = \emptyset$.

Theorem 7.1. In any game with strictly bounded cooperation always exists a coalition structure with empty value of its domination structure Δ. Moreover, in such games, any coalition structure is effective from below, and it is effective from above if and only if it is stable. In symbols, $\mathbf{K}_{\text{ef}} = \mathbf{K}$, $\{\mathscr{K} \in \mathbf{K} : \Delta(\mathscr{K}) = \emptyset\} \neq \emptyset$,

$$(7.1) \quad \mathbf{S} = \mathbf{K}_{\text{ef}}^{\text{ef}} = \{\mathscr{K} : \sum_{K \in \mathscr{K}} v(K) \geqq \sum_{J \in \mathscr{J}} v(J), \; \textit{for all } \mathscr{J} \in \mathbf{K}\} = \{\mathscr{K} \in \mathbf{K} : \Delta(\mathscr{K}) = \emptyset\} .$$

Proof. Let $\mathscr{K}_1, \mathscr{K}_2, \ldots, \mathscr{K}_m$ be all coalition structures in $\mathbf{K}$, $\mathbf{K} = \{\mathscr{K}_1, \ldots, \mathscr{K}_m\}$, ordered in such a way that for any $r = 1, \ldots, m - 1$ the coalition structures $\mathscr{K}_{r+1}, \ldots$ $\ldots, \mathscr{K}_m$ are subpartitions of $\mathscr{K}_r$, and there is no $\mathscr{K} \in \mathbf{K}$ which would be a subpartition of $\mathscr{K}_m$. It is an immediate consequence of (1.1) that $\mathscr{K}_r \in \mathbf{K}_{\text{ef}}$ for all $r = 1, \ldots, m$. We define the sets

$$X_r^{(r)} = X(\mathscr{K}_r),$$
$$X_s^{(r)} = \{\mathbf{x} \in X(\mathscr{K}_r) : \sum_{i \in K} x_i \geqq v(K) \quad for\ all \quad K \in \bigcup_{p=r}^{s} \mathscr{K}_p\},$$

for all $r = 1, \ldots, m$, $s = r + 1, \ldots, m$. The condition (1.1) implies that $X_s^{(r)} \neq \emptyset$ for all r and s. Consequently, by Remark 3.2, $\mathscr{K}_r \in \mathbf{K}_{\text{ef}}$ for $r = 1, \ldots, m$, and $\mathbf{K}_{\text{ef}} = \mathbf{K}$. It follows from Definition 2 that $\mathscr{K}_1 \in \mathbf{K}_{\text{ef}}^{\text{ef}}$. It implies that $\mathscr{K}_1 \,\sigma\, \mathbf{K}_{\text{ef}}$, as $X_m^{(1)} \neq \emptyset$. According to Lemma 5.3, $\Delta(\mathscr{K}_1) = \emptyset$. It implies, by Theorem 5.2, that $\mathbf{S} = \{\mathscr{K} \in \mathbf{K} :$ $: \Delta(\mathscr{K}) = \emptyset\}$. If for some $\mathscr{K} \in \mathbf{K}$

$$\sum_{K \in \mathscr{K}} v(K) < \sum_{K \in \mathscr{K}_1} v(K),$$

then $\mathscr{K} \notin \mathbf{K}_{\text{ef}}^{\text{ef}}$, and $\Delta(\mathscr{K}) \neq \emptyset$, as follows from Remark 5.1. By Lemma 5.2 also $\mathscr{K} \notin \mathbf{S}$. It means that (7.1) is proved.

8. STABILITY OF CONFIGURATIONS

The 5th section of this paper was devoted to the description of the Bargaining Solution on the class of all coalition structures, which was defined in the last paragraphs of that section. The Bargaining Solution on the class of all coalition structures depends, by means of Definition 3, on the sets of imputations, corresponding to coalition structures. In this section we introduce some notions and statements, necessary for the description of general Bargaining Solution, including also the imputations, respectively the configurations. The concept of the general Bargaining Solution is specified and discussed in the next section and some its special properties are investigated also in the remaining sections of the presented work.

It is obvious that even for the coalition structures, belonging to the Bargaining Solution on the class of all coalition structures, i.e. to the class

$$\mathbf{S} \cap \mathbf{K}_{\text{ef}}^{\text{ef}},$$

some of their imputations are more suitable to act as a final result of bargaining, and some of their imputations are less suitable, or quite unconvenient to this purpose.

