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Jiří Cochlar

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Optimum Stopping Rules on the Sequence of Statistically Dependent Vectors

JIŘÍ COCHLAR

The paper deals with the problem of determining the optimum stopping rules on the sequence of statistically dependent vectors. The theory of optimum stopping rules on Markov sequences is used for a solution of this problem. For this purpose, there is a case of statistically dependent vectors transferred to the case of homogeneous Markov sequence in chapter 2. Then in chapter 3 general equations determining the optimum stopping rule are developed. Chapter 4 deals with an application of the obtained theory to the problem of the sequential test of the finite number of hypotheses for statistically dependent observations. The complete solution of obtained equations for a special case of conditionally uniformly distributed observations is introduced in chapter 5.

1. INTRODUCTION

We shall deal with the following problem. The sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ of statistically dependent random vectors is given, where $\mathbf{x}_n$ is an element of M-dimensional Euclidean space for every integer $n > 0$. Further for every integer $n > 0$ a random variable y_n is given by the relation

$$y_n \triangleq g_n(\mathbf{x}_1, \ldots, \mathbf{x}_n)$$

where g_n is a given function. We shall interpret the value $y_n = g_n(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ as a gain which we obtain by interrupting the process of observation the random vectors $\mathbf{x}_i$ in the n-th step. By a stopping rule τ we shall understand some positive integer random variable which "does not depend on the future", i.e., an event $\{\tau = n\}$ can depend only on events concerning the values $\mathbf{x}_1, \ldots, \mathbf{x}_n$. (Exact definitions will be given later.) By the optimum stopping rule τ^* we shall then understand a stopping rule (if such a stopping rule exists) for which it holds

$$M(y_{\tau^*}) = \sup_{\tau \in \mathscr{C}} M(y_\tau)$$

where $\mathscr{C}$ is the set of all stopping rules for which $M(y_\tau)$ exists, M is the expected value. Our paper will deal with the problems of existence and description of the optimum stopping rules.

Chapters 2 and 3 deal with the general solution of the mentioned problem using a theory of the optimum stopping rules on the Markov sequences. In chapters 4 and 5 general results are applied to the problem of the sequential test of a finite number of disjoint hypotheses when observations are statistically dependent.

Further we shall give exact definitions of concepts we shall use in the following chapters. Let

$$N \triangleq \{1, 2, \ldots\}$$

and let a measurable space $(E_n, \mathscr{B}_n)$ be given for every $n \in N$. We shall assume that it holds

$$E_n \triangleq E\,, \quad \mathscr{B}_n \triangleq \mathscr{B}$$

for every $n \in N$, where E is M-dimensional Euclidean space ($M > 0$ is fixed) and $\mathscr{B}$ is a σ-algebra of Borel sets in E. For any $n \in N$ we shall denote

$$E^n \triangleq \prod_{\{1 \leqq t \leqq n\}} E_t \triangleq E_1 \times \ldots \times E_n$$

$$\mathscr{B}^n \triangleq \bigotimes_{\{1 \leqq t \leqq n\}} \mathscr{B}_t \triangleq \mathscr{B}_1 \otimes \ldots \otimes \mathscr{B}_n\,.$$

We shall define now a measurable space $(\Omega, \mathscr{F})$ by the relations

$$\Omega \triangleq \prod_N E_n \triangleq E_1 \times E_2 \times \ldots$$

$$\mathscr{F} \triangleq \bigotimes_N \mathscr{B}_n \triangleq \mathscr{B}_1 \otimes \mathscr{B}_2 \otimes \ldots$$

Let the vector $\mathbf{x}_n$ be defined on the space $(\Omega, \mathscr{F})$ for every $n \in N$ by the relation

$$\mathbf{x}_n(\omega) \triangleq \mathbf{e}_n \quad \text{for} \quad \omega = (\mathbf{e}_1, \ldots, \mathbf{e}_n, \ldots)\,, \quad \mathbf{e}_i \in E_i$$

and analogously, let the n-tuple $\mathscr{X}_n$ be given by the relation

$$\mathscr{X}_n(\omega) \triangleq (\mathbf{e}_1, \ldots, \mathbf{e}_n) \quad \text{for} \quad \omega = (\mathbf{e}_1, \ldots, \mathbf{e}_n, \ldots) \quad \mathbf{e}_i \in E_i\,.$$

It is clear that the vector $\mathbf{x}_n$ and the n-tuple $\mathscr{X}_n$, respectively, are $\mathscr{F}/\mathscr{B}_n$ and $\mathscr{F}/\mathscr{B}^n$-measurable transformations on $(\Omega, \mathscr{F})$ with values in E_n and E^n, respectively. It holds $\mathscr{X}_n(\omega) = (\mathbf{x}_1(\omega), \ldots, \mathbf{x}_n(\omega))$.

Let us assume that the joint probability density function $\mathrm{w}_n(\mathscr{X}_n)$ of an n-tuple $\mathscr{X}_n$ is given and also the conditional probability density function $\mathrm{w}_{k|n}(\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{n+k} \mid \mathscr{X}_n)$ of a k-tuple $(\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{n+k})$ is givem for an arbitrary $k \in N$ and for the given n-tuple $\mathscr{X}_n$. Let us assume that functions w_n and $\mathrm{w}_{k|n}$ satisfy the conditions allowing to define transition probabilities.

As a consequence of the Tulcea's theorem (see [1], chapter V., Corollary 2 of Theorem V.1.1), there exists one and only one probability measure $\boldsymbol{P}$ on the space

$(\Omega, \mathscr{F})$ induced by the mentioned probability density functions. Everywhere below, we shall interpret both vectors $\mathbf{x}_n$ and the n-tuple $\mathscr{X}_n$ as random elements defined on the basic probabilistic space $(\Omega, \mathscr{F}, \boldsymbol{P})$.

According to [1] we shall every set A for which it holds

$$A \triangleq \prod_N B_n \triangleq \{\omega = (\mathbf{e}_1, \ldots, \mathbf{e}_n, \ldots) : \mathbf{e}_n \in B_n;\ n \in N\} \quad B_n \in \mathscr{B}_n$$

denote as the measurable rectangle in Ω, whilst we assume that the sets B_n differ from E_n only for the finite set of values $n \in N$, i.e. for every measurable rectangle A in Ω there exists such $n_A \in N$ that for all $n > n_A$ it holds $B_n = E_n$. For an arbitrary measurable rectangle A in Ω then it holds (see [1])

$$\boldsymbol{P}(A) = \int_{B_1} \mathrm{d}\mathbf{x}_1 \ldots \int_{B_{n_A}} \mathrm{d}\mathbf{x}_{n_A}\, \mathrm{w}_{n_A}(\mathbf{x}_1, \ldots, \mathbf{x}_{n_A}) .$$

Let an arbitrary set $B^n \in \mathscr{B}^n$ be given. The set

$$B \triangleq B^n \times \prod_{\{k > n\}} E_k$$

will be called the measurable cylinder B in Ω with the basis B^n. For every $t \in N$ let us define now $\mathscr{F}_t \subset \mathscr{F}$ as a sub-σ-algebra of the σ-algebra $\mathscr{F}$, which is created by all possible measurable cylinders with bases from $\mathscr{B}^t$. Evidently it holds $\mathscr{F}_t \subset \mathscr{F}_s$ for $t \leqq s;\ t, s \in N$.

Definition 1.1. *The stopping rule* τ on the sequence $\{\mathbf{x}_n\}$ is every integer random variable defined on $(\Omega, \mathscr{F}, \boldsymbol{P})$, with values in N, for which it holds

$$\{\omega : \tau(\omega) = n\} \in \mathscr{F}_n \quad \text{for every} \quad n \in N . \tag{1.1}$$

We shall define now a new random variable y_n on the probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$ for every $n \in N$ by the relation

$$y_n(\omega) \triangleq \mathrm{g}_n(\mathscr{X}_n(\omega)) , \quad n \in N \tag{1.2}$$

where g_n is some given $\mathscr{B}^n$-measurable function for every $n \in N$.

We shall introduce a notation $\mathrm{f}^+ \triangleq \max[\mathrm{f}, 0]$, $\mathrm{f}^- \triangleq \max[-\mathrm{f}, 0]$, where f is an arbitrary real function. Everywhere below we shall assume that functions g_n satisfy the condition

$$\boldsymbol{M}(\sup_{n \in N} \mathrm{g}_n^+(\mathscr{X}_n)) < \infty \tag{1.3}$$

where $\boldsymbol{M}$ is the expected value on $(\Omega, \mathscr{F}, \boldsymbol{P})$.

Since it holds

$$y_\tau \leqq \sup_{n \in N} \mathrm{g}_n^+(\mathscr{X}_n) ,$$

according to (1.2), for an arbitrary stopping rule τ, it must hold, according to (1.3), for arbitrary τ

$$-\infty \leqq M(y_\tau) \leqq M(\sup_{n\in N} g_n^+(\mathscr{X}_n)) < \infty$$

and thus there always exists $M(y_\tau)$ for an arbitrary stopping rule τ, but it can hold $M(y_\tau) = -\infty$.

Definition 1.2. *The optimum stopping rule* on the sequence $\{\mathbf{x}_n\}$ is the stopping rule τ^* for which it holds (if there exists some stopping rule with such a property):

$$M(y_{\tau^*}) = \sup_{\tau\in\mathscr{C}} M(y_\tau) \tag{1.4}$$

where $\mathscr{C}$ is the set of all stopping rules τ on the sequence $\{\mathbf{x}_n\}$.

