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DUALITY THEORY IN MATHEMATICAL PROGRAMMING AND OPTIMAL CONTROL

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The supplement deals with the duality theory in extremal problems. It is divided into three parts according to the nature of the original problem (convex, smooth nonconvex and nonsmooth locally Lipschitz problems). From various aspects of the duality theory preferably the possibility of replacing the original (primal) problem by another easier (dual) problem is discussed. Of course, also other advantages of this theory like an easy deriving of optimality conditions or proving the existence of solutions are demonstrated. Most dual schemes are derived in a unified way using the perturbational theory of duality. Single dualisations are illustrated mostly on constrained optimal control problems.

PART I: CONVEX PROBLEMS

0. Introduction

Various duality principles or dual relations can be disclosed in different branches of mathematics like in logic, geometry, algebraic topology etc. Generally, *duality* is a certain relation between two classes of objects of the appropriate theory. This relation provides us with a better insight into the nature of the objects in question which brings considerable advantages for both the theoretical investigations and numerical solution of concrete problems.

The present paper deals with duality in extremal problems, where, generally, to some original minimization problem, a certain dual maximization problem is assigned, and their solutions and optimal cost values are related. The construction of a suitable dual problem (or a family of dual problems) may help us in

(i) the proof of the existence of a solution of the original (primal) problem;

(ii) the solution of the original problem by way of the dual provided the dual problem is easily solvable;

(iii) providing us with various important characteristics like dual estimates of the cost, estimates of the sensitivity, necessary and sufficient optimality conditions etc.

In the sequel preferably the item (ii) will be persuaded. As locally convex spaces are especially suitable for the formulation of most extremal problems, the duality theory of linear topological spaces (cf. e.g. [7]) gives us the necessary framework.

The paper is divided into three parts, devoted to convex problems, smooth nonconvex problems and locally Lipschitz problems, respectively. In the convex case the duality theory is quite natural because closed convex sets as well as lower semicontinuous functions possess dual representations in conjugate spaces. Indeed, closed convex sets in locally convex spaces are intersections of closed halfspaces containing them so that we may associate to each convex set A the dual object — its polar A^0. Similarly, a lower semicontinuous proper convex function f on a locally convex space is the pointwise supremum of all affine functions not exceeding it. The corresponding dual object is the conjugate or polar function f^*, cf. Sec. 1. Thus, the essence of the duality theory for convex extremal problems lies in the bipolar theorem and the Fenchel-Moreau theorem (Prop. 1.4) specifying the assumptions under which $A^{00} = A$ and $f^{**} = f$, respectively. Based on these general considerations, the *perturbational theory of duality* has been developed and enables to assign to most convex extremal problems in a unified way their various dual problems and to examine the appropriate dual relations. For this versatile and extremely fruitful theory we have to thank to W. Fenchel, J. J. Moreau and especially R. T. Rockafellar. Most duality results which have been obtained previously can be derived within the framework of the perturbational theory in an easy and elegant way.

In the first part of this paper we collect at the beginning those basic results of convex analysis which are needed for all further considerations (Sec. 1). The fundamentals of the perturbational theory of duality can be found in Sec. 2. Further sections contain then the descriptions of most important duality schemes accompanied with illustrative examples from mathematical programming and optimal control.

The excellent success of the convex perturbational theory has forced the mathematicians to investigate similar approaches even in the nonconvex case. Several such theories have been developed and we shall describe them briefly in the introductory section of the second part of the sequel. However, one of them we have chosen as a general pattern for an easy and elegant construction of dual problems to a nonconvex original problem. It is the perturbational theory of Lindberg [14], [15], and it will be explained in Sec. 8. Besides various dual schemes derived with the help of this nonconvex perturbational theory for smooth extremal problems, we include to the second part of the sequel also some special duality results, exploiting the specific nature of the problem. Herewith we mean e.g. the duality theory of Toland ([36], [37]).

Of course, the Gâteaux and Fréchet derivatives in smooth problems and subdifferentials in convex problems are very important for the construction of dual problems and their subsequent solution. Fortunately, we have now at our disposal convex local approximation sets substituting the gradients or the subdifferentials in the case of nonsmooth or even noncontinuous functions without the convexity requirement. In the third part of the sequel we shall deal with the locally Lipschitz case where the local approximation set – the generalized gradient of Clarke – has been advantageously exploited already in the computational praxis. We exploit again the perturbation theory of Lindberg to get an important class of nonsmooth dual problems, the properties of which will be examined in detail.

Throughout the whole sequel the single duality schemes (dualisations) are applied to one of the following three extremal problems:

(i) general minimization problem ($\mathscr{P}\,1$):

$$f(x) \to \inf$$

subj. to

$$x \in \boldsymbol{A} \subset \boldsymbol{X},$$

where $f[\boldsymbol{X} \to \mathbb{R}_{cl}]$;

(ii) mathematical programming problem ($\mathscr{P}$ 2):

$$f(x) \to \inf$$

subj. to

$$x \in A \subset \boldsymbol{X},$$

$$-q(x) \in D \subset \boldsymbol{Z},$$

where $f[\boldsymbol{X} \to \mathbb{R}_{cl}]$, $q[\boldsymbol{X} \to \boldsymbol{Z}]$;

(iii) abstract optimal control problem ($\mathscr{P}$ 3):

$$J(u, y) \to \inf$$

subj. to

$$y = \Lambda u,$$

$$u \in M \subset \boldsymbol{U},$$

$$y \in N \subset Y,$$

where $J[\boldsymbol{U} \times \boldsymbol{Y} \to \mathbb{R}_{cl}]$, $\Lambda[\boldsymbol{U} \to \boldsymbol{Y}]$.

In the above problem formulations $\boldsymbol{X}, \boldsymbol{Z}, \boldsymbol{U}, \boldsymbol{Y}$ are linear topological spaces. In the whole first part of the paper besides Sec. 6 it is assumed that Λ is an affine operator, i.e.

$$\Lambda u = \Pi u + y_0,$$

where $\Pi \in \mathscr{L}[\boldsymbol{U}, \boldsymbol{Y}]$, the space of linear continuous operators mapping $\boldsymbol{U}$ into $\boldsymbol{Y}$.