Definition 6. We say that the configuration $(\mathscr{K}, \mathbf{x})$, $\mathscr{K} \in \mathbf{K}$, $\mathbf{x} \in X(\mathscr{K})$, is stable, iff

$$\mathscr{K} \in \mathbf{K}_{\text{ef}}^{\text{ef}} \cap \mathbf{S},$$

$$(\mathscr{K}, \mathbf{x}) \quad \text{is rational},$$

if

$$\sum_{i \in M} x_i < v(M) \quad \text{for some} \quad M \in \{J : J \in \mathscr{J}, \mathscr{J} \in \mathbf{K}_{\text{ef}}\}$$

then there exists $\mathscr{M} \in \Delta(\mathscr{K})$ such that $M \in \mathscr{M}$.

Theorem 8.1. If $\mathscr{K}$ is a stable and effective coalition structure, i.e.

$$\mathscr{K} \in \mathbf{S} \cap \mathbf{K}_{\text{ef}}^{\text{ef}},$$

then there exists an imputation $\mathbf{x} \in X(\mathscr{K})$ such that the configuration $(\mathscr{K}, \mathbf{x})$ is stable.

Proof. Let $\mathscr{K}$ be an arbitrary, fixed coalition structure from $\mathbf{S}$, where $\mathbf{S} \neq \emptyset$ by Theorem 5.1. Let us denote for any $\mathbf{M} \subset \mathbf{K}_{\text{ef}}$ the set of imputations

$$Y_{\mathbf{M}}(\mathscr{K}) = \{\mathbf{x} \in X(\mathscr{K}) : (\mathscr{K}, \mathbf{x}) \textit{ is rational, and } \sum_{i \in K} x_i \geqq v(K)$$
$$\textit{for all } K \in \{M : M \in \mathscr{M}, \mathscr{M} \in \mathbf{M}\}\}.$$

It is an immediate consequence of Definition 3 that

$$Y_{\mathbf{M}}(\mathscr{K}) \neq \emptyset \Leftrightarrow \mathscr{K} \,\sigma\, \mathbf{M},$$

and that

$$Y_{\mathbf{M}}(\mathscr{K}) \neq \emptyset$$

for at least one $\mathbf{M} \subset \mathbf{K}_{\text{ef}}$ (e.g. for empty class), because $\mathscr{K} \in \mathbf{S} \subset \mathbf{K}_{\text{ef}}$ (c.f. Remark 5.2). Let us choose some $\mathscr{J} \in \mathbf{K}_{\text{ef}}$ and $\mathbf{M} \subset \mathbf{K}_{\text{ef}}$, such that

$$Y_{\mathbf{M}}(\mathscr{K}) \neq \emptyset.$$

Then

$$Y_{(\mathbf{M} \cup \{\mathscr{J}\})}(\mathscr{K}) \subset Y_{\mathbf{M}}(\mathscr{K}).$$

If

$$Y_{(\mathbf{M} \cup \{\mathscr{J}\})}(\mathscr{K}) = \emptyset$$

then $\mathscr{J} \in \Delta(\mathscr{K})$, and $\mathscr{J} \in \mathbf{S}$, by Lemma 5.4.

There are two possibilities for the coalition structure $\mathscr{K}$. Either

$$Y_{\mathbf{M}}(\mathscr{K}) \neq \emptyset \quad \text{even for} \quad \mathbf{M} = \mathbf{K}_{\text{ef}},$$

or there exists $\mathbf{L} \subset \mathbf{K}_{\text{ef}}$ such that

$$Y_{\mathbf{L}}(\mathscr{K}) = \emptyset.$$

In the former case $\Delta(\mathscr{K}) = \emptyset$, by Lemma 5.1, and $(\mathscr{K}, \mathbf{x})$ is obviously stable for any

$$\mathbf{x} \in Y_{\mathbf{K}_{\text{ef}}}(\mathscr{K}).$$

In the latter case we can choose at least one (but, generally, not only one) class $\mathbf{M} \subset \mathbf{K}_{ef}$, for which

$$Y_{\mathbf{M}}(\mathscr{K}) \neq \emptyset ,$$

and

$$Y_{(\mathbf{M} \cup \{\mathscr{J}\})}(\mathscr{K}) = \emptyset \quad \text{for any} \quad \mathscr{J} \in \mathbf{K}_{ef} - \mathbf{M} .$$

It means that for any $\mathscr{J} \in \mathbf{K}_{ef} - \mathbf{M}$ is $\mathscr{J} \in \Delta(\mathscr{K})$, and, consequently, for any

$$\mathbf{x} \in Y_{\mathbf{M}}(\mathscr{K}) \neq \emptyset$$

the configuration $(\mathscr{K}, \mathbf{x})$ fulfills the last two conditions of Definition 3. Because of the general validity of the previous consideration for all $\mathscr{K} \in \mathbf{S}$, it is also true for all coalition structures being stable and effective. Consequently, for any

$$\mathscr{K} \in \mathbf{S} \cap \mathbf{K}_{ef}^{ef}$$

there exists $\mathbf{x} \in X(\mathscr{K})$ such that the configuration $(\mathscr{K}, \mathbf{x})$ is stable.