Several works (see e.g. [2], [3]) deal with the problems of an existence and a construction of the optimum stopping rules for various concrete choices of functions g_n. Reference [4] deals with an important special case when a sequence $\{\mathbf{x}_n, \mathscr{F}_n, P_\mathbf{x}\}$ is a homogeneous Markov sequence, where $P_\mathbf{x}$ is a probability measure on $(\Omega, \mathscr{F})$ corresponding to the initial stage $\mathbf{x}$. There is understood in [4] by the optimum stopping rule τ^* such a stopping rule, for which it holds:

$$M_\mathbf{x}(y_{\tau^*}) = \sup_{\tau\in\mathscr{C}} M_\mathbf{x}(y_\tau)$$

for all $\mathbf{x} \in E$, where $M_\mathbf{x}$ is the expected value on $(\Omega, \mathscr{F}, P_\mathbf{x})$.

In this paper we shall solve the problem of an existence and a construction of the optimum stopping rule in the sense of the Definition 1.2 using the theory developed in [4]. For this reason, we shall transform the sequence $\{\mathbf{x}_n, \mathscr{F}_n\}$ to a homogeneous Markov sequence $\{\mathscr{X}_t^\sim, \mathscr{F}_t^\sim, P_{\mathscr{X}^\sim}\}$ in the next chapter.

2. TRANSFORMATION OF THE SEQUENCE OF THE GENERALLY STATISTICALLY DEPENDENT VECTORS TO THE HOMOGENEOUS MARKOV SEQUENCE

The following lemma will be an entry point for the next discussions.

Lemma 2.1. Let the arbitrary fixed $n \in N$ and $\mathscr{X} \in E^n$ be given. Then the only probability measure $P_{n,\mathscr{X}}$ exists on $(\Omega, \mathscr{F})$, the value of which is given on an arbitrary measurable rectangle $\prod_N B_t$ by the relation

$$P_{n,\mathscr{X}}(\prod_N B_t) = I_{B^n}(\mathscr{X}) \int_{B_{n+1}} d\mathbf{x}_{n+1} \dots \int_{B_{n+T}} d\mathbf{x}_{n+T}\, w_{n+T|n}(\mathbf{x}_{n+1}, \dots, \mathbf{x}_{n+T} \mid \mathscr{X}) \tag{2.1}$$

where I_{B^n} is the indicator of the set $B^n \triangleq \prod_{\{1 \leqq t \leqq n\}} B_t$ and T is a positive integer so large that it holds $B_{n+t} = E_{n+t}$ for all $t > T$. It holds

(2.2)

$$\int_\Omega \boldsymbol{P}_{n,\mathscr{X}}(\mathrm{d}\omega')\, Y(\omega') = \int_{E_{n+1}} \mathrm{d}\mathbf{x}_{n+1} \ldots \int_{E_{n+T}} \mathrm{d}\mathbf{x}_{n+T}\, \mathrm{w}_{n+T|n}(\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{n+T} \mid \mathscr{X})\, Y(\mathbf{x}_1, \ldots \ldots, \mathbf{x}_{n+T})$$

for $\mathscr{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and for any nonnegative real random variable Y on $(\Omega, \mathscr{F})$, depending only on coordinates with indexes not exceeding the value $n + T$. In addition, an integral

$$\int_\Omega \boldsymbol{P}_{n,\mathscr{X}}(\mathrm{d}\omega')\, Y(\omega')$$

is a $\mathscr{B}^n$-measurable function of $\mathscr{X}$ for every nonnegative real random variable Y on $(\Omega, \mathscr{F})$.

Proof. We obtain the assertion of Lemma 2.1 immediately from Tulcea's theorem (see [1], chapter V., Theorem V.1.1) if we replace transition probabilities of [1] by corresponding conditional densities $\mathrm{w}_{k|n}$ and if we put the measurable space $(E_0, \mathscr{F}_0)$ of [1] equal to the measurable space $(E^n, \mathscr{B}^n)$. Spaces $(E_t, \mathscr{F}_t)$ of [1] are equal to $(E_{n+t}, \mathscr{B}_{n+t})$ for $t \in N$.

For next discussions we shall use the following property of the probability measure $\boldsymbol{P}_{n,\mathscr{X}}$.

Lemma 2.2. For an arbitrary $n \leqq t$; $n, t \in N$ and for a nonnegative real random variable Y it holds

$$\int_\Omega \boldsymbol{P}_{n,\mathscr{X}}(\mathrm{d}\omega')\, Y(\omega') = \int_\Omega \boldsymbol{P}_{n,\mathscr{X}}(\mathrm{d}\omega') \int_\Omega \boldsymbol{P}_{t,\mathscr{X}_t(\omega')}(\mathrm{d}\omega'')\, Y(\omega'') . \tag{2.3}$$

For an arbitrary $n \in N$, a nonnegative real random variable Y and a nonnegative $\mathscr{F}_n$-measurable real random variable Z it holds

$$\int_\Omega \boldsymbol{P}_{n,\mathscr{X}}(\mathrm{d}\omega')\, Y(\omega')\, Z(\omega') = Z(\mathbf{x}_1, \ldots, \mathbf{x}_n) \int_\Omega \boldsymbol{P}_{n,\mathscr{X}}(\mathrm{d}\omega')\, Y(\omega') \tag{2.4}$$

for $\mathscr{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$.

The proof is analogous to the proof of Corollary 1 to Theorem V.1.1 in [1].

Lemma 2.3. For an arbitrary $n \leqq t$; $n, t \in N$ and for an arbitrary $B \in \mathscr{F}$ it holds

$$\boldsymbol{P}_{n,\mathscr{X}}(B \mid \mathscr{F}_t) = \boldsymbol{P}_{t,\mathscr{X}_t(\omega)}(B) \qquad \boldsymbol{P}_{n,\mathscr{X}} \quad \text{a.s.} \tag{2.5}$$

Proof. We must prove that for an arbitrary $A \in \mathscr{F}_t$ it holds

$$P_{n,\mathscr{X}}(A \cap B) = \int_A P_{t,\mathscr{X}_t(\omega')}(B)\, P_{n,\mathscr{X}}(\mathrm{d}\omega') . \tag{2.6}$$

Let us define a nonnegative real random variable $Y(\omega) \triangleq \mathrm{I}_A(\omega)\, \mathrm{I}_B(\omega)$ where I_A is a $\mathscr{F}_t$-measurable indicator of a set A and I_B is a $\mathscr{F}$-measurable indicator of a set B According to the relation (2.3) it now holds

$$P_{n,\mathscr{X}}(A \cap B) = \int_\Omega P_{n,\mathscr{X}}(\mathrm{d}\omega')\, Y(\omega') = \tag{2.7}$$

$$= \int_\Omega P_{n,\mathscr{X}}(\mathrm{d}\omega') \int_\Omega P_{t,\mathscr{X}_t(\omega')}(\mathrm{d}\omega'')\, \mathrm{I}_A(\omega'')\, \mathrm{I}_B(\omega'') .$$

According to (2.4) the inner integral on the right side satisfies the equation

$$\int_\Omega P_{t,\mathscr{X}_t(\omega')}(\mathrm{d}\omega'')\, \mathrm{I}_A(\omega'')\, \mathrm{I}_B(\omega'') = \mathrm{I}_A(\mathbf{e}'_1, \ldots, \mathbf{e}'_t) \int_\Omega P_{t,\mathscr{X}_t(\omega')}(\mathrm{d}\omega'')\, \mathrm{I}_B(\omega'') =$$

$$= \mathrm{I}_A(\mathbf{e}'_1, \ldots, \mathbf{e}'_t)\, P_{t,\mathscr{X}_t(\omega')}(B) \quad \text{for} \quad \omega' = (\mathbf{e}'_1, \ldots, \mathbf{e}'_t, \ldots) .$$

After a substitution into (2.7) we obtain

$$P_{n,\mathscr{X}}(A \cap B) = \int_\Omega P_{n,\mathscr{X}}(\mathrm{d}\omega')\, \mathrm{I}_A(\mathbf{e}'_1, \ldots, \mathbf{e}'_t)\, P_{t,\mathscr{X}_t(\omega')}(B) =$$

$$= \int_A P_{n,\mathscr{X}}(\mathrm{d}\omega')\, P_{t,\mathscr{X}_t(\omega')}(B)$$

By this Lemma 2.3 is proved.

We shall now prove an auxiliary assertion which we shall use in our next discussions. Let be given measurable spaces $(\Omega_n, \mathscr{M}_n)$ for every $n \in N$, where Ω_n is some nonempty set and $\mathscr{M}_n$ is a σ-algebra of subsets Ω_n. Denote Ω' the set of all possible pairs (n, z) such that $z \in \Omega_n$ for $n \in N$. Further we shall denote

$$(n, A_n) \triangleq \{(n, z) \in \Omega' : z \in A_n\} \subset \Omega' ,$$

$$(n, \emptyset) \triangleq \emptyset$$

for an arbitrary $n \in N$, $A_n \subset \Omega_n$. Then it holds

Lemma 2.4. Let $\mathscr{M}'$ be a system of all possible subsets M of the set Ω' such that it holds

$$M = \bigcup_N (n, A_n)$$

where $A_n \in \mathscr{M}_n$ for every $n \in N$ (some of sets A_n can be empty). Then $\mathscr{M}'$ is σ-algebra, Proof. Let $M = \bigcup_N (n, A_n)$ be an element of $\mathscr{M}'$. We shall prove that also the complement of M, i.e. the set $\Omega' \setminus M$, belongs to $\mathscr{M}'$. Evidently, it holds

$$\Omega' \setminus M = \bigcup_N (n, B_n), \quad B_n = \Omega_n \setminus A_n \quad \text{for} \quad n \in N.$$

Since $A_n \in \mathscr{M}_n$, $\Omega_n \in \mathscr{M}_n$, then also $B_n \in \mathscr{M}_n$ for all $n \in N$ and thus $\Omega' \setminus M \in \mathscr{M}'$. Similarly we can prove that also disjunction and conjunction of an arbitrary at least countable system of sets from $\mathscr{M}'$ belong to $\mathscr{M}'$. The Lemma 2.4 is then proved.