We shall employ the following notation: $\mathbb{N}$ is the set of natural numbers, $\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ is the set of rationals, $\mathbb{R}$ is the real line, $\mathbb{R}_{cl}$ is the extended real line, ϑ is the zero vector, A^* is the conjugate of a linear operator A, x^j is the j-th coordinate of a vector x, $|\cdot|_n$ is a norm in $\mathbb{R}^n$ (the Euclidean n-dimensional space), $\|\cdot\|$ is a norm in an infinite-dimensional normed space, $\nabla f(x)$ and $\nabla^G f(x)$ are the Fréchet and Gâteaux derivative of a functional (operator) f at x, respectively. The sign $\leqq$ in $\mathbb{R}^n$ means that the inequality is valid for all coordinates, cl A denotes the closure of a set A and $\langle \cdot, \cdot \rangle$ is a bilinear form (pairing). If more different pairings appear in one expression, we add the notation of one appropriate space as a subscript. $\langle \cdot, \cdot \rangle_n$ is the inner product in $\mathbb{R}^n$. $\mathfrak{U}_{Y,\tau}(x)$ denotes the filter of all neighbourhoods of x in a linear topological space (Y, τ). The index corresponding to the topology will be omitted if there is not any possibility of confusion. $B^\varepsilon_{\boldsymbol{Y}}(a) = \{y \in \boldsymbol{Y} \mid \|y - a\| \leqq \varepsilon\}$ for $\boldsymbol{Y}$ being a normed linear space, in $\mathbb{R}^n$ we write simply $B^\varepsilon_n(a)$. Finally, the directional derivative of a functional $f[\boldsymbol{X} \to \mathbb{R}_{cl}]$ at $x_0 \in \boldsymbol{X}$ in the direction $h \in \boldsymbol{X}$ will be denoted as follows

$$D f(x_0; h) = \lim_{\lambda \to 0_+} \frac{f(x_0 + \lambda h) - f(x_0)}{\lambda}.$$

We say that f is directionally differentiable at x_0 if for all h $D f(x_0; h) \in \mathbb{R}_{cl}$. For the purpose of its computation we consider $\infty - \infty = 0, (-\infty) - (-\infty) = 0$. epi f denotes the epigraph of a functional f.

1. Convex analysis

The aim of this preparatory section is to provide a reader with a brief review of the main definitions and assertions of convex analysis which will be extensively used throughout the whole sequel. The proofs cannot be presented but the corresponding references will be given. We suppose the knowledge of elementary concepts like convexity of functions and sets, lower semicontinuity of functions, indicatory function of a set A which will be denoted by δ_A, support function of a set A which will be denoted by δ_A^*, interior of a set A denoted int A, projection of a vector x on a set A denoted x^A as well as a basic knowledge of functional analysis.

Let V and V^* be two linear topological spaces and $\langle \cdot, \cdot \rangle$ be a bilinear form defined on $V \times V^*$. We shall require that this bilinear form is a *pairing*, i.e. it has the separation property: if $\langle u^*, u \rangle = 0\, \forall u^* \in V^*$ then $u = \vartheta$; if $\langle u^*, u \rangle = 0\ \forall u \in V$, then $u^* = \vartheta$. Topologies on V, V^* are *compatible* with the pairing (duality) if the linear functions $\langle \cdot, u \rangle$, $\langle u^*, \cdot \rangle$ are all continuous and every continuous linear function on V, V^* can be represented in this form for some $v^* \in V^*$, $v \in V$ respectively. When we say that V is *placed in duality* or *paired* with V^* by a pairing $\langle \cdot, \cdot \rangle$, we mean that V and V^* have been equipped with compatible topologies. Compatible topologies render the spaces V, V^* locally convex and Hausdorff. Among them we may distinguish the weak topologies denoted $w(V, V^*)$, $w(V^*, V)$ generated by the families of seminorms $|\langle v^*, \cdot \rangle|$, $|\langle \cdot, v \rangle|$ respectively; they are the coarsest topologies compatible with the pairing. The most important example of a pairing is the *canonical pairing*: V is a Hausdorff locally convex linear topological space endowed with a topology τ, V^* is its topological dual and $\langle v^*, v \rangle = v^*(v)$; the separation property is now a consequence of the Hahn-Banach theorem.

Let us still remark that for V, V^* being paired by a pairing $\langle \cdot, \cdot \rangle$ and for convex sets of V (V^*) the property to be closed is independent of the choice of the compatible topology.

Throughout the first part of the paper the most general structure which will be used is the Hausdorff locally convex linear topological space denoted consistently l.c.s. We start now with the continuity properties of convex functions.

Proposition 1.1. Every lower semicontinuous (l.s.c.) convex function $f[V \to \mathbb{R}_{cl}]$ remains l.s.c. when V is endowed by $w(V, V^*)$ topology.

Definition 1.1. A function $f[V \to \mathbb{R}_{cl}]$ will be termed *proper convex* if

(i) f is convex;

(ii) $f(u) > -\infty$ for all $u \in V$;

(iii) the *effective domain* of f denoted $\operatorname{dom} f := \{u \in V \mid f(u) < +\infty\}$ is nonempty.

A function $g[V \to \mathbb{R}_{cl}]$ will be termed *proper concave* if $-g$ is proper convex.

Proposition 1.2. Let $f[V \to \mathbb{R}_{cl}]$ be a convex function. The following statements are equivalent to each other:

(i) there exists a nonempty open set $\mathcal{O}$ on which f is not everywhere equal to $-\infty$ and is bounded above by a constant $a < +\infty$;
(ii) f is proper convex and continuous over int (dom f) which is nonempty.