We shall introduce a new symbol. For any $\mathscr{K} \in \mathbf{K}$ we denote the set of imputations

(8.1) $$X^*(\mathscr{K}) = \{\mathbf{x} \in X(\mathscr{K}) : (\mathscr{K}, \mathbf{x}) \textit{ is stable}\} .$$

Remark 8.1. It is an immediate consequence of Definition 6 and the previous Theorem 8.1 that for any $\mathscr{K} \in \mathbf{K}$

$$X^*(\mathscr{K}) \neq \emptyset \Leftrightarrow \mathscr{K} \in \mathbf{S} \cap \mathbf{K}_{ef}^{ef} ,$$

and consequently,

(8.2) $$\bigcup_{\mathscr{K} \in \mathbf{K}} X^*(\mathscr{K}) = \bigcup_{\mathscr{K} \in \mathbf{S}} X^*(\mathscr{K}) = \bigcup_{\mathscr{K} \in \mathbf{K}_{ef}{}^{ef}} X^*(\mathscr{K}) = \bigcup_{\mathscr{K} \in \mathbf{S} \cap \mathbf{K}^{ef}{}_{ef}} X^*(\mathscr{K}) .$$

Remark 8.2. Definitions 1 and 6 imply that for any $\mathscr{K} \in \mathbf{S} \cap \mathbf{K}_{ef}^{ef}$

(8.3) $$X^*(\mathscr{K}) = \{\mathbf{x} \in X(\mathscr{K}) : \sum_{i \in K} x_i \geqq v(K) \quad \textit{for all } K \in \mathfrak{A}(\mathscr{K})\} ,$$

where

(8.4) $$\mathfrak{A}(\mathscr{K}) = \{K : K \in \mathscr{L} \in \mathbf{K}_{ef} \Rightarrow \mathscr{L} \in \mathbf{K}_{ef} - \Delta(\mathscr{K})\} \cup$$
$$\cup \{J : J \in \mathscr{J} \textit{ for some } \mathscr{J} \in \mathbf{K}, \mathscr{J} \textit{ is a subpartition of } \mathscr{K}\} .$$

Lemma 8.1. The set $X^*(\mathscr{K})$ is convex for any $\mathscr{K} \in \mathbf{K}$.

Proof. Let $\mathscr{K} \in \mathbf{K}$, let k be the number of elements (coalitions) in $\mathscr{K}$, and let n be the number of all players on the set I. Then $X(\mathscr{K})$ is an $(n - k)$-dimensional subspace of the n-dimensional Euclidian space (c.f. Section 2). If $\mathscr{K} \notin \mathbf{S} \cap \mathbf{K}_{ef}^{ef}$ then $X^*(\mathscr{K})$ is empty. If $\mathscr{K} \in \mathbf{S} \cap \mathbf{K}_{ef}^{ef}$, then

$$X^*(\mathscr{K}) = \{\mathbf{x} \in X(\mathscr{K}) : \sum_{i \in K} x_i \geqq v(K) \text{ for all } K \in \mathfrak{A}(\mathscr{K})\} ,$$

where $\mathfrak{A}(\mathscr{K})$ is given by (8.4). It means that $X^*(\mathscr{K})$ is a subset of the Euclidian subspace $X(\mathscr{K})$, restricted by finite many linear inequalities. Consequently, it is a convex subset of $X(\mathscr{K})$.

Lemma 8.2. If the coalition structure $\mathscr{K}_0 = \{\{i\}\}_{i \in I}$, containing exactly all one-element coalitions, is admissible in the considered game $\Gamma = (I, \mathbf{K}, v)$, i.e. if $\mathscr{K}_0 \in \mathbf{K}$, then for any $\mathscr{K} \in \mathbf{K}$ the set $X^*(\mathscr{K})$ is a compact convex subset of an Euclidian space.