We shall now define some new objects. Let $\tilde{E}$ be the set of all possible pairs $(n, \mathscr{X})$, where $n \in N$, $\mathscr{X} \in E^n$. Further let $\mathscr{B}^\sim$ be a system of subsets of $\tilde{E}$ created by all possible sets B of the type

$$B = \bigcup_N (n, B_n) \quad \text{where} \quad B_n \in \mathscr{B}^n \quad \text{for} \quad n \in N.$$

It is clear that $\mathscr{B}^\sim$ contains all one-point sets. According to Lemma 2.4, $\mathscr{B}^\sim$ is a σ-algebra, thus $(\tilde{E}, \mathscr{B}^\sim)$ is a phase space. According to [4] p. 32, we assume by a phase space every measurable space, σ-algebra of which contains all one-point sets.

Further let $\tilde{\Omega} = N \times \Omega$ and let $\mathscr{F}^\sim$ be the system of all such subsets $A \in \tilde{\Omega}$ that it holds

$$A = \bigcup_N (n, A_n) \quad \text{where} \quad A_n \in \mathscr{F} \quad \text{for} \quad n \in N.$$

According to Lemma 2.4, $\mathscr{F}^\sim$ is a σ-algebra and $(\tilde{\Omega}, \mathscr{F}^\sim)$ is thus a measurable space.

Let $N_0 \triangleq \{0, 1, \ldots\}$. For every $t \in N_0$ we shall define $\mathscr{F}_t^\sim$ as a system of all such sets $B \subset \tilde{\Omega}$, that

$$B = \bigcup_N (n, B_n) \quad \text{where} \quad B_n \in \mathscr{F}_{n+t} \quad \text{for} \quad n \in N.$$

It is evident that it holds

$$\mathscr{F}_t^\sim \subset \mathscr{F}_s^\sim \subset \mathscr{F}^\sim \quad \text{for} \quad t \leqq s; \quad t, s \in N_0$$

Further, for $\tilde{\omega} = (n, \omega) \in \tilde{\Omega}$ and for every $t \in N_0$, we define a random element $\mathscr{X}_t^\sim(\tilde{\omega})$ on $\tilde{\Omega}$ with a value in $(\tilde{E}, \mathscr{B}^\sim)$ by the relation:

$$\mathscr{X}_t^\sim(\tilde{\omega}) \triangleq (t + n, \mathscr{X}_{t+n}(\omega)). \tag{2.8}$$

We shall show that it holds for every $t \in N_0$, $C \in \mathscr{B}^\sim$:

$$\{\tilde{\omega} : \mathscr{X}_t^\sim(\tilde{\omega}) \in C\} \in \mathscr{F}_t^\sim$$

and thus the random process $\mathscr{X}_t^\sim$ is matched with the system $\{\mathscr{F}_t^\sim\}$ of σ-algebras (definition of matching see [4], p. 15). It holds

$$\{\tilde{\omega} : X_t^\sim(\tilde{\omega}) \in C\} = \{(n, \omega) : (n + t, \mathscr{X}_{n+t}(\omega)) \in C\} = \bigcup_N (n, B_n) \quad B_n \subset \Omega.$$

If $C = \bigcup_N (n, C_n)$, $C_n \in \mathscr{B}^n$ (there can exist such n-s for which $C_n = \emptyset$), then

$$B_n = \{\omega : \mathscr{X}_{n+t}(\omega) \in C_{n+t}\} \in \mathscr{F}_{n+t}$$

and the statement is proved.

Let us define a probability measure $P_{\mathscr{X}}$ on the measurable space $(\tilde{\Omega}, \mathscr{F}^{\sim})$ for an arbitrary $\mathscr{X}^{\sim} = (n, \mathscr{X}) \in \tilde{E}$ by the relation

$$P^{\sim}_{\mathscr{X}^{\sim}}(A) \triangleq P_{n,\mathscr{X}}(A_n) \quad \text{where} \quad A = \bigcup_N (k, A_k) \in \mathscr{F}^{\sim} . \tag{2.9}$$

It is easy to show that $P^{\sim}_{\mathscr{X}^{\sim}}(A)$ is really a probability measure and our definition is thus correct. We shall prove the following theorem:

Theorem 2.1. The system $\tilde{X} \triangleq (\mathscr{X}^{\sim}_t, \mathscr{F}^{\sim}_t, P^{\sim}_{\mathscr{X}^{\sim}})$ for $t \in N_0$ and $\mathscr{X}^{\sim} \in \tilde{E}$ is a homogeneous Markov sequence with values in the phase space $(\tilde{E}, \mathscr{B}^{\sim})$.

Proof. According to [4], cf. Def. 1, p. 32, the following conditions must be satisfied:

1. $P^{\sim}_{\mathscr{X}^{\sim}}(A)$ must be a $\mathscr{B}^{\sim}$-measurable function of $\mathscr{X}^{\sim}$ for every $A \in \mathscr{F}^{\sim}$.

2. For all $\mathscr{X}^{\sim} \in \tilde{E}$, $B \in \mathscr{B}^{\sim}$; $s, t \in N_0$ it must hold:

$$P^{\sim}_{\mathscr{X}^{\sim}}(\mathscr{X}^{\sim}_{t+s} \in B \mid \mathscr{F}^{\sim}_t) = P^{\sim}_{\mathscr{X}^{\sim}_t(\tilde{\omega})}(\mathscr{X}^{\sim}_s \in B) \qquad P^{\sim}_{\mathscr{X}^{\sim}} \quad \text{a.s.}$$

3. For every $\mathscr{X}^{\sim} \in \tilde{E}$ it must hold $P^{\sim}_{\mathscr{X}^{\sim}}(\mathscr{X}^{\sim}_0 = \mathscr{X}^{\sim}) = 1$.

4. For every $t \in N_0$ and $\tilde{\omega} \in \tilde{\Omega}$ there exists one and only one $\tilde{\omega}' \in \tilde{\Omega}$ such that it holds

$$\mathscr{X}^{\sim}_s(\tilde{\omega}') = \mathscr{X}^{\sim}_{t+s}(\tilde{\omega})$$

for all $s \in N_0$.

We shall verify validity of these conditions.

1. According to relation (2.9) it holds

$$P^{\sim}_{\mathscr{X}^{\sim}}(A) = P_{n,\mathscr{X}}(A_n) \quad \text{where} \quad A_n \in \mathscr{F} .$$

Let $I_{A_n}(\omega)$ be an indicator of a set A_n. Then it holds:

$$P_{n,\mathscr{X}}(A_n) = \int_{\Omega} P_{n,\mathscr{X}}(d\omega')\, I_{A_n}(\omega')$$

and according to Lemma 2.1, $P_{n,\mathscr{X}}(A_n)$ is a $\mathscr{B}^n$-measurable function of $\mathscr{X}$. $\mathscr{B}^{\sim}$-measurability of $P^{\sim}_{\mathscr{X}^{\sim}}(A)$ then follows immediately from the definition of a σ-algebra $\mathscr{B}^{\sim}$.

2. According to the definition of a conditional probability we must prove that it holds for an arbitrary $A \in \mathscr{F}_t^{\sim}$:

$$(2.10) \qquad \boldsymbol{P}_{\mathscr{X}^{\sim}}^{\sim}(\{\mathscr{X}_{t+s}^{\sim} \in B\} \cap A) = \int_A \boldsymbol{P}_{\mathscr{X}^{\sim}{}_t(\tilde{\omega}')}^{\sim}(\mathscr{X}_s^{\sim} \in B)\, \boldsymbol{P}_{\mathscr{X}^{\sim}}^{\sim}(\mathrm{d}\omega') .$$

In agreement with the definition of the measure $\boldsymbol{P}_{\mathscr{X}^{\sim}}^{\sim}$, the integral on the right side of (2.10) is understood as an integral over the set A_n according the measure $\boldsymbol{P}_{n,\mathscr{X}}$ where $\mathscr{X}^{\sim} = (n, \mathscr{X})$ and $A = \bigcup_N (k, A_k) \in \mathscr{F}_t^{\sim}$. It holds (see also [1], section 1 of chapter III.)

$$\boldsymbol{P}_{\mathscr{X}^{\sim}}^{\sim}(\{\mathscr{X}_{t+s}^{\sim}(\tilde{\omega}) \in B\} \cap A) = \boldsymbol{P}_{n,\mathscr{X}}(\{\mathscr{X}_{t+s}^{\sim}(\tilde{\omega}) \in B\}_{(n)} \cap A_n)$$

where for an arbitrary $\Gamma \subset \tilde{\Omega}$ we define

$$\Gamma_{(n)} \triangleq \{\omega \in \Omega : (n, \omega) \in \Gamma\} .$$

It evidently holds $A_{(n)} = A_n \in \mathscr{F}_{n+t}$. Further

$$\{\mathscr{X}_{t+s}^{\sim}(\tilde{\omega}) \in B\}_{(n)} = \{\omega : \mathscr{X}_{t+s+n}(\omega) \in B_{t+s+n}\}$$

for $B = \bigcup_N (k, B_k)$, $B_k \in \mathscr{B}^k$ for $k \in N$. According to the Lemma 2.3, it holds

$$\boldsymbol{P}_{n,\mathscr{X}}(\{\mathscr{X}_{t+s+n} \in B_{t+s+n}\} \cap A_n) = \int_{A_n} \boldsymbol{P}_{t+n,\mathscr{X}_{t+n}(\omega')}(\{\mathscr{X}_{t+n+s} \in B_{t+n+s}\})\, \boldsymbol{P}_{n,\mathscr{X}}(\mathrm{d}\omega')$$

and thus according to the definition of the measure $\boldsymbol{P}_{\mathscr{X}^{\sim}}^{\sim}$

$$\boldsymbol{P}_{\mathscr{X}^{\sim}}^{\sim}(\{\mathscr{X}_{t+s}^{\sim}(\tilde{\omega}) \in B\} \cap A) = \int_A \boldsymbol{P}_{\mathscr{X}^{\sim}{}_t(\tilde{\omega})}^{\sim}(\mathscr{X}_s^{\sim} \in B)\, \boldsymbol{P}_{\mathscr{X}^{\sim}}^{\sim}(\mathrm{d}\tilde{\omega}')$$

by which the condition 2 is checked.