Corollary 1.2.1. Let f be a proper convex function on a normed space. The following properties are equivalent to each other:
(i) there exists a nonempty open set on which f is bounded above;
(ii) int (dom f) $\neq \emptyset$ and f is locally Lipschitz there.

Corollary 1.2.2. Every convex l.s.c. function over a barrelled space (in particular a Banach space) is continuous over the interior of its effective domain.

Remark. A *barrel* in V is a closed convex balanced absorbing subset of V. V is *barrelled* if every barrel is a neighborhood of ϑ.

It is an easy consequence of the convexity that every convex function is directionally differentiable everywhere on V.
We turn now our attention to the important notion of conjugate or polar functions.

Definition 1.2. The set of functions $f[V \to \mathbb{R}_{cl}]$ which are pointwise supremum of a family of continuous affine functions is denoted by $\Gamma(V)$. $\Gamma_0(V)$ is the subset of $f \in \Gamma(V)$ other than the constants $+\infty$ and $-\infty$.

Proposition 1.3. The following properties are equivalent to each other:
(i) $f \in \Gamma(V)$;
(ii) f is a convex l.s.c. function from V into $\mathbb{R}_{cl}$, and if f takes the value $-\infty$, then $f \equiv -\infty$.

For a function $f[V \to \mathbb{R}_{cl}]$ its largest minorant in $\Gamma(V)$ is termed the *Γ-regularization* of f.

Definition 1.3. Let $f[V \to \mathbb{R}_{cl}]$ and $f^*[V^* \to \mathbb{R}_{cl}]$ be given by

$$f^*(u^*) = \sup_{u \in V} [\langle u^*, u \rangle - f(u)] . \tag{1.1}$$

Then f^* is called the *conjugate* (or *polar*) function of f in the convex sense and the function $^*f[V^* \to \mathbb{R}_{cl}]$ given by

$$^*f(u^*) = \inf_{u \in V} [\langle u^*, u \rangle - f(u)] \tag{1.2}$$

is called the *conjugate* (or *polar*) of f in the concave sense.

One can easily see that the support function of a set A is conjugate in convex sense to the indicatory function of A. This justifies our notation.
We can repeat the process which leads to the biconjugate $f^{**}[V \to \mathbb{R}_{cl}]$ given by

$$f^{**}(u) = \sup_{u^* \in V^*} [\langle u^*, u \rangle - f^*(u^*)] . \tag{1.3}$$

Proposition 1.4. Let $f[V \to \mathbb{R}_{cl}]$. Then its biconjugate f^{**} is actually its Γ-regularization. Hence, if $f \in \Gamma(V), f^{**} = f$.

Generally, of course, $f^{**} \leqq f$.

One often needs to compute the conjugate of a sum or of an integral functional. For this purpose we recall here the notion of *infimal convolution*. For n proper convex functions $f_i[V \to \mathbb{R}_{cl}]$, $i = 1, 2, \ldots, n$, it is the convex function

$$(1.4) \qquad (\mathop{\square}_{i=1}^{n} f_i)(u) = (f_1 \square f_2 \square \ldots \square f_n)(u) = \inf \{ \sum_{i=1}^{n} f_i(u_i) \mid \sum_{i=1}^{n} u_i = u \} .$$

Proposition 1.5. Let $f_1, \ldots, f_n[V \to \mathbb{R}_{cl}]$ be proper convex functions and there exists a point $u_0 \in \bigcap_{i=1}^{n} \operatorname{dom} f_i$, where all f_i with the exception of, may be, one of them are continuous. Then,

$$(f_1 + f_2 + \ldots + f_n)^* = f_1^* \square f_2^* \square \ldots \square f_n^* .$$

Proposition 1.6. Let f be a function from $[0, 1] \times \mathbb{R}^n$ to $(-\infty, +\infty]$,

$$I_f(u) = \int_0^1 f(t, u(t)) \, \mathrm{d}t$$

be an integral functional defined on the space V, and

$$I_{f^*}(u^*) = \int_0^1 f^*(t, u^*(t)) \, \mathrm{d}t$$

be an integral functional defined on the space V^* (the conjugation of f is taken only with respect to the second variable). Let the space V, V^* be decomposable, f be a normal convex integrand such that $f(t, u(t))$ is summable in t for at least one $u \in V$ and $f^*(t, u(t))$ be summable in t for at least one $u^* \in V^*$. Then I_f on V and I_{f^*} of V^* are proper convex functions conjugate to each other.

Remark. For the proof of the preceding assertion as well as definitions of "decomposable spaces" and "normal convex integrand" see [28]. We remark here only that L_p spaces $(1 \leqq p < \infty)$ are decomposable, and if

(i) $f(t, x)$ is a convex function of $x \in \mathbb{R}^n$ for each $t \in [0, 1]$;
(ii) $f(t, x)$ is measurable in t for each fixed x;
(iii) for each $t, f(t, x)$ is l.s.c. in x and $\operatorname{int} \{x \mid f(t, x) < +\infty\} \neq \emptyset$,
then f is a normal convex integrand.

In [28] the strong properties of so called proximal mapping are used in the proof of Prop. 1.6.

Definition 1.4. Let $\boldsymbol{H}$ be a Hilbert space and $f \in \Gamma_0(\boldsymbol{H})$. The mapping which assigns to each $x \in \boldsymbol{H}$ the unique minimum $\bar{u}$ of the function $f(u) + \frac{1}{2}\|u - x\|_{\boldsymbol{H}}^2$ is termed the *proximal mapping* (with respect to f) and we write

$$\bar{u} = \text{prox}_f\, x\,.$$

Some of many useful properties of this mapping are collected in two following assertions:

Proposition 1.7. Let x, y, z be three arbitrary elements of $\boldsymbol{H}$ and $f \in \Gamma_0(\boldsymbol{H})$. The following statements are equivalent to each other:

(i) $$z = x + y \qquad f(x) + f^*(y) = \langle x, y\rangle\,;$$

(ii) $$x = \text{prox}_f\, z \qquad y = \text{prox}_{f^*}\, z\,.$$

Proposition 1.8. At every point $z \in \boldsymbol{H}$ the function

$$\varphi(z) = \inf_{u \in \boldsymbol{H}} \left[f(u) + \tfrac{1}{2}\|u - z\|^2\right] = \left(f \,\square\, \tfrac{1}{2}\|\cdot\|^2\right)(z)$$

is Fréchet differentiable, and

$$\nabla\varphi(z) = \text{prox}_{f^*}\, z\,.$$

The introduction of conjugate functions enables us to find very fruitful relations between various important properties like "continuity" and "boundedness", or "differentiability" and "rotundity".