Proof. For any $\mathscr{K} \notin \mathbf{S} \cap \mathbf{K}_{ef}^{ef}$, the set $X^*(\mathscr{K})$ is empty, and for any $\mathscr{K} \in \mathbf{S} \cap \mathbf{K}_{ef}^{ef}$ the set $X^*(\mathscr{K})$ is given by (8.3) and (8.4). For all $i \in I$ the coalition $\{i\} \in \mathfrak{A}(\mathscr{K})$ (c.f. (8.4)). Hence, $X^*(\mathscr{K})$ is a bounded subset of $(n - k)$-dimensional subspace $X(\mathscr{K})$ of the n-dimensional Euclidian space, (where n is the number of players in I and k is the number of coalitions in $\mathscr{K}$), restricted by finite many linear inequalities, among which are also the inequalities

$$x_i \geqq v(\{i\}) \quad \text{for all} \quad i \in I .$$

Moreover, the form of inequalities implies that the set $X^*(\mathscr{K})$ is closed (as the inequalities are not strict). Consequently, $X^*(\mathscr{K})$ is a compact and covex subset of $X(\mathscr{K})$.

Lemma 8.2. Let $\mathscr{K}, \mathscr{L} \in \mathbf{K}$ be such that $\Delta(\mathscr{K}) = \Delta(\mathscr{L}) = \emptyset$. Then

$$X^*(\mathscr{K}) = X^*(\mathscr{L}) .$$

Proof. By Lemma 5.5,

$$\sum_{K \in \mathscr{K}} v(K) = \sum_{L \in \mathscr{L}} v(L) .$$

Lemma 5.1 implies that $\mathscr{K}\ \sigma\ \mathbf{K}_{ef}$ and $\mathscr{L}\ \sigma\ \mathbf{K}_{ef}$. Let $\mathbf{x} \in X^*(\mathscr{K})$. Then

$$\sum_{i \in J} x_i \geqq v(J) \quad \text{for all} \quad J \in \{M : M \in \mathscr{M}, \mathscr{M} \in \mathbf{K}_{ef}\} . \tag{8.5}$$

It means that also

$$\sum_{i \in L} x_i \geqq v(L) \quad \text{for all} \quad L \in \mathscr{L} ,$$

and, consequently,

$$\sum_{i \in L} x_i = v(L) \quad \text{for all} \quad L \in \mathscr{L} , \tag{8.6}$$

since

$$\sum_{K \in \mathscr{K}} v(K) = \sum_{i \in I} x_i = \sum_{L \in \mathscr{L}} \sum_{i \in L} x_i \geqq \sum_{L \in \mathscr{L}} v(L) .$$

The equality (8.6) implies that $\mathbf{x} \in X(\mathscr{L})$, and, as (8.5) is true, also $\mathbf{x} \in X^*(\mathscr{L})$. Hence, $\mathbf{x} \in X^*(\mathscr{K})$ implies $\mathbf{x} \in X^*(\mathscr{L})$. The same procedure proves that $\mathbf{x} \in X^*(\mathscr{L})$ implies $\mathbf{x} \in X^*(\mathscr{K})$.

Theorem 8.2. If there exists at least one coalition structure $\mathscr{K}$ with empty value of domination structure, $\Delta(\mathscr{K}) = \emptyset$, then

$$(8.7) \qquad \bigcup_{\mathscr{M} \in \mathbf{K}} X^*(\mathscr{M}) = \bigcup_{\mathscr{M} \in \mathbf{S} \cap \mathbf{K}_{\mathrm{ef}}{}^{\mathrm{ef}}} X^*(\mathscr{M}) = \bigcap_{\mathscr{M} \in \mathbf{S}} X^*(\mathscr{M}) = X^*(\mathscr{K})$$

for all $\mathscr{K} \in \mathbf{K}$, and the set (8.7) is a convex set.

If for all $\mathscr{K} \in \mathbf{K}$ is $\Delta(\mathscr{K}) \neq \emptyset$ then the set

$$\bigcup_{\mathscr{K} \in \mathbf{K}} X^*(\mathscr{K})$$

is a union of finite many convex sets.

Proof. Let $\mathscr{K}, \mathscr{L} \in \mathbf{K}$ and let $\Delta(\mathscr{K}) = \Delta(\mathscr{L}) = \emptyset$. Then, by Lemma 8.3, $X^*(\mathscr{K}) = X^*(\mathscr{L})$. Let us denote

$$\mathbf{B} = \{\mathscr{K} : \mathscr{K} \in \mathbf{K}, \Delta(\mathscr{K}) = \emptyset\}.$$

According to Theorem 5.2 and Remark 5.1,

$$(8.8) \qquad \mathbf{B} = \mathbf{S} \cap \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}} = \mathbf{S} \Leftrightarrow \mathbf{B} \neq \emptyset.$$