3) According to Lemma 2.1 and definition of $\boldsymbol{P}_{\mathscr{X}^{\sim}}^{\sim}$, it holds for $\mathscr{X}^{\sim} = (n, \mathscr{X})$:

$$\boldsymbol{P}_{\mathscr{X}^{\sim}}^{\sim}(\mathscr{X}_0^{\sim} = \mathscr{X}^{\sim}) = \boldsymbol{P}_{n,\mathscr{X}}(\{\mathscr{X}_0^{\sim}(\tilde{\omega}) = \mathscr{X}^{\sim}\}_{(n)}) = \boldsymbol{P}_{n,\mathscr{X}}(\mathscr{X}_n = \mathscr{X}) = 1 .$$

4. According to the definition, it holds for $\tilde{\omega} = (n, \omega)$ and $\tilde{\omega}' = (n', \omega')$

$$\mathscr{X}_{s+t}^{\sim}(\tilde{\omega}) = (s + t + n, \mathscr{X}_{s+t+n}(\omega)) = (s + n', \mathscr{X}_{s+n'}(\omega')) = \mathscr{X}_s^{\sim}(\tilde{\omega}') .$$

This equation is valid for $\tilde{\omega}' = (t + n, \omega)$; so condition 4 is verified and the proof of Theorem 2.1 is finished.

Theorem 2.1 solves the problem of a transformation of the sequence of statistically dependent vectors $\mathbf{x}_n$, $n \in N$ to the homogeneous Markov sequence $\mathscr{X}_t^{\sim}$, $t \in N_0$. It is easy to show that the system $X \triangleq (\mathscr{X}_t, \mathscr{F}_t, \boldsymbol{P}_{n,\mathscr{X}})$ is a nonhomogeneous Markov sequence. We can get our system $\hat{X} = (\mathscr{X}_t^{\sim}, \mathscr{F}_t^{\sim}, \boldsymbol{P}^{\sim}{}_{\mathscr{X}^{\sim}})$ from this sequence by the method, described in [5], chapter 4.

3. THE OPTIMUM STOPPING RULE ON A SEQUENCE OF STATISTICALLY DEPENDENT VECTORS

Using the results of the preceding chapter, we shall now transform our task, formulated in chapter 1, to the problem of the optimum stopping rule on the homogeneous Markov sequence.

Definition 3.1. *The stopping rule* on the homogeneous Markov sequence $\hat{X} = (\mathcal{X}_t^{\sim}, \mathcal{F}_t^{\sim}, P_{\mathcal{X}^\sim}^{\sim})$, $t \in N_0$ is every integer random variable $\tilde{\tau}$, defined on $(\tilde{\Omega}, \mathcal{F}^\sim)$ with values in N_0, for which it holds for an arbitrary $t \in N_0$

$$\{\tilde{\omega} : \tilde{\tau}(\tilde{\omega}) = t\} \in \mathcal{F}_t^{\sim} .$$

Let us define random variables $\tilde{y}_t$ by the relation

$$\tilde{y}_t(\tilde{\omega}) \triangleq g^\sim(\mathcal{X}_t^{\sim}(\tilde{\omega})) \tag{3.1}$$

for all $t \in N_0$ on the space $(\tilde{\Omega}, \mathcal{F}^\sim)$, where $g^\sim(\mathcal{X}^\sim) \triangleq g_n(\mathcal{X})$ for an arbitrary $\mathcal{X}^\sim = (n, \mathcal{X}) \in \tilde{E}$. It evidently holds

$$\tilde{y}_t(\tilde{\omega}) = y_{n+t}(\omega) \quad \text{for} \quad \tilde{\omega} = (n, \omega) \in \tilde{\Omega}\,, \quad t \in N_0\,, \quad n \in N\,. \tag{3.2}$$

Definition 3.2. *The optimum stopping rule* on a homogeneous Markov sequence $\hat{X} = (\mathcal{X}_t^{\sim}, \mathcal{F}_t^{\sim}, P_{\mathcal{X}^\sim}^{\sim})$ (for a given function g) is a stopping rule $\tilde{\tau}^*$ on $\hat{X}$, for which it holds for all $\mathcal{X}^\sim \in \tilde{E}$

$$M_{\mathcal{X}^\sim}^{\sim}(\tilde{y}_{\tilde{\tau}^*}) = \sup_{\tilde{\tau} \in \mathcal{C}^\sim} M_{\mathcal{X}^\sim}^{\sim}(\tilde{y}_{\tilde{\tau}}) \tag{3.3}$$

where $\mathcal{C}^\sim$ is the set of all stopping rules $\tilde{\tau}$ on $\hat{X}$, for which there exists the mean value $M_{\mathcal{X}^\sim}^{\sim}(\tilde{y}_{\tilde{\tau}})$ on $(\tilde{\Omega}, \mathcal{F}^\sim, P_{\mathcal{X}^\sim}^{\sim})$.

In Definitions 3.1 and 3.2 there we have narrowed general concepts of the stopping rule and the optimum stopping rule, used in [4], chap. II, to our concrete case of Markov sequence $\hat{X}$ and a function of a gain $g^\sim$, which is defined by the relation (3.1).

It follows from the definition of a probability measure $P_{\mathcal{X}^\sim}^{\sim}$ and from the relation (3.2) that for $\mathcal{X}^\sim = (n, \mathcal{X})$ and an arbitrary $\tilde{\tau} \in \mathcal{C}^\sim$

(3.4)
$$M_{\mathcal{X}^\sim}^{\sim}(\tilde{y}_{\tilde{\tau}}) = \int_{\tilde{\Omega}} P_{\mathcal{X}^\sim}^{\sim}(\mathrm{d}\tilde{\omega}')\, \tilde{y}_{\tilde{\tau}(\tilde{\omega}')}(\tilde{\omega}') = \int_{\Omega} y_{n+\tau_n(\omega')}(\omega')\, P_{n,\mathcal{X}}(\mathrm{d}\omega') = M_{n,\mathcal{X}}(y_{n+\tau_n})$$

where $\tau_n(\omega) \triangleq \tilde{\tau}(\tilde{\omega})$ for $\tilde{\omega} = (n, \omega)$.

$M_{n,\mathcal{X}}$ is the expected value on the space $(\Omega, \mathcal{F}, P_{n,\mathcal{X}})$. It follows from the relations (3.2) and (3.4) that supremum on the right side of the equation (3.3) is given by the relation

$$\sup_{\tilde{\tau} \in \mathcal{C}^\sim} M_{\mathcal{X}^\sim}^{\sim}(\tilde{y}_{\tilde{\tau}}) = \sup_{\tau_n \in \mathcal{C}_n} M_{n,\mathcal{X}}(y_{n+\tau_n}) \quad \text{for} \quad \mathcal{X}^\sim = (n, \mathcal{X}) \in \tilde{E} \tag{3.5}$$

where τ_n is an integer random variable defined for $n \in N$ on $(\Omega, \mathscr{F}, \boldsymbol{P}_{n,\mathscr{X}})$, with values from N_0, for which it holds for every $s \in N_0$

$$\{\omega : \tau_n(\omega) = s\} \in \mathscr{F}_{n+s} \tag{3.6}$$

and $\mathscr{C}_n$ is the of all possible τ_n with the property (3.6) for which $\boldsymbol{M}_{n,\mathscr{X}}(y_{n+\tau_n})$ exists for every $\mathscr{X} \in E^n$.

Let us assume for a moment, that there exists the optimum stopping rule $\tilde{\tau}^*$ on $\mathscr{X}^\sim$ in the sense of our Definition 3.2. It is clear from relations (3.3) till (3.5) that then for every $n \in N$ there exists $\tau_n^* \in \mathscr{C}_n$ given by the relation

$$\tau_n^*(\omega) \triangleq \tilde{\tau}^*(\tilde{\omega}) \quad \text{for} \quad \tilde{\omega} = (n, \omega) \tag{3.7}$$

for which it holds for all $\mathscr{X} \in E^n$

$$\boldsymbol{M}_{n,\mathscr{X}}(y_{n+\tau_n^*}) = \sup_{\tau_n \in \mathscr{C}_n} \boldsymbol{M}_{n,\mathscr{X}}(y_{n+\tau_n}) \,. \tag{3.8}$$

According to Corollary 2 of Theorem V.1.1 in [1] and in accordance with properties of a conditional mean value it holds for arbitrary $\tau_n \in \mathscr{C}_n$

$$\boldsymbol{M}(y_{n+\tau_n}) = \boldsymbol{M}(\boldsymbol{M}_{n,\mathscr{X}_n(\omega)}(y_{n+\tau_n})) \,. \tag{3.9}$$

We get then from relations (3.8) and (3.9) for $n = 1$ and an arbitrary $\tau_1 \in \mathscr{C}_1$

$$\boldsymbol{M}(y_{1+\tau_1^*}) = \boldsymbol{M}(\boldsymbol{M}_{1,\mathscr{X}_1(\omega)}(y_{1+\tau_1^*})) = \tag{3.10}$$

$$= \int_\Omega \boldsymbol{M}_{1,\mathscr{X}_1(\omega')}(y_{1+\tau_1^*}) \, \boldsymbol{P}(\mathrm{d}\omega') \geqq \int_\Omega \boldsymbol{M}_{1,\mathscr{X}_1(\omega')}(y_{1+\tau_1}) \, \boldsymbol{P}(\mathrm{d}\omega') = \boldsymbol{M}(y_{1+\tau_1}) \,.$$

There exists a one-one correspondence between the sets $\mathscr{C}$ and $\mathscr{C}_1$ using relations $\tau = \tau_1 + 1$ for $\tau \in \mathscr{C}$, $\tau_1 \in \mathscr{C}_1$. Then it follows from this correspondence and from the relation (3.10)

$$\boldsymbol{M}(y_{1+\tau_1^*}) = \sup_{\tau \in \mathscr{C}} \boldsymbol{M}(y_\tau) = \boldsymbol{M}(y_{\tau^*}) \,. \tag{3.11}$$

It is clear from (3.11) that if the optimum stopping rule $\tilde{\tau}^*$ in the sense of Definition 3.2 exists then the optimum stopping rule τ^* on the sequence $\{\mathbf{x}_n\}$ in the sense of Definition 1.2 also exists and it is given by the relation

$$\tau^*(\omega) = \tau_1^*(\omega) + 1 \,. \tag{3.12}$$

Relations (3.7) and (3.12) and the corresponding discussions then enable us to use the theory of optimum stopping rules on Markov sequences, developed in chapter II of [4], for a solution of our problem.