Definition 1.5. A set A in $\boldsymbol{V}$ is *bounded* if $\delta_A^*(u^*) < +\infty$ for all $u^* \in \boldsymbol{V}^*$. A is *equicontinuous* if δ_A^* is continuous on $\boldsymbol{V}^*$ in $w(\boldsymbol{V}^*, \boldsymbol{V})$.

An important theorem of functional analysis asserts that if A is closed, convex and equicontinuous, then it is in fact $w(\boldsymbol{V}, \boldsymbol{V}^*)$ compact. This is used in the assertion below.

Proposition 1.9. Let $f[\boldsymbol{V} \to \mathbb{R}_{cl}]$ be finitely bounded above on a neighbourhood of ϑ. Then the level sets

$$\text{lev}_\beta f^* := \left\{u^* \in \boldsymbol{V}^* \,\middle|\, f^*(u)^* \leqq \beta\right\}, \quad \beta \in \mathbb{R}$$

are all equicontinuous (also closed, convex and hence $w(\boldsymbol{V}^*, \boldsymbol{V})$ compact). Conversely, if one of these level sets for $\beta > \inf f^*$ is equicontinuous, then f^{**} is continuous at ϑ.

Definition 1.6. Let $f[\boldsymbol{V} \to \mathbb{R}_{cl}]$ and $\boldsymbol{V}$ be endowed with a Hausdorff topology τ. We shall say that f is *τ-rotund* at a given $u \in \boldsymbol{V}$ relative to a given $u^* \in \boldsymbol{V}^*$ if $f(u)$ is finite and for every $\mathcal{O} \in \mathfrak{U}_{\boldsymbol{V},\tau}(\vartheta)$ there exists a $\delta > 0$ such that

$$\left\{v \,\middle|\, f(u + v) - f(u) - \langle u^*, v\rangle \leqq \delta\right\} \subset \mathcal{O}\,.$$

It can be seen that f is τ-rotund at v relative to v^* iff the infimum of the functional $f - \langle v^*, \cdot\rangle$ over $\boldsymbol{V}$ is attained at v *strongly* with respect to τ, i.e. every sequence

$\{y_i\} \subset V$ with

$$\lim_{i\to\infty} [f(y_i) - \langle v^*, y_i\rangle] = \inf_{y\in V} [f(y) - \langle v^*, y\rangle]$$

is τ-convergent to v.

Proposition 1.10. Let $f[V \to \mathbb{R}_{cl}]$ be $w(V, V^*)$ l.s.c. at u and have the value $+\infty$ everywhere outside of a certain bounded subset of V. Then f^* is Gâteaux-differentiable at u^* with $\nabla^G f^*(u^*) = u$ iff f is $w(V, V^*)$-rotund at u relative to u^*. If V is a normed space and V^* its normed dual and the norm topologies are compatible with the duality, then f^* is Fréchet-differentiable at u^* with $\nabla f^*(u^*) = u$ iff f is norm-rotund at u relative to u^*.

Proposition 1.11. Let f and f^* be proper convex functions conjugate to each other on V V^*, respectively. Then f^* is Gâteaux-differentiable at u^* with $\nabla^G f^*(u^*) = u$ iff f is $w(V, V^*)$-rotund at u relative to u^*. If V is a normed space and V^* its normed dual, then f^* is Fréchet-differentiable at u^* with $\nabla f^*(u^*) = u$ iff f is norm-rotund at u relative to u^*.

To investigate local properties of convex functions, the concept of subgradient (subdifferential) seems to be quite natural and extremely useful.

Definition 1.7. A function $f[V \to \mathbb{R}_{cl}]$ is said to be *subdifferentiable* at a point $u \in V$ if it has a continuous affine minorant which is exact at u. The slope $u^* \in V^*$ of such a minorant is called a *subgradient* of f at u, and the set of all subgradients at u is called the *subdifferential* at u and is denoted by $\partial f(u)$.

Proposition 1.12. Let $f[V \to \mathbb{R}_{cl}]$. Then $u^* \in \partial f(u)$ iff

$$f(u) + f^*(u^*) = \langle u^*, u\rangle \,. \tag{1.4}$$

The above equation implies in particular that the set $\partial f(u)$ (possibly empty) is convex and $w(V^*, V)$-closed in V^*. Furthermore,

$$u^* \in \partial f(u) \Rightarrow u \in \partial f^*(u^*)\,, \tag{1.5}$$

and for $f \in \Gamma(V)$ the converse of (1.5) is also true.

Proposition 1.13. Let $f[V \to \mathbb{R}_{cl}]$ be a convex function, finite and continuous at a point $u \in V$. Then $\partial f(v) \neq \emptyset$ for all $v \in \operatorname{int}(\operatorname{dom} f)$, and in particular $\partial f(u) \neq \emptyset$.

If $f[V \to \mathbb{R}]$ is convex, finite and continuous at $u \in V$, then $\partial f(u) = \{u^* \in V^* \mid \langle u^*, h\rangle \leqq Df(u; h)\, \forall h \in V\}$.

Subdifferentials of significant functions play mostly also an important role in convex analysis. E.g. in the case of the indicatory function of a convex set $A \subset V$ we obtain for

$$\partial\delta_A(u) = \{u^* \in V^* \mid \langle u^*, z - u\rangle \leqq 0\ \forall z \in A\}$$

which is the *normal cone* of A at u. This cone will be denoted $N_A(u)$.