Consequently,

$$(8.9) \qquad \bigcup_{\mathscr{M} \in \mathbf{K}} X^*(\mathscr{M}) = \bigcup_{\mathscr{M} \in \mathbf{B}} X^*(\mathscr{M}) = \bigcap_{\mathscr{M} \in \mathbf{B}} X^*(\mathscr{M}) = X^*(\mathscr{K})$$

$$\text{for all} \quad \mathscr{K} \in \mathbf{S} = \mathbf{B}.$$

The relations (8.8) and (8.9) prove the relation (8.7). Lemma 8.1 implies that the set (8.9) is a convex set. If $\mathbf{B} = \emptyset$ then the set

$$\bigcup_{K \in \mathbf{K}} X^*(\mathscr{K})$$

is a union of finite many sets $X^*(\mathscr{K})$, which are convex, by Lemma 8.1.

Corollary. Let the coalition structure $\mathscr{K}_o = \{\{i\}\}_{i \in I}$, belongs to the class $\mathbf{K}$, and let n be the number of players in the set I. Then Theorem 8.2 and Lemma 8.2 imply that

$$\bigcup_{\mathscr{K} \in \mathbf{K}} X^*(\mathscr{K})$$

is a union of finite many convex compact sets in n-dimensional Euclidian space; if, moreover, there exists $\mathscr{K} \in \mathbf{K}$ with $\Delta(\mathscr{K}) = \emptyset$ then the set

$$\bigcup_{\mathscr{K} \in \mathbf{K}} X^*(\mathscr{K}) = \bigcap_{\mathscr{K} \in \mathbf{S}} X^*(\mathscr{K})$$

is a convex compact set in n-dimensional Euclidian space.

Remark 8.3. If $\{\mathscr{K} : \mathscr{K} \in \mathbf{K}, \Delta(\mathscr{K}) = \emptyset\} \neq \emptyset$ then the set

$$\bigcup_{\mathscr{K} \in \mathbf{K}} X^*(\mathscr{K}) = \bigcap_{\mathscr{K} \in \mathbf{S}} X^*(\mathscr{K})$$

is a subset of $(n - \lambda)$-dimensional subspace of n-dimensional Euclidian space, where n is the number of players in I, and λ is the number of coalitions in the set $\{L : L \in \mathscr{L}, \mathscr{L} \in \mathbf{K}, \Delta(\mathscr{L}) = \emptyset\}$, as follows from (8.6) for all $\mathscr{L} \in \mathbf{S}$, $L \in \mathscr{L}$.

We have specified the properties of imputations, which may occur among possible results of bargaining, and we are able to formulate the complete and general Bargaining Solution.

9. THE BARGAINING SOLUTION

The concept of the Bargaining Solution on the class of all coalition structures was already introduced in the 5th Section of this paper. Now, we are going to formulate the general Bargaining Solution of the coalition-game, i.e. the Bargaining Solution on the set of all admissible configurations.

We say that the configuration $(\mathscr{K}, \mathbf{x})$, $\mathscr{K} \in \mathbf{K}$, $\mathbf{x} \in X(\mathscr{K})$, belongs to the *Bargaining Solution of the coalition-game*

$$\Gamma = (I, \mathbf{K}, v)$$

if and only if $(\mathscr{K}, \mathbf{x})$ is stable in the sense of Definition 6.

It is obvious that the Bargaining Solution, defined in this way, corresponds with the intuitive idea of an expected bargaining result. The configuration $(\mathscr{K}, \mathbf{x})$ belongs to the Bargaining Solution, if the coalition structure $\mathscr{K}$ belongs to the Bargaining Solution on the class of all coalition structures $\mathbf{K}$, if $(\mathscr{K}, \mathbf{x})$ is rational, and if it can not be broken by any coalition structure (with its imputations) which does not belong to the value of the domination structure Δ for $\mathscr{K}$. According to Lemma 5.2, the last condition means that the configuration $(\mathscr{K}, \mathbf{x})$, belonging to the Bargaining Solution, can not be broken by any coalition structure, which is not stable in the sense of Definition 5.

The following section contains some special formulations of the general results, obtained in the 8th Section, which correspond with the special situations in coalition-games with strictly bounded cooperation and in games with effective subpartitions of effective coalition structures.

The coalition-games, in which effective subpartitions of effective coalition structures exist, were investigated in Section 6. The following Theorem describes the properties of sets $X^*(\mathscr{L})$ and $X^*(\mathscr{K})$ (c.f. (8.1)), if $\mathscr{K}$ and $\mathscr{L}$ fulfil the condition (6.1).