Let us denote

$$\mathrm{s}_n(\mathscr{X}) \triangleq \sup_{\tau_n \in \mathscr{C}_n} \boldsymbol{M}_{n,\mathscr{X}}(y_{n+\tau_n}) \quad \text{for} \quad n \in N \,, \quad \mathscr{X} \in E^n \tag{3.13}$$

(3.14) $$\hat{s} \triangleq \sup_{\tau \in \mathscr{C}} M(y_\tau)$$

It follows from relations (3.9) till (3.11)

(3.15) $$\hat{s} = M(s_1(\mathscr{X}_1)) = \int_{E_1} s_1(\mathscr{X}_1)\, w_1(\mathscr{X}_1)\, d\mathscr{X}_1 .$$

The following theorem holds:

Theorem 3.1. Let the gain, corresponding to the stopping of observations in a step $n \in N$, be given by the relation

$$y_n(\omega) = g_n(\mathscr{X}_n(\omega))$$

where g_n is a $\mathscr{B}^n$-measurable function for every $n \in N$ and let it hold

(3.16) $$M(\sup_{n \in N} g_n^+(\mathscr{X}_n)) < \infty .$$

Then it holds

(3.17) $$s_n(\mathscr{X}_n) = \max\left[g_n(\mathscr{X}_n);\ M_{n,\mathscr{X}_n}(s_{n+1}(\mathscr{X}_{n+1}))\right]$$

(3.18) $$s_n(\mathscr{X}_n) = \lim_{b \to \infty} \lim_{a \to -\infty} \lim_{k \to \infty} Q^k[g_n(a, b, \mathscr{X}_n)]$$

where

(3.19) $$g_n(a, b, \mathscr{X}_n) \triangleq \begin{cases} b & g_n(\mathscr{X}_n) > b \\ g_n(\mathscr{X}_n) & a \leqq g_n(\mathscr{X}_n) \leqq b \\ a & g_n(\mathscr{X}_n) < a \end{cases}$$

for $a \leqq 0$, $b \geqq 0$ and Q^k is the k-th power of an operator Q defined by the relation

(3.20) $$Q[f_n(\mathscr{X}_n)] \triangleq \max\left[f_n(\mathscr{X}_n);\ M_{n,x_n}(f_{n+1}(\mathscr{X}_{n+1}))\right] .$$

The proof is very easy, since except of the notations, our theorem 3.1 is equivalent with statements 3 and 4 of Theorem 7 in chapter II of work [4], applied to the Markov sequence $\hat{X} = (\mathscr{X}_t^{\sim}, \mathscr{F}_t^{\sim}, P_{\mathscr{X}^{\sim}})$. For a transformation to our notation we have used relations (3.4) and (3.13). There is needed in the above mentioned Theorem 7 to hold for functions g_n for every $n \in N$ and $\mathscr{X}_n \in E^n$

(3.21) $$M_{n,\mathscr{X}}(\sup_{t \in N_0} g_{n+t}^+(\mathscr{X}_{n+t})) < \infty .$$

According to the Corollary 2 of Theorem V.1.1 of [1], $M_{n,\mathscr{X}_n(\omega)}(Y)$ is the everywhere defined variant of the conditional mean value $M(Y \mid \mathscr{F}_n)$ for an arbitrary nonnegative random variable Y. If it holds $M(Y) < \infty$, then according to the Radon-Nikodym's theorem (see e.g. [1], Theorem IV.1.4 of chapter IV), $M(Y \mid \mathscr{F}_n)$ is a finite random variable on the space $(\Omega, \mathscr{F}_n, P)$. Since it holds

$$0 \leqq \sup_{t \in N_0} g^+_{n+t}(\mathscr{X}_{n+t}) \leqq \sup_{n \in N} g^+_n(\mathscr{X}_n)$$

relation (3.21) follows from (3.16) and the proof of Theorem 3.1 is completed.

Theorem 3.2. Let g_n satisfy condition (3.16) and let

$$\tau^* = \inf_{n \in N} \{n : s_n(\mathscr{X}_n) = g_n(\mathscr{X}_n)\} . \tag{3.22}$$

Then τ^* is the optimum stopping rule on the sequence $\{\mathbf{x}_n\}$ if it holds for every $n \in N$, $\mathscr{X} \in E^n$ either

1. $P_{n,\mathscr{X}}(\tau^* < \infty) = 1$ or
2. $\lim_{t \to \infty} g_{n+t}(\mathscr{X}_{n+\cdot}) = -\infty \quad P_{n,\mathscr{X}}$ a.s.

Proof. Theorem 3.2 is a transcription of statements 3 and 4 of Theorem 8 in chapter II of work [4] and proof is thus easy. A substitution of the assumption (3.21) used in Theorem 8 of [4] by the assumption (3.16) is explained in the proof of our Theorem 3.1.

Theorems 3.1 and 3.2 solve our problem formulated in chapter 1. The statements contained in our Theorems 3.1 and 3.2 are in fact equivalent to the results of section 3 of [3]. Since we utilize the theory of optimum stopping rule on Markov sequences [4], we can define a cost $s_n(\mathscr{X})$ for an arbitrary (not random!) $\mathscr{X} \in E^n$ by the relation (3.13) in difference from a relation

$$s_n(\mathscr{X}_n(\omega)) \triangleq \operatorname{ess\,sup}_{\tau \in \mathscr{C}_n} M(y_{n+\tau} \mid \mathscr{F}_n)$$

which is used in other notation in [3].

Comparing to [2], in our results we need not assume the existence of an expansion

$$y_n \triangleq y'_n - y''_n \triangleq y^*_n - y^{**}_n$$

with some assumed properties of components. But our equation (3.18) is more complex then equations (24) and (51) which have the similar sense in [2] as (3.18) has in our work.

4. SEQUENTIAL TEST OF THE FINITE NUMBER OF DISJOINT HYPOTHESES FOR STATISTICALLY DEPENDENT OBSERVATIONS

In this chapter we shall apply the results of the previous chapter to the problem of determining the optimum sequential test of H mutually disjoint hypotheses ($H > 1$, fixed). To transfer this problem to the problem of optimum stopping rules we shall use the method described in [6].

Let $\mathscr{H} \triangleq \{1, 2, \ldots, H\}$ be a set of hypotheses (a set of nature states) and let $\mathscr{A} \triangleq \{1, 2, \ldots, H\}$ be a set of possible decisions of a statistician. Further let be given a matrix of looses $\mathbf{L}$, elements L_{ij} of it represent a loss of the statistician if $i \in \mathscr{H}$ is a true hypothesis and $j \in \mathscr{A}$ is a decision of the statistician. We shall assume $0 < L_{ij} < \infty$ for $i \neq j$, and $L_{ii} = 0$ for $i \in \mathscr{H}$, $j \in \mathscr{A}$. The statistician selects his decisions $j \in \mathscr{A}$ as a result of an observation of vectors $\mathbf{x}_n$ sequentially for $n = 1, 2, \ldots$. We assume that the cost, which the statistician will pay in the n-th step due to the observation of $\mathbf{x}_n$, is equal to $c_n(\mathscr{X}_n) \geqq 0$, i.e. it depends on the whole course of observations of $\mathscr{X}_n = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ until the n-th step. The process of statistician's observations terminates by selecting a decision.

Let d be *a rule of a terminal decision*, i.e., an arbitrary function on $(\tilde{E}, \mathscr{B}^{\sim})$ with values in $\mathscr{A}$, for which it holds for every $a \in \mathscr{A}$

$$\{(n, \mathscr{X}) \in \tilde{E} : d(n, \mathscr{X}) = a\} \in \mathscr{B}^{\sim} . \tag{4.1}$$

Definition 4.1. By *the sequential test of a set of hypotheses* $\mathscr{H}$ we shall understand every pair (d, τ) where d is an arbitrary rule of a terminal decision and τ is an arbitrary stopping rule on a sequence $\{\mathbf{x}_n\}$, if the statistician selects a decision $a = d(n, \mathscr{X}_n(\omega))$ then and only then, if $\tau(\omega) = n$.