The relation between subdifferentiability and Gâteaux-differentiability is expressed in the following assertion:

Proposition 1.14. Let $f[V \to \mathbb{R}_{cl}]$ be convex. If f is Gâteaux-differentiable at $u \in V$, it is subdifferentiable at u and $\partial f(u) = \nabla^G f(u)$. Conversely, if at a point $u \in V$, f is continuous and finite and has only one subgradient, then f is Gâteaux-differentiable at u and $\partial f(u) = \nabla^G f(u)$.

One often needs to compute subgradients of sums or composite functions. The appropriate rules belong to so called subdifferential calculus.

Proposition 1.15. Let $f_1, f_2[V \to \mathbb{R}_{cl}]$. At every point $u \in V$, we have

$$\partial(f_1 + f_2)(u) \subset \partial f_1(u) + \partial f_2(u).$$

If $f_1, f_2 \in \Gamma(V)$ and if there is a point $\bar{u} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ where f_1 is continuous, we have for all $u \in V$

$$\partial(f_1 + f_2)(u) = \partial f_1(u) + \partial f_2(u).$$

Proposition 1.16. Let V, V^* and Y, Y^* be two pairs of l.c.s. placed in duality by pairings $\langle \cdot, \cdot \rangle_V$, $\langle \cdot, \cdot \rangle_Y$, respectively. Let $\Lambda \in \mathscr{L}[V, Y]$ and $f \in \Gamma(Y)$. Then, clearly, $f \circ \Lambda \in \Gamma(V)$. If, moreover, there exists a point $\Lambda\bar{u}$ where f is continuous and finite, then for all $u \in V$

$$\partial(f \circ \Lambda)(u) = \Lambda^* \partial f(\Lambda u).$$

Let now $f[V_1 \times V_2 \to \mathbb{R}_{cl}]$ be convex on the cartesian product of l.c.s. V_1, V_2. It is natural to denote, for each $u \in V_1$, the subdifferential of the function $v \mapsto f(u, v)$ by $\partial_v f(u, v)$; similarly we define $\partial_u f(u, v)$.

It can easily be seen that the rotundity of a functional $f[V \to \mathbb{R}_{cl}]$ at a point $u_0 \in V$ may be examined merely with respect to functionals $u^* \in \partial f(u_0)$.

Lemma 1.1. Let V be normed, $f[V \to \mathbb{R}_{cl}]$ and $u^* \in \partial f(u)$. If there exists a function $\varphi_u[\mathbb{R}_+ \to \mathbb{R}_+]$ such that $\inf_{[\varepsilon, +\infty)} \varphi_u > 0$ for all $\varepsilon > 0$, and

$$f(v) \geqq f(u) + Df(u; v - u) + \varphi_u(\|v - u\|),$$

then f is norm-rotund at u with respect to u^*.

Besides the subdifferential, another set with rather strong properties is frequently used in convex analysis, namely the ε-**subdifferential**.

Definition 1.8. Let $f[V \to \mathbb{R}_{cl}]$ and $\varepsilon > 0$. The set

$$\{u^* \in V^* \mid \langle u^*, u \rangle \leqq f(u) + f^*(u^*) \leqq \langle u^*, u \rangle + \varepsilon\}$$

is called the ε-*subdifferential* of f at u and denoted $\partial^\varepsilon f(u)$.

For $f \in \Gamma_0(V)$ we obtain

$$u^* \in \partial^\varepsilon f(u) \Leftrightarrow u \in \partial^\varepsilon f^*(u^*) .$$

The sets $\partial^\varepsilon f(u)$ decrease with ε and their intersection for $\varepsilon > 0$ is the subdifferential $\partial f(u)$.

We close the first section with an important existence result concerning the general extremal problem ($\mathscr{P}$ *1*). Of course, it can easily be extended to other problems like ($\mathscr{P}$ *2*) or ($\mathscr{P}$ *3*) if necessary.

Proposition 1.17. Let V be a reflexive Banach space, $f[V \to \mathbb{R}_{cl}]$ be proper convex and l.s.c., and A be a nonempty closed convex subset of V. Let, in addition, either the set A be founded, or the function f be *coercive* on A, i.e.

$$\lim f(u) = +\infty \quad \text{for} \quad u \in A , \quad \|u\| \to \infty .$$

Then the problem ($\mathscr{P}$ *1*) has at least one solution. It has a unique solution if the function f is strictly convex over A.

A few important definitions and assertions collected in this section by far do not exhaust the large number of useful results of convex analysis for the theory of extremal problems in general and convex perturbation theory of duality in particular.

It is merely an absolute minimum, necessary for a basic understanding of further sections. The proofs of Props. 1.1–1.5 and 1.13–1.17 can be found in [8], 1.6 in [28], 1.7, 1.8 in [19], 1.9 in [25] and 1.10, 1.11 in [1]. The proof of Lemma 1.1 is evident. We refer the interested reader to these works containing many more important and convenient results.

2. Perturbation theory of duality

In this section we show an efficient and elegant way how to associate (under certain assumptions) to the general minimization problem ($\mathscr{P}$ *1*) a dual maximization problem ($\mathscr{D}$ *1*) and we shall examine the relationship between these two problems. With respect to the limited extent of the sequel, we shall present only main ideas of the most proofs; however, exact references will be given.