Theorem 10.1. Let $\mathscr{K}$, $\mathscr{L} \in \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}$ be effective coalition structures, such that $\mathscr{K}$ is a subpartition of $\mathscr{L}$. Then any imputation $\mathbf{x}$ forms a stable configuration with $\mathscr{K}$, if it forms a stable configuration with $\mathscr{L}$; in symbols

$$X^*(\mathscr{L}) \subset X^*(\mathscr{K}) .$$

Proof. Theorem 6.1 implies that

$$\mathscr{L} \in \mathbf{S} \cap \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}} \Rightarrow \mathscr{K} \in \mathbf{S} \cap \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}} .$$

Consequently, if $X^*(\mathscr{L}) \neq \emptyset$ then $X^*(\mathscr{K}) \neq \emptyset$, and

$$X^*(\mathscr{K}) = \{\mathbf{x} \in X(\mathscr{K}) : \sum_{i \in M} x_i \geqq v(M) \quad \textit{for all} \quad M \in \mathfrak{A}(\mathscr{K})\} ,$$

$$X^*(\mathscr{L}) = \{\mathbf{x} \in X(\mathscr{L}) : \sum_{i \in M} x_i \geqq v(M) \quad \textit{for all} \quad M \in \mathfrak{A}(\mathscr{L})\} ,$$

where

$$\mathfrak{A}(\mathscr{K}) = \{J : J \in \mathscr{J} \in \mathbf{K}_{\mathrm{ef}} \Rightarrow \mathscr{J} \in \mathbf{K}_{\mathrm{ef}} - \Delta(\mathscr{K})\} \cup$$
$$\cup \{J : J \in \mathscr{J} \textit{ for some } \mathscr{J} \in \mathbf{K}, \mathscr{J} \textit{ is a subpartition of } \mathscr{K}\} ,$$

$$\mathfrak{A}(\mathscr{L}) = \{J : J \in \mathscr{J} \in \mathbf{K}_{\mathrm{ef}} \Rightarrow \mathscr{J} \in \mathbf{K}_{\mathrm{ef}} - \Delta(\mathscr{L})\} \cup$$
$$\cup \{J : J \in \mathscr{J} \textit{ for some } \mathscr{J} \in \mathbf{K}, \mathscr{J} \textit{ is a subpartition of } \mathscr{L}\} ,$$

(c.f. Remark 8.2).

Because $\mathscr{K}$ is a subpartition of $\mathscr{L}$,

$$\{J : J \in \mathscr{J} \textit{ for some } \mathscr{J} \in \mathbf{K}, \mathscr{J} \textit{ is a subpartition of } \mathscr{K}\} \subset$$
$$\subset \{J : J \in \mathscr{J} \textit{ for some } \mathscr{J} \in \mathbf{K}, \mathscr{J} \textit{ is a subpartition of } \mathscr{L}\} ,$$

and

$$\{J : J \in \mathscr{J} \in \mathbf{K}_{\mathrm{ef}} \Rightarrow \mathscr{J} \in \mathbf{K}_{\mathrm{ef}} - \Delta(\mathscr{K})\} \subset$$
$$\subset \{J : J \in \mathscr{J} \in \mathbf{K}_{\mathrm{ef}} \Rightarrow \mathscr{J} \in \mathbf{K}_{\mathrm{ef}} - \Delta(\mathscr{L})\} ,$$

because Lemma 6.3 implies that

$$\mathbf{K}_{ef} - \Delta(\mathscr{L}) \supset \mathbf{K}_{ef} - \Delta(\mathscr{K}) .$$

It means that

$$\mathfrak{A}(\mathscr{L}) \supset \mathfrak{A}(\mathscr{K}) ,$$

and, consequently,

$$X^*(\mathscr{L}) \subset X^*(\mathscr{K}) .$$

Also the coalition-games with strictly bounded cooperation were already investigated, namely in the 7th section of this paper. The stable configurations in coalition-games with strictly bounded cooperation fulfil the statement of the following Theorem.

Theorem 10.2. Let the considered coalition-game $\Gamma = (I, \mathbf{K}, v)$ be a game with strictly bounded cooperation, and let $\mathscr{M} \in \mathbf{K}$ be the coalition structure, for which any $\mathscr{K} \in \mathbf{K}$ is a subpartition of $\mathscr{M}$. Then the imputation $\mathbf{x}$ forms a stable configuration with some coalition structure $\mathscr{K} \in \mathbf{K}$, if and only if $\mathbf{x}$ forms a rational configuration with $\mathscr{M}$; in symbols

$$\bigcup_{\mathscr{K} \in \mathbf{K}} X^*(\mathscr{K}) = X^*(\mathscr{M}) = \{\mathbf{x} \in X(\mathscr{M}) : (\mathscr{M}, \mathbf{x}) \text{ is rational}\} .$$