Let $\pi \triangleq ({}^1\pi, \ldots, {}^H\pi)$, ${}^i\pi \geqq 0$, $\sum_{i=1}^{H} {}^i\pi = 1$ be a priori probability distribution on $\mathscr{H}$, on which we assume that it exists and it is given. Further let ${}^i\mathrm{w}_n(\mathscr{X}_n)$, $i \in \mathscr{H}$, $n \in N$ be the probability density of n-tuple $\mathscr{X}_n$ when the hypothesis i is true. We define the probability density w_n by the relation

$$\mathrm{w}_n(\mathscr{X}_n) = \sum_{i=1}^{H} {}^i\pi \, {}^i\mathrm{w}_n(\mathscr{X}_n) \quad \text{for} \quad n \in N , \quad \mathscr{X}_n \in E^n . \tag{4.2}$$

Analogously, let ${}^i\mathrm{w}_{k|n}(\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{n+k} \mid \mathscr{X}_n)$ be the conditional probability density of a k-tuple $(\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{n+k})$ when the n-tuple $\mathscr{X}_n$ is given and when the hypothesis $i \in \mathscr{H}$ is true. We define

$$\mathrm{w}_{k|n}(\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{n+k} \mid \mathscr{X}_n) = \sum_{i=1}^{H} {}^i\pi_n(\mathscr{X}_n) \, {}^i\mathrm{w}_{k|n}(\mathbf{x}_{n+1}, \ldots, \mathbf{x}_{n+k} \mid \mathscr{X}_n) \quad k, n \in N \; ; \quad \mathscr{X}_n \in E^n \; ; \quad \mathbf{x}_t \in E_t \tag{4.3}$$

where $\pi_n(\mathscr{X}_n) \triangleq ({}^1\pi_n(\mathscr{X}_n), \ldots, {}^H\pi_n(\mathscr{X}_n))$ be a posteriori probability distribution on $\mathscr{H}$ when n-tuple $\mathscr{X}_n$ is given. It holds

$$(4.4) \qquad {}^i\pi_n(\mathscr{X}_n) = \frac{{}^i\pi\ {}^i\mathrm{w}_n(\mathscr{X}_n)}{\sum_{k=1}^{H} {}^k\pi\ {}^k\mathrm{w}_n(\mathscr{X}_n)} \quad n \in N\ ; \quad i \in \mathscr{H}\ ; \quad \mathscr{X}_n \in E^n\ .$$

We shall assume that densities ${}^i\mathrm{w}_n$ and ${}^i\mathrm{w}_{k|n}$, respectively, exist and they are given for all $i \in \mathscr{H}$ and every k, $n \in N$.

Further let $\boldsymbol{P}_{ij}(d, \tau)$ be the probability of decision $j \in \mathscr{A}$ of the sequential test (d, τ) if the true hypothesis is $i \in \mathscr{H}$. For the probability $\boldsymbol{P}_{ij}(d, \tau)$ it holds

$$(4.5) \qquad \boldsymbol{P}_{ij}(d, \tau) = {}^i\boldsymbol{P}(\bigcup_{k=1}^{\infty} \{\omega : \tau(\omega) = k,\ d(k, \mathscr{X}_k(\omega)) = j\})$$

where ${}^i\boldsymbol{P}$ is a probability measure on the space $(\Omega, \mathscr{F})$ for every $i \in \mathscr{H}$. This measure is given by the same method as that $\boldsymbol{P}$ in chapter 1, only w_n and $\mathrm{w}_{k|n}$ will be replaced by ${}^i\mathrm{w}_n$ and ${}^i\mathrm{w}_{k|n}$, respectively.

By the risk of the statistician, we shall assume a value $\mathrm{r}(\pi, d, \tau)$ for which it holds

$$(4.6) \qquad \mathrm{r}(\pi, d, \tau) = \sum_{i=1}^{H} \sum_{j=1}^{H} \boldsymbol{P}_{ij}(d, \tau)\, L_{ij}\, {}^i\pi + {}^i\boldsymbol{M}(\sum_{k=1}^{\tau} c_k(\mathscr{X}_k))$$

where ${}^i\boldsymbol{M}$ is the expected value on the probability space $(\Omega, \mathscr{F}, {}^i\boldsymbol{P})$ for $i \in \mathscr{H}$.

Let Δ be the set of all possible sequential tests (d, τ), for which $\mathrm{r}(\pi, d, \tau)$ is defined for every π by the relation (4.6). Let us denote

$$(4.7) \qquad \varrho(\pi) \triangleq \inf_{(d,\tau)\in\Delta} \mathrm{r}(\pi, d, \tau)\,.$$

Definition 4.2. We shall say that *the sequential test* $(d^*, \tau^*) \in \Delta$ *is optimum in Bayesian sense*, if it holds

$$(4.8) \qquad \varrho(\pi) = \mathrm{r}(\pi, d^*, \tau^*)\,.$$

It is known (see e.g. [6]) that there exists such a rule of the terminal decision d^* that it holds for every $(d, \tau) \in \Delta$

$$(4.9) \qquad \mathrm{r}(\pi, d^*, \tau) \leqq \mathrm{r}(\pi, d, \tau)\,.$$

This rule is defined by the relation (for $n \in N$, $\mathscr{X} \in E^n$):

$$(4.10) \quad d^*(n, \mathscr{X}) \triangleq \min_{j\in\mathscr{A}} [j : \sum_{i=1}^{H} L_{ij}\, {}^i\pi_n(\mathscr{X}) \leqq \sum_{i=1}^{H} L_{ik}\, {}^i\pi_n(\mathscr{X}) \quad \text{for all} \quad k \in \mathscr{A}]$$

Let a $\mathscr{F}_n$-measurable random variable y_n be now given on $(\Omega, \mathscr{F}, \boldsymbol{P})$ for every $n \in N$ by the relation

$$(4.11) \qquad y_n(\omega) \triangleq y_n(\mathscr{X}_n(\omega)) \triangleq -\mathrm{h}(\pi_n(\mathscr{X}_n(\omega))) - \sum_{k=1}^{n} c_k(\mathscr{X}_k(\omega))$$

where

$$(4.12) \qquad \mathrm{h}(\mathbf{t}) \triangleq \min_{j \in \mathscr{A}} \left[\sum_{i=1}^{H} L_{ij} \, {}^i t \right]$$

for every $\mathbf{t} \triangleq ({}^1t, \ldots, {}^Ht)$, ${}^it \geqq 0$, $\sum_{i=1}^{H} {}^it = 1$.

It is evident from the Definition 4.2 and from the relations (4.9) till (4.12) that the problem of determining the Bayes optimum sequential test is equivalent to determining such a stopping rule τ^*, for which it holds

$$(4.13) \qquad \boldsymbol{M}(y_{\tau^*}) = \sup_{\tau \in \mathscr{C}} \boldsymbol{M}(y_\tau)$$

where $\boldsymbol{M}$ is the expected value on the space $(\Omega, \mathscr{F}, \boldsymbol{P})$. We can easy show that it holds

$$(4.14) \qquad \boldsymbol{M}(\sup_{n \in N} y_n^+) = 0 < \infty$$

and thus $\mathscr{C}$ is the set of all possible stopping rules on the sequence $\{\mathbf{x}_n\}$ in a sense of Definition 1.1.

Definition 4.3. The stopping rule τ^*, for which it (4.13) holds, will be called *the Bayes optimum stopping rule.*

Remark 4.1. It is evident from the previous discussions that the sequential test $(d, \tau) \in \Delta$ is the Bayes optimum test then and only then, if it holds $d = d^*$ and $\tau = \tau^*$, where d^* is the rule of the terminal decision given by (4.10) and τ^* is the Bayes optimum stopping rule.

According to results of the preceding chapter we can now formulate Theorem 4.1, which together with Remark 4.1 will solve the problem of determining the Bayes optimum sequential test in the sense of Definition 4.2.

Theorem 4.1. Let $\pi_n(\mathscr{X}_n)$ be defined for all $n \in N$ and further let be

$$(4.15) \qquad \mathrm{h}(\pi) > \varrho(\pi)$$

where $\varrho(\pi)$ is defined below by the relation (4.22) and for every $n \in N$, $\mathscr{X}_n \in E^n$ let $c_n(\mathscr{X}_n)$ be a $\mathscr{B}^n$-measurable function, $0 \leqq c_n(\mathscr{X}_n) \leqq \infty$ and let

$$(4.16) \qquad \lim_{t \to \infty} \sum_{k=1}^{t} c_k(\mathscr{X}_k(\omega)) = +\infty \quad \boldsymbol{P}_{n,\mathscr{X}_n} \text{ a.s.}$$

Then it holds for the Bayes optimum stopping rule

$$\tau^* = \inf_{n \in N} \{n : r_n(\mathscr{X}_n) = h(\pi_n(\mathscr{X}_n))\} . \tag{4.17}$$

The next relations hold for the function $r_n(\mathscr{X}_n)$ for every $n \in N$, $\mathscr{X}_n \in E^n$

$$r_n(\mathscr{X}_n) = \min [h(\pi_n(\mathscr{X}_n)); \boldsymbol{M}_{n,\mathscr{X}_n}(r_{n+1}(\mathscr{X}_{n+1}) + c_{n+1}(\mathscr{X}_{n+1}))] \tag{4.18}$$

$$r_n(\mathscr{X}_n) = \lim_{a \to \infty} \lim_{t \to \infty} \tilde{Q}^t[G_n(a, \mathscr{X}_n)] \tag{4.19}$$

where for $a > 0$ it holds

$$G_n(a, \mathscr{X}_n) \triangleq \begin{cases} h(\pi_n(\mathscr{X}_n)) & \text{for} \quad h(\pi_n(\mathscr{X}_n)) + \sum_{k=1}^{n} c_k(\mathscr{X}_k) \leqq a \\ a - \sum_{k=1}^{n} c_k(\mathscr{X}_k) & \text{in the other cases} \end{cases} \tag{4.20}$$

and $\tilde{Q}^t$ is the t-th power of the operator $\tilde{Q}$ defined by the relation

$$\tilde{Q}[f_n(\mathscr{X}_n)] \triangleq \min [f_n(\mathscr{X}_n); \boldsymbol{M}_{n,\mathscr{X}_n}(f_{n+1}(\mathscr{X}_{n+1}) + c_{n+1}(\mathscr{X}_{n+1}))] . \tag{4.21}$$

The following relation holds for the risk $\varrho(\pi)$

$$\varrho(\pi) = \int_{E_1} (r_1(\mathscr{X}_1) + c_1(\mathscr{X}_1)) \, w_1(\mathscr{X}_1) \, d\mathscr{X}_1 . \tag{4.22}$$

Proof. If the condition (4.15) is not satisfied, then the least risk brings a deciding without observations, i.e. selecting a decision d^* according to a priori distribution π. This case is not interesting from the point of view of practical applications, thus we shall not deal with it any more.