Let X be a l.c.s. placed in duality with X^* by a pairing $\langle \cdot, \cdot \rangle_X$. Furthermore, let V, V^* be two other l.c.s. placed in duality by a pairing $\langle \cdot, \cdot \rangle_V$. We will consider a suitable function $\Phi[X \times V \to \mathbb{R}_{cl}]$ termed the *perturbed essential objective* such that

$$\Phi(x, \vartheta) = f(x) + \delta_A(x) , \tag{2.1}$$

and for every $p \in V$ we will examine the extremal problem

$$(\mathscr{P}\ 1_p) \qquad \inf_{x \in X} \Phi(x, p) .$$

Clearly for $p = \vartheta$ ($\mathscr{P}$ 1_p) is the same as ($\mathscr{P}$ *1*). In what follows ($\mathscr{P}$ 1_p) will be termed the *perturbed primal problem*. The function $h[V \to \mathbb{R}_{cl}]$ defined by

$$h(p) = \inf_{x \in X} \Phi(x, p) \tag{2.2}$$

is called *extremal-value function*, and evidently $\inf_{x \in A} f(x) = h(\vartheta)$. Thus,

$$\alpha \equiv h(\vartheta) \geqq h^{**}(\vartheta) = \sup_{p^* \in V^*} - h^*(p^*) \equiv \beta\,. \tag{2.3}$$

We might try to solve the maximization problem $\sup_{p^* \in V^*} - h^*(p^*)$ instead of $(\mathscr{P}\,1)$ provided $\alpha = \beta$, but it would not be very realistic for nonconvex h. Therefore, throughout the whole first part of the paper we shall assume that h is a convex function. This is trivially fulfilled if $\Phi \in \Gamma_0(X \times V)$ ([8], III. Prop. 1.1), hence for all reasonably defined convex programs. However, h may be convex even in some nonconvex problems provided a suitable perturbation is involved; this will be demonstrated in Sec. 6. By definition

$$h^*(p^*) = \sup_{p \in V} [\langle p^*, p \rangle_V - h(p)] = \sup_{p \in V} [\langle p^*, p \rangle_V - \inf_{x \in X} \Phi(x, p)] =$$
$$= \Phi^*(\vartheta, p^*)\,,$$

and the maximization problem

$$(\mathscr{D}\,1) \qquad \begin{array}{c} -\Phi^*(\vartheta, p^*) \to \sup \\ \text{subj. to} \\ p^* \in V^* \end{array}$$

will be termed *dual* of $(\mathscr{P}\,1)$ with respect to Φ (or with respect to the given perturbations).

The technique used to form the dual of $(\mathscr{P}\,1)$ can be easily applied also to $(\mathscr{D}\,1)$: We introduce the *perturbed dual problem*

$$(\mathscr{D}\,1_p) \qquad \begin{array}{c} -\Phi^*(x^*, p^*) \to \sup \\ \text{subj. to} \\ p^* \in V^*\,, \end{array}$$

the dual *extremal-value* function $g[X^* \to \mathbb{R}_{cl}]$

$$g(x^*) = \inf_{p^* \in V^*} \Phi^*(x^*, p^*)\,, \tag{2.4}$$

and note that $\beta = -g(\vartheta)$. (The sign was changed in (2.4) to keep g convex.) In such a way

$$\beta = -g(\vartheta) \leqq -g^{**}(\vartheta) = \inf_{x \in X} g^*(x) =$$
$$= \inf_{x \in X} \sup_{x^* \in X^*} [\langle x^*, x \rangle_X - \inf_{p^* \in V_*} \Phi^*(x^*, p^*)] = \inf_{x \in X} \Phi^{**}(x, \vartheta)\,.$$

The minimization problem

$$(\mathscr{B}\,1) \qquad \Phi^{**}(x, \vartheta) \to \inf$$

subj. to

$$x \in X$$

will be termed *bidual* of $(\mathscr{P}\ 1)$ with respect to Φ. If $(\mathscr{B}\ 1)$ is identical to $(\mathscr{P}\ 1)$, each of problems $(\mathscr{P}\ 1)$ and $(\mathscr{D}\ 1)$ is found to be dual of the other and there is thus a complete symmetry between primal and dual problems. This will certainly be the case if $\Phi \in \Gamma_0(X \times V)$.

Concerning $(\mathscr{D}\ 1)$ and its relation to $(\mathscr{P}\ 1)$, we face three important problems, namely

(i) under what conditions vanishes the *duality gap* $\alpha - \beta$;
(ii) the existence of a vector $\hat{p}^*$ solving $(\mathscr{D}\ 1)$;
(iii) how $\hat{p}^*$ is related to a solution $\hat{x}$ of $(\mathscr{P}\ 1)$ (if it exists);

Before we try to answer them, we introduce three important notions.

Definition 2.1. 1) The problem $(\mathscr{P}\ 1)$ is *inf-stable* (or *normal*) (with respect to Φ) if $h(\vartheta)$ is finite and h is l.s.c. at ϑ.
2) The problem $(\mathscr{P}\ 1)$ is *inf-dif stable*, if $h(\vartheta)$ is finite and h is subdifferentiable at ϑ.
3) The problem $(\mathscr{P}\ 1)$ is *stable* if $h(\vartheta)$ is finite and h is continuous at ϑ.

Proposition 2.1. The following two conditions are equivalent to each other:
(i) $(\mathscr{P}\ 1)$ is inf-stable;
(ii) $\alpha = \beta$ and this number is finite.
If $\Phi \in \Gamma_0(X \times V)$, then the following condition is equivalent to the preceding two:
(iii) $(\mathscr{D}\ 1)$ is inf-stable.

Proposition 2.2. The following two conditions are equivalent to each other:
(i) $(\mathscr{P}\ 1)$ is inf-dif stable;
(ii) $(\mathscr{P}\ 1)$ is inf-stable and the set $\mathscr{B}$ of solutions of $(\mathscr{D}\ 1)$ is nonempty.

The proofs of the both assertions above follow from Prop. 1.12 and the fact that $-h^*$ is the dual objective, $\mathscr{B} = \partial h(\vartheta)$ and $\beta = h^{**}(\vartheta)$. For details see [2], 2, Thms 2.1, 2.1', 2.2.

In the convex case Prop. 2.2 attains the following form:

Proposition 2.3. If $\Phi \in \Gamma_0(X \times V)$, the three following conditions are equivalent to each other:
(i) $(\mathscr{P}\ 1)$ and $(\mathscr{D}\ 1)$ are inf-stable and have some solutions;
(ii) $(\mathscr{P}\ 1)$ and $(\mathscr{D}\ 1)$ are inf-dif stable;
(iii) $(\mathscr{P}\ 1)$ is inf-dif stable and has some solutions.