Proof. Let us denote $\mathbf{B} = \{\mathscr{K} \in \mathbf{K} : \Delta(\mathscr{K}) = \emptyset\}$. Then, by Theorem 7.1,

$$\emptyset \neq \mathbf{B} = \mathbf{S} = \mathbf{K}_{ef}^{ef} = \{\mathscr{K} : \sum_{K \in \mathscr{K}} v(K) \geqq \sum_{J \in \mathscr{J}} v(J) \text{ for all } \mathscr{J} \in \mathbf{K}\} .$$

It means, that $\mathscr{M} \in \mathbf{B}$, and, by Theorem 8.2,

$$\bigcup_{\mathscr{K} \in \mathbf{K}} X^*(\mathscr{K}) = X^*(\mathscr{M}) .$$

$$X^*(\mathscr{M}) = \{\mathbf{x} \in X(\mathscr{M}) : \sum_{i \in K} x_i \geqq v(K) \text{ for all } K \in \mathfrak{A}(\mathscr{M})\}$$

where

$$\mathfrak{A}(\mathscr{M}) = \bigcup_{\mathscr{K} \in \mathbf{K}} \mathscr{K} ,$$

(c.f. Remark 8.2).

It means, by Remark 3.2, that

$$X^*(\mathscr{M}) = \{\mathbf{x} \in X(\mathscr{M}) : (\mathscr{M}, \mathbf{x}) \text{ is rational}\} .$$

11. BARGAINING SOLUTION AND CORE

The concept of Core is well-known and often used in the coalition-games theory. It is interesting for us to verify, if, and how, corresponds our model of bargaining with that important notion.

If $\Gamma = (I, \mathbf{K}, v)$ is a coalition-game then the Core of the game Γ is the following set of imputations

$$C = \{\mathbf{x} = (x_i)_{i\in I} : \mathbf{x} \in X(\mathscr{K}) \ for\ some\ \mathscr{K} \in \mathbf{K}, \ \sum_{i\in K} x_i \geqq v(K) \ for\ all\ K \in \bigcap_{\mathscr{K}\in\mathbf{K}} \mathscr{K}\} .$$

The correspondence between the Core and the Bargaining Solution exists and it is very strong.

Theorem 11.1. Let $\Gamma = (I, \mathbf{K}, v)$ be a coalition-game, let C be the Core of Γ, and let us denote by $\mathbf{B}$ the class of all coalition structures with empty value of domination structure Δ, $\mathbf{B} = \{\mathscr{K} \in \mathbf{K} : \Delta(\mathscr{K}) = \emptyset\}$. Then the Core is non-empty, if and only if there exists at least one coalition structure with empty value of domination structure, and, moreover, any imputation $\mathbf{x}$ belongs to the core, if and only if it forms a stable configuration with some coalition structure from $\mathbf{B}$; in symbols

$$C = \emptyset \qquad \text{iff} \quad \mathbf{B} = \emptyset ,$$
$$C = \bigcap_{\mathscr{K}\in\mathbf{B}} X^*(\mathscr{K}) \neq \emptyset \quad \text{iff} \quad \mathbf{B} \neq \emptyset .$$

Proof. Let $C = \emptyset$. Then for any

$$\mathbf{x} \in \bigcup_{\mathscr{K}\in\mathbf{K}} X(\mathscr{K})$$

there exists a coalition $K \in \{M : M \in \mathscr{M}, \mathscr{M} \in \mathbf{K}\}$, such that

$$v(K) > \sum_{i\in K} x_i .$$

It means that there exists a real vector $\mathbf{y} = (y_i)_{i\in K}$ such that

$$\sum_{i\in K} y_i = v(K), \ y_i > x_i \quad \text{for all} \quad i \in K .$$

In this situation, either $K \in \{M : M \in \mathscr{M}, \mathscr{M} \in \mathbf{K}_{\mathrm{ef}}\}$ or $K \in \{M : M \in \mathscr{M}\ for\ some\ \mathscr{M} \in \mathbf{K}\} - \{M : M \in \mathscr{M}\ for\ some\ \mathscr{M} \in \mathbf{K}_{\mathrm{ef}}\}$, and then for any such $\mathbf{y} = (y_i)_{i\in K}$ there exists a coalition $J \in \{M : M \in \mathscr{M}\ for\ some\ \mathscr{M} \in \mathbf{K}_{\mathrm{ef}}\}$, such that $J \subset K$

$$v(J) > \sum_{i\in J} y_i > \sum_{i\in J} x_i .$$

Consequently, if $C = \emptyset$ then for any

$$\mathbf{x} \in \bigcup_{\mathscr{K}\in\mathbf{K}} X(\mathscr{K})$$

there exists $L \in \{M : M \in \mathscr{M}\ for\ some\ \mathscr{M} \in \mathbf{K}_{\mathrm{ef}}\}$, such that

$$v(L) > \sum_{i\in L} x_i .$$

By Lemma 5.1 (and its Corollary), there exists no $\mathscr{K} \in \mathbf{K}$ such that $\Delta(\mathscr{K}) = \emptyset$. It means that $\mathbf{B} = \emptyset$.