According to the fact that it holds for an arbitrary $n \in N$ and $\mathscr{X} \in E^n$

$$0 \leqq h(\pi_n(\mathscr{X})) \leqq \max_{i,j} L_{ij} < \infty$$

from the relation (4.16), it follows the validity of the condition 2 of our Theorem 3.2. The relation (4.14) guarantees the satisfaction of the condition (3.16) from Theorem 3.1 and then the statements (4.17) and (4.18) are evident consequences of Theorems 3.1 and 3.2 taking into account substitution of a function $s_n(\mathscr{X}_n)$ by a function $r_n(\mathscr{X}_n)$ according to the relation

$$r_n(\mathscr{X}_n) \triangleq -s_n(\mathscr{X}_n) - \sum_{k=1}^{n} c_k(\mathscr{X}_k) . \tag{4.23}$$

Relations (4.19) and (4.20) follow from (3.18), (3.19) and from the fact that $y_n(\mathscr{X}) \leqq 0$ for all $\mathscr{X} \in E^n$. Relation (4.21) is a consequence of (4.23) and (3.20) and the relation (4.22) follows from (4.23) and (3.15). The proof of Theorem 4.1 is completed.

Remark 4.2. Besides the relation (4.16) let it hold for all $n \in N$ and all $\mathscr{X}_n \in E^n$

$$c_n(\mathscr{X}_n) \leqq C_n < \infty \tag{4.24}$$

where C_n is some finite positive constant for every $n \in N$. Then it immediately follows from (4.19) and (4.20) that

$$\mathrm{r}_n(\mathscr{X}_n) = \lim_{t \to \infty} \tilde{Q}^t[\mathrm{h}(\pi_n(\mathscr{X}_n))] . \tag{4.25}$$

Remark 4.3. Let there exist such $n \in N$ and $\mathscr{X}'_n \in E^n$ that

$$\boldsymbol{M}_{n,\mathscr{X}_n'}(c_{n+1}(\mathscr{X}_{n+1})) = +\infty . \tag{4.26}$$

Then it is clear from relations (4.17) and (4.18) that for all $\omega \in \Omega' \subset \Omega$ it holds

$$\tau^*(\omega) \leqq n \tag{4.27}$$

where $\Omega' \triangleq \{\omega: \mathscr{X}_n(\omega) = \mathscr{X}'_n\} \neq \emptyset$.

Theorem 4.1 in principle solves the problem of Bayes optimum sequential test of a finite number of disjoint hypotheses for dependent observations under the assumption that a cost c for the n-th observation $\boldsymbol{x}_n$ (i.e. $c = c_n(\mathscr{X}_n)$) is a function of the n-tuple of observations $\mathscr{X}_n$. The assumption (4.16) of Theorem 4.1 well coresponds to practical applications. The statement of Remark 4.3 enables us to analyse a case of so called "cutted" sequential tests, i.e. of tests, for which it holds $\boldsymbol{P}\{\tau^* \leqq n\} = 1$ for some n. This case is very important in practice.

The finding of the constructive methods for determining functions $\mathrm{r}_n(\mathscr{X}_n)$ for $n = 1, 2, \ldots$ remains an open problem. A solution of this problem for some simple concrete example is introduced in the following chapter. The next works will deal with the general solution of this problem.

5. SEQUENTIAL TEST OF TWO HYPOTHESES FOR CONDITIONALLY UNIFORM PROBABILITY DISTRIBUTION OF OBSERVATIONS

We shall introduce an example of a sequential test of two hypotheses for conditionally uniform probability distribution of observations for an illustration of how to use statement of Theorem 4.1 for a construction of Bayes optimum sequential tests. Our example corresponds to the case $H = 2$, $\mathscr{H} \triangleq \{1, 2\}$. We shall assume that observation x_n for $n \in N$ is a real random variable, thus $M = 1$.

Let a_1, a_2 be given fixed real constants for which it holds $0 < a_1 < a_2$. Let us define sets ${}^iL_n \subset E^n$ for $i = 1, 2$; $n \in N$ by the relation

$${}^iL_n \triangleq \{(x_1, \ldots, x_n) : |x_1| \leqq \tfrac{1}{2}a_i, |x_2 - x_1| \leqq \tfrac{1}{2}a_i, \ldots, |x_n - x_{n-1}| \geqq \tfrac{1}{2}a_i\} \tag{5.1}$$

It evidently holds ${}^1L_n \subset {}^2L_n$.

For the true hypothesis $\boldsymbol{H}_i$, $i = 1, 2$ let it hold

$$(5.2) \qquad {}^i\mathrm{w}_1(x_1) \triangleq \begin{cases} a_i^{-1} & \text{for } |x_1| \leqq \frac{1}{2}a_i \\ 0 & \text{for } |x_1| > \frac{1}{2}a_i \end{cases} \qquad i = 1, 2$$

and for an arbitrary $\mathscr{X}_n = (x_1, \dots, x_n) \in {}^2L_n$ let it hold

$$(5.3) \qquad {}^i\mathrm{w}_{1|n}(x_{n+1} \mid \mathscr{X}_n) \triangleq \begin{cases} a_i^{-1} & \text{for } |x_{n+1} - x_n| \leqq \frac{1}{2}a_i \\ 0 & \text{for } |x_{n+1} - x_n| > \frac{1}{2}a \end{cases} \qquad i = 1, 2$$

${}^1\mathrm{w}_{1|n}(x_{n+1} \mid \mathscr{X}_n)$ is not defined if $\mathscr{X}_n \in {}^2L_n \setminus {}^1L_n$.

Let us denote

$$(5.4) \qquad \pi_n(\mathscr{X}_n) \triangleq {}^1\pi_n(\mathscr{X}_n)\,; \quad a \triangleq L_{12}\,, \quad b \triangleq L_{21}$$

Then

$${}^2\pi_n(\mathscr{X}_n) = 1 - \pi_n(\mathscr{X}_n)$$

and

$$(5.5) \qquad \mathrm{h}(t) \triangleq \min\left[at; b(1-t)\right] \quad \text{for} \quad 0 \leqq t \leqq 1\,.$$

Let us further assume that the cost of observation $c_n(\mathscr{X}_n)$ is finite and constant, i.e.

$$c_n(\mathscr{X}_n) = c > 0\,, \quad n \in N\,, \quad \mathscr{X}_n \in E^n\,.$$

Our problem is to find the Bayes optimum sequential test of hypotheses $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ for the given value π of a priori probability that $\boldsymbol{H}_1$ is true. It holds $0 < \pi < 1$.

From relations (5.1) until (5.4) it follows for $\mathscr{X}_n \in {}^2L_n$

$$(5.6) \qquad \pi_n(\mathscr{X}_n) = \begin{cases} p_n & \text{for } \mathscr{X}_n \in {}^1L_n \\ 0 & \text{for } \mathscr{X}_n \in {}^2L_n \setminus {}^1L_n \end{cases}$$

where

$$p_n = \frac{1}{1 + \eta^n \dfrac{1-\pi}{\pi}}\,, \quad \eta \triangleq a_1/a_2 < 1\,.$$

The value of $\pi_n(\mathscr{X}_n)$ is not defined for $\mathscr{X}_n \notin {}^2L_n$. We can easy show that according to (4.3) and (5.6) it holds for $\mathrm{w}_{1|n}$

$$(5.7) \qquad \mathrm{w}_{1|n}(x_{n+1} \mid \mathscr{X}_n) = \begin{cases} \dfrac{\pi_n(\mathscr{X}_n)}{a_1} + \dfrac{1 - \pi_n(\mathscr{X}_n)}{a_2} & \text{for } |x_{n+1} - x_n| \leqq \frac{1}{2}a_1 \\ \dfrac{1 - \pi_n(\mathscr{X}_n)}{a_2} & \text{for } \frac{1}{2}a_1 < |x_{n+1} - x_n| \leqq \frac{1}{2}a_2 \\ 0 & \text{for } \frac{1}{2}a_2 < |x_{n+1} - x_n| \end{cases}$$

where $\mathscr{X}_n = (x_1, \dots, x_n) \in {}^2L_n$.

We shall prove that it holds in our example for the k-th power $\tilde{Q}^k$ of the operator $\tilde{Q}$

(5.8) $$\tilde{Q}^k[\mathrm{h}(\pi_n(\mathscr{X}_n))] = \min\,[a\pi_n(\mathscr{X}_n);\; b(1-\pi_n(\mathscr{X}_n));\; {}^1\alpha_n(\mathscr{X}_n);\; \ldots;\; {}^k\alpha_n(\mathscr{X}_n)]$$

where

$$ {}^k\alpha_n(\mathscr{X}_n) \triangleq \eta^k b(1-\pi_n(\mathscr{X}_n)) + c(1 + \sum_{l=1}^{k-1}(\pi_n(\mathscr{X}_n) + \eta^l(1-\pi_n(\mathscr{X}_n)))) \quad k \in N\,.$$

We shall do the proof by induction. Let $\mathscr{X}_n = (x_1, \ldots, x_n) \in {}^2L_n$ and $\mathscr{X}_{n+1} = (x_1, \ldots, x_n, x_{n+1})$. From (5.5) until (5.7) it follows that

(5.9) $$M_{n,\mathscr{X}_n}(\mathrm{h}(\pi_{n+1}(\mathscr{X}_{n+1}))) =$$
$$= \int_{|x_{n+1}-x_n| \leqq \frac{1}{2}a_1} \min\,[a\pi_{n+1}(\mathscr{X}_{n+1});\; b(1-\pi_{n+1}(\mathscr{X}_{n+1}))]\left(\frac{\pi_n(\mathscr{X}_n)}{a_1} + \frac{1-\pi_n(\mathscr{X}_n)}{a_2}\right)\mathrm{d}x_{n+1} = \min\,[a\pi_n(\mathscr{X}_n);\; \eta b(1-\pi_n(\mathscr{X}_n))]\,.$$