For the proof consider the complete symmetry of $(\mathscr{P}\ 1)$ and $(\mathscr{D}\ 1)$ in this case and the preceding assertions. Cf. [8], III, Cor. 2.1.

Proposition 2.4. Let $\hat{x} \in X$ and $\hat{p}^* \in V^*$ satisfy one of the following (equivalent) conditions:

(i) $\Phi^{**}(\hat{x}, \vartheta) + \Phi^*(\vartheta, \hat{p}^*) = 0$;

(ii) $(\hat{x}, \vartheta) \in \partial\Phi^*(\vartheta, \hat{p}^*)$;

(iii) $(\vartheta, \hat{p}^*) \in \partial\Phi^{**}(\hat{x}, \theta)$.

Then $\hat{x}$ is a solution of $(\mathscr{B}\, 1)$ and $\hat{p}^*$ is a solution of $(\mathscr{D}\, 1)$. If $\Phi \in \Gamma_0(X \times V)$, then clearly $\Phi^{**}(\hat{x}, \vartheta) = \Phi(\hat{x}, \vartheta)$ and $\hat{x}$ is a solution of $(\mathscr{P}\, 1)$.

The proof exploits again the facts remarked to the proof of Props. 2.1, 2.2, cf. [2], 2, Thms. 2.3, 2.3'.

As we have seen, the inf-dif stability plays a crucial role in the above considerations. By Prop. 1.13 it is implied by the stability and hence

$$\left.\begin{array}{l}\vartheta \in \operatorname{int} \operatorname{dom} h\\ h \in \Gamma_0(V)\end{array}\right\} \Rightarrow (\mathscr{P}\, 1) \text{ is inf-dif stable} \tag{2.5}$$

provided V is a barrelled space (Cor. 1.2.2).

The following three stability criteria are frequently used in applications:

Proposition 2.5. Each of the following conditions is sufficient for the convex function h to be continuous at ϑ and hence for $(\mathscr{P}\, 1)$ to be stable:

(i) $h(\vartheta)$ is finite and there exists $x_0 \in X$ such that $p \mapsto \Phi(x_0, p)$ is finite and continuous at $\vartheta(\in V)$;

(ii) V and V^* are both Banach spaces (in the designated compatible topologies), $h(\vartheta)$ is finite, $\Phi \in \Gamma_0(X \times V)$ and $\vartheta \in \operatorname{int} \operatorname{dom} h$;

(iii) $h(\vartheta)$ is finite, $\Phi \in \Gamma_0(X \times V)$ and there exist a neighbourhood $N \in \mathfrak{U}_{X^*}(\vartheta)$ and a number β such that the set

$$\{p^* \in V^* \mid \exists x^* \in N,\ \Phi^*(x^*, p^*) < \beta\} \tag{2.6}$$

is nonempty and equicontinuous.

Proof. (i) is a consequence of Prop. 1.2 and the boundedness of h from above on some $\mathcal{O} \in \mathfrak{U}_X(\vartheta)$.

(ii) We denote

$$\gamma_p(x^*) = \inf_{p^* \in V^*} [\Phi^*(x^*, p^*) - \langle p^*, p\rangle_V], \quad q_n(p^*) = \inf_{x^* \in B^n_{X^*}(\vartheta)} \Phi^*(x^*, p^*)$$

and observe that

$$h(p) = \inf_{x \in X} \sup_{\substack{x^* \in X^*\\ p^* \in V^*}} [\langle x^*, x\rangle_X + \langle p^*, p\rangle_V - \Phi^*(x^*, p^*)] = \inf_{x \in X} \gamma_p^*(x) =$$

$$= -\gamma_p^{**}(\vartheta) = \inf_{\varepsilon > 0} \sup_{\substack{x^* \in B^\varepsilon_{X^*}(\vartheta)\\ p^* \in V^*}} [\langle p^*, p\rangle_V - \Phi^*(x^*, p^*)] \leqq q_1^*(p) \leqq$$

$$\leqq \Phi(x, p) + \sup_{x^* \in B^1_{X^*}(\vartheta)} [-\langle x^*, x\rangle_X] = \Phi(x, p) + \|x\|_X .$$

Hence, $h(p) \leqq q_1^*(p) \leqq \inf_{x \in X} [\Phi(x, p) + \|x\|]$ for all $p \in V$ which yields int (dom h) = int (dom q_1^*). q_1^* is continuous on int (dom q_1^*) (Cor. 1.2.2) and hence the assertion of Prop. 1.2 completes the proof.

(iii) Clearly $h^{**}(\vartheta) < +\infty$. Like in (ii) we prove for $q_N(p^*) = \inf_{x^* \in N} \Phi^*(x^*, p^*)$ that $q_N^* \geqq h$. Hence, q_N^* is proper convex and cl epi q_N = epi q_N^{**}. Therefore,

$$\{p^* \in V^* \mid q_N^{**} \leqq \beta\} = \text{cl}\,\{p^* \in V^* \mid q_N \leqq \beta\}$$

and these sets are equicontinuous due to (2.6). We may now apply Prop. 1.9 and conclude that q_N^* is bounded above on some $\mathcal{O} \in \mathfrak{A}_{V^*}(\vartheta)$. As $h \leqq q_N^*$, the assertion has been proved. □

In most cases it is not possible to evaluate $\Phi^*(\vartheta, p^*)$ explicitly. It is the reason for introducing the *Lagrangian* function $L(x, p^*)$ defined on $X \times V^*$ by

$$L(x, p^*) = -\sup_{p \in V} [\langle p^*, p \rangle_V - \Phi(x, p)] \tag{2.7}$$

so that ($\mathscr{D}$ *1*) attains the form

$$\sup_{p^* \in V^*} \inf_{x \in X} L(x, p^*) .$$

If the function $p \mapsto \Phi(x, p)$ denoted henceforth $\Phi_x(p)$ belongs to $\Gamma_0(V)\ \forall x \in X$ then

$$\Phi_x^{**}(p) = \Phi(x, p) = \sup_{p^* \in V^*} [\langle p^*, p \rangle_V - \Phi_x^*(p^*)] = \sup_{p^* \in V^*} [\langle p^*, p \rangle_V + L(x, p^*)] .$$