Let $C \neq \emptyset$. Then there exists

$$\mathbf{x} \in \bigcup_{\mathscr{K} \in \mathbf{K}} X(\mathscr{K})$$

such that

$$\sum_{i \in K} x_i \geqq v(K) \quad \text{for all} \quad K \in \bigcup_{\mathscr{L} \in \mathbf{K}} \mathscr{L} \supset \bigcup_{\mathscr{L} \in \mathbf{K}_{ef}} \mathscr{L} .$$

Lemma 5.1 implies that there exists $\mathscr{K} \in \mathbf{K}$ such that $\Delta(\mathscr{K}) = \emptyset$, and, consequently, $\mathbf{B} \neq \emptyset$.

If $\mathbf{x} \in C$, then, by the previous step of this proof, there exists $\mathscr{K} \in \mathbf{K}$ such that $(\mathscr{K}, \mathbf{x})$ is stable, so that $\mathscr{K} \in \mathbf{S} = \mathbf{B}$ and $\mathbf{x} \in X^*(\mathscr{K})$. According to Theorem 8.2, $\mathbf{x} \in X^*(\mathscr{L})$ for all $\mathscr{L} \in \mathbf{L}$; it means that

$$C \subset \bigcap_{\mathscr{K} \in \mathbf{B}} X^*(\mathscr{K}) .$$

If $\mathbf{x} \notin C$ then there exists $K \in \{M : M \in \mathscr{M} \textit{ for some } \mathscr{M} \in \mathbf{K}\}$, such that

$$v(K) > \sum_{i \in K} x_i . \tag{11.1}$$

In the previous part of this proof, we have shown that (11.1) is true also for at least one $L \in \{M : M \in \mathscr{M} \textit{ for some } \mathscr{M} \in \mathbf{K}_{ef}\}$, and by Lemma 5.1 (and its Corollary), if $\mathscr{K} \in \mathbf{B}$ and $\mathbf{x} \in X(\mathscr{K})$ then $\mathbf{x} \notin X^*(\mathscr{K})$. It means that $C \supset X^*(\mathscr{K})$. Theorem 8.2 implies that

$$C \supset \bigcap_{\mathscr{K} \in \mathbf{B}} X^*(\mathscr{K}) .$$

CONCLUSIVE COMMENTS

The method, described in this work, enables us to find out the stable coalition structures and corresponding configurations in the following way.

First of all we construct the sets $X(\mathscr{K})$ for any coalition structure $\mathscr{K} \in \mathbf{K}$ (c.f. (2.1)), and then we find all coalition structures, being effective from below, effective from above, and effective.

Then we find out for any $\mathscr{K} \in \mathbf{K}_{ef}$ and for any class $\mathbf{M} \subset \mathbf{K}_{ef} - \{\mathscr{K}\}$, whether

$$\mathscr{K}\ \sigma\ \mathbf{M} \quad \text{or} \quad \mathscr{K}\ \text{non}\ \sigma\ \mathbf{M} .$$

After that, we construct the mapping Δ, i.e. the sets $\Delta(\mathscr{K})$ for all $\mathscr{K} \in \mathbf{K}_{ef}$, and we find the classes

$$\mathbf{S} \quad \text{and} \quad \mathbf{S} \cap K_{ef}^{ef} ,$$

using Definitions and Theorems, introduced in the 5th Section, respectively also the Theorems, introduced in the 6th or 7th Section.

Finally, we find out the set of coalitions $\mathfrak{A}(\mathcal{K})$ and the set of imputations $X^*(\mathcal{K})$ (c.f. (8.3) and (8.4)) for any

$$\mathcal{K} \in \mathbf{K}_{cf}^{cf} \cap \mathbf{S} .$$

Results, obtained by the proposed method include the concept of Core, as their special case. Moreover, it is not difficult to verify that for 3-persons coalition-games the results, obtained by this method, correspond with the results, introduced in [3]. The correspondence between our results, i.e. our Bargaining Solution, and the intuitive idea of the rational bargaining result, was discussed in the Section 9.

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