After the substitution of (5.9) into (4.21) we obtain

$$\tilde{Q}[\mathrm{h}(\pi_n(\mathscr{X}_n))] = \min\,[a\,\pi_n(\mathscr{X}_n);\; b(1-\pi_n(\mathscr{X}_n));\; \eta b(1-\pi_n(\mathscr{X}_n)) + c]$$

and it is proved the validity of (5.8) for $k = 1$. Let now (5.8) hold for some $k \in N$. Then for $\tilde{Q}^{k+1}$ we obtain

(5.10) $$\tilde{Q}^{k+1}[\mathrm{h}(\pi_n(\mathscr{X}_n))] = \tilde{Q}[\tilde{Q}^k[\mathrm{h}(\pi_n(\mathscr{X}_n))]] =$$
$$= \min\,[a\pi_n(\mathscr{X}_n),\; b(1-\pi_n(\mathscr{X}_n));\; {}^1\alpha_n(\mathscr{X}_n);\; \ldots;\; {}^k\alpha_n(\mathscr{X}_n);\; M_{n,\mathscr{X}_n}(\min\,[a\,\pi_{n+1}(\mathscr{X}_{n+1});$$
$$b(1-\pi_{n+1}(\mathscr{X}_{n+1}));\; {}^1\alpha_{n+1}(\mathscr{X}_{n+1});\; \ldots;\; {}^k\alpha_{n+1}(\mathscr{X}_{n+1})]) + c]\,.$$

Let us determine the expected value $M_{n,\mathscr{X}_n}$ from the right side of (5.10)

(5.11) $$M_{n,\mathscr{X}_n}(\min\,[a\pi_{n+1}(\mathscr{X}_{n+1}),\; b(1-\pi_{n+1}(\mathscr{X}_{n+1}));$$
$${}^1\alpha_{n+1}(\mathscr{X}_{n+1});\; \ldots;\; {}^k\alpha_{n+1}(\mathscr{X}_{n+1})]) =$$
$$= \int_{|x_{n+1}-x_n| \leqq \frac{1}{2}a_1} \min\,[a\pi_{n+1}(\mathscr{X}_{n+1});\; \ldots;\; {}^k\alpha_{n+1}(\mathscr{X}_{n+1})]\cdot\left(\frac{\pi_n(\mathscr{X}_n)}{a_1} + \frac{1-\pi_n(\mathscr{X}_n)}{a_2}\right)\mathrm{d}x_{n+1} = \min\,[a\pi_n(\mathscr{X}_n);\; \eta b(1-\pi_n(\mathscr{X}_n));\; {}^2\alpha_n(\mathrm{X}_n) - c;\; \ldots;$$
$${}^{k+1}\alpha_n(\mathscr{X}_n) - c]\,.$$

Substituting (5.11) into (5.10) and after an easy arrangement we obtain (5.8) for $k + 1$. By this, the validity of (5.8) is proved for all $k \in N$.

We shall now show that for every $n \in N$ and $\mathscr{X}_n \in {}^2L_n$ there exists such $k_0 = k_0(n) \in N$ that for all $k \geqq k_0$ it holds

$$\tilde{Q}^k[\mathrm{h}(\pi_n(\mathscr{X}_n))] = Q^{k_0}[\mathrm{h}(\pi_n(\mathscr{X}_n))] , \quad k \in N , \quad k \geqq k_0 . \tag{5.12}$$

After an easy arrangement we can prove that the inequality

$$^{k+1}\alpha_n(\mathscr{X}_n) \geqq {}^k\alpha_n(\mathscr{X}_n) \quad k \in N \tag{5.13}$$

is equivalent to the inequality

$$p_{n+k} \geqq 1 - \frac{c}{b(1 - \eta)} \tag{5.14}$$

for $\mathscr{X}_n \in {}^1L_n$. It follows from (5.6) that $p_{n+1} > p_n$, $\lim_{n \to \infty} p_n = 1$. It is thus clear that there must exist such $k_0 = k_0(n)$ that (5.14) holds for all $k \geqq k_0$. For such k then also (5.13) holds which together with (5.8) proves the relation (5.12) for $\mathscr{X}_n \in {}^1L_n$. Since for $\mathscr{X}_n \in {}^2L_n \setminus {}^1L_n$ (5.12) holds for an arbitrary $k_0 \in N$ the proof of the validity (5.12) is completed.

According (4.25) and using (5.12) we can write

$$\mathrm{r}_n(\mathscr{X}_n) = \tilde{Q}^{k_0(n)}[\mathrm{h}(\pi_n(\mathscr{X}_n))] . \tag{5.15}$$

We shall prove that there exists such $n_0 \in N$ that it holds for all $n \geqq n_0$, $n \in N$

$$\mathrm{r}_n(\mathscr{X}_n) = \mathrm{h}(\pi_n(\mathscr{X}_n)) \quad \mathscr{X}_n \in {}^2L_n . \tag{5.16}$$

It is clear from (5.5), (5.6), (5.8) and (5.15) that the equation (5.16) holds for every $n_0 \in N$ when $\mathscr{X}_n \in {}^2L_n \setminus {}^1L_n$. Then we shall continue the proof only for $\mathscr{X}_n \in {}^1L_n$. Let us check, for which $n \in N$ ti holds

$$b(1 - p_n) \leqq {}^1\alpha_n(\mathscr{X}_n) \quad \mathscr{X}_n \in {}^1L_n . \tag{5.17}$$

Inequality (5.17) is equivalent to the inequality

$$p_n \geqq 1 - \frac{c}{b(1 - \eta)} . \tag{5.18}$$

By the same manner, as in the discusion of (5.14), we can easy show that there exists such n_0 that (5.17) holds for $n \geqq n_0$. It follows from the discussion of (5.13) and (5.14) that for such $n \geqq n_0$ it holds

$$b(1 - p_n) \leqq {}^k\alpha_n(\mathscr{X}_n) \quad \mathscr{X}_n \in {}^1L_n \tag{5.19}$$

also for an arbitrary $k \in N$. Relations (5.17) and (5.19) prove the validity of (5.16) for $\mathscr{X}_n \in {}^1L_n$ and by this the proof of validity (5.16) is completed for all $\mathscr{X}_n \in {}^2L_n$.

Taking into account Theorem 4.1 and Remark 4.2, we can summarize our discussions as follows:

If the inequality

$$\pi \geqq 1 - \frac{c}{b(1-\eta)}$$

holds, then the test without observations $(d, 0)$ brings the least risk. This test selected decision d according to the relation

$$d = \begin{cases} 1 & \text{for} \quad a\pi > b(1-\pi) \\ 2 & \text{for} \quad a\pi \leqq b(1-\pi) . \end{cases}$$

Everywhere below we shall assume that it holds

$$\pi < 1 - \frac{c}{b(1-\eta)} . \tag{5.20}$$

We shall note that inequality $(1 - \eta) b > c$ follows from (5.20). In accordance with (5.18) let us define n_0 by the relation

$$n_0 \triangleq \min_{n \in N} \left[n \colon p_n \geqq 1 - \frac{c}{b(1-\eta)} \right] . \tag{5.21}$$

If it holds $ap_{n_0} \leqq b(1 - p_{n_0})$, then the test without observations (2,0) reaches the least risk. This test always accepts H_2. Let then (5.20) hold and simultaneously

$$ap_{n_0} > b(1 - p_{n_0}) .$$

Then the Bayes optimum sequential test (d^*, τ^*) of hypotheses H_1, H_2 is defined by the relations

$$\tau^* = \min \left[n_0 ; \inf_{n \in N} \{ n \colon \mathscr{X}_n \in {}^2L_n \setminus {}^1L_n \} \right] \tag{5.22}$$

$$d^* = \begin{cases} 1 & \text{for} \quad \tau^* = n_0 , \quad \mathscr{X}_{n_0} \in {}^1L_n \\ 2 & \text{for} \quad \tau^* \leqq n_0 , \quad \mathscr{X}_{\tau^*} \in {}^2L_n \setminus {}^1L_n \end{cases}$$

Note, that the infimum of the empty set is defined to be $+\infty$ in (5.22).

The Bayes optimum sequential test of hypotheses H_1, H_2 is thus a test which makes sequentially observations $x_1, \ldots, x_{n_0}$. If it holds $\mathscr{X}_n = (x_1, \ldots, x_n) \in {}^2L_n \setminus {}^1L_n$ for some $n \leqq n_0$, then the test is stopped, no more observation is made and H_2 is accepted. If the test continued without stopping until the observation x_{n_0} and simultaneously it holds $\mathscr{X}_{n_0} \in {}^1L_{n_0}$, then the test also stops and H_1 is accepted.

Test (d^*, τ^*), which we obtained for our concrete example by solution of (4.17) till (4.21), is in good agreement with our intuitive ideas about an "reasonable" testing of hypotheses $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$. Note that it is „cutted" test, since $\tau^* \leqq n_0$ always holds. This circumstance is not caused due to the dependence of the cost c on the serial number n and the n-tuple of observation $\mathscr{X}_n$ in the sense of Remark 4.3, but it is given directly by an essence of the problem, since $\boldsymbol{H}_2$ we can accept with a certainty for $\mathscr{X}_n \in {}^2L_n \setminus {}^1L_n$.

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Ing. Jiří Cochlar, CSc., katedra radioelektronických zařízení a soustav elektrotechnické fakulty ČVUT (Czech. Technical University — Department of Radioeletronic Devices and Systems), Suchbátarova 2, 166 27 Praha 6. Czechoslovakia.