Thus

$$\Phi(x, \vartheta) = \sup_{p^* \in V^*} L(x, p^*) ,$$

and ($\mathscr{P}$ *1*) can alternatively be written in the form

$$\inf_{x \in X} \sup_{p^* \in V^*} L(x, p^*) .$$

The properties of L are given by

Lemma 2.1. For all $x \in X$ the function

$$L_x : p^* \mapsto L(x, p^*)$$

is a concave u.s.c. function $[V^* \to \mathbb{R}_{cl}]$. If Φ is convex, then for all $p^* \in V^*$ the function

$$L_{p^*} : x \mapsto L(x, p^*)$$

is a convex function $[X \to \mathbb{R}_{cl}]$.

Proof. The first assertion is clear. The second is a direct consequence of the definition of L, cf. [8], 3, Lemma 3.1. □

Proposition 2.6. Let $(\hat{x}, \hat{p}^*)$ be a saddle point of $L(x, p^*)$. Then $\hat{p}^*$ is a solution of ($\mathscr{D}$ *1*), $\hat{x}$ is a solution of ($\mathscr{P}$ *1*) and $\Phi^{**}(\hat{x}, \vartheta) = -\Phi^*(\vartheta, \hat{p}^*)$.

In the convex case, the above assertion may be strengthened as follows:

Proposition 2.7. Under the hypothesis that $\Phi \in \Gamma_0(X \times V)$, the following two conditions are equivalent to each other:
(i) $(\hat{x}, \hat{p}^*)$ is a saddle point of L;
(ii) $\hat{x}$ is a solution of $(\mathscr{P}\ 1)$, $\hat{p}^*$ is a solution of $(\mathscr{D}\ 1)$ and $\alpha = \beta$.

In the proof of Prop. 2.7 one utilizes the convex-concave structure of L, cf. [8], 3, Prop. 3.1. Prop. 2.6 is a consequence of Prop. 2.7 and the fact that $(\hat{x}, \hat{p}^*)$ is a saddle point of the Lagrangian corresponding to $(\mathscr{B}\ 1)$ as well (note that $\Phi^{***} = \Phi^*$).

When attacking $(\mathscr{P}\ 1)$ by way of $(\mathscr{D}\ 1)$, it is certainly of a great help if the dual objective $G(p^*) = -\Phi^*(\vartheta, p^*)$ is differentiable in some sense possibly on the whole space V^*. As $G = -h^*$, it suffices to investigate the rotundity of h according to Props. 1.10, 1.11 or Lemma 1.1. Alternatively, Prop. 1.14 may also be applied.

There is still one important problem concerning the case in which we solve numerically $(\mathscr{D}\ 1)$ instead of $(\mathscr{P}\ 1)$ and find an approximate solution p_a^* satisfying e.g. the inequality

$$G(p_a^*) = -\Phi(\vartheta, p_a^*) \geqq \beta - \varepsilon \tag{2.8}$$

for some $\varepsilon \geqq 0$.

Proposition 2.8. Let p_a^* satisfy Ineq. (2.8) and $\hat{x}$ be a solution of problem $(\mathscr{P}\ 1)$ which is inf-stable. Then,

$$(\hat{x}, \vartheta) \in \partial^\varepsilon \Phi^*(\vartheta, p_a^*) . \tag{2.9}$$

Proof. Evidently, Ineq. (2.8) implies that

$$0 \leqq \Phi(\hat{x}, \vartheta) + \Phi^*(\vartheta, p_a^*) - \langle \vartheta, \hat{x} \rangle_X - \langle p_a^*, \vartheta \rangle_V \leqq \varepsilon$$

so that $(\vartheta, p_a^*) \in \partial^\varepsilon \Phi(\hat{x}, \vartheta)$ and of course also Incl. (2.9) holds because of Prop. 1.12. □

Incl. (2.9) enables us in concrete cases to estimate the quality of the approximate solution $x_a \in \arg\min_{x \in X} L(x, p_a^*)$.

3. Fenchel dualisation

Among many types of important perturbations the prominent place belongs to perturbations of "Fenchel" and "Lagrange type". This section is devoted to the former one.

Let in $(\mathscr{P}\ 1)$ $A = X$ and $f = J_1 - J_2$, where functions $J_1, -J_2$ $[X \to \mathbb{R}_{cl}]$ are convex. We set $V = X$ and

$$\Phi(x, p) = J_1(x) - J_2(x - p) . \tag{3.1}$$

It is then a trivial exercise to show that $(\mathscr{D}\ 1)$ attains the form

$$^*J_2(p^*) - J_1^*(p^*) \to \sup \tag{3.2}$$

subj. to

$$p^* \in X^* .$$

Problem (3.2) is mostly referred as the Fenchel dual. Many works have been devoted to the perturbation (3.1) and various generalizations, let us mention at least [27], [32]. We take here as the primal problem rather the optimal control problem ($\mathscr{P}$ 3) which seems to be very suitable to study various aspects of this dualisation. We can then also illustrate the theory on a simple example from constrained control problems with a parabolic system.

Throughout this section it will be assumed that U is a reflexive Banach space, paired with its topological dual U^* by the canonical pairing $\langle \cdot, \cdot \rangle_U$, and Y, Y^* are two paired l.c.s. with respect to some pairing $\langle \cdot, \cdot \rangle_Y$. Furthermore, we assume that

(i) $J \in \Gamma_0(U \times Y)$ with $M \times N \subset \operatorname{dom} J$;
(ii) M, N are closed convex subsets of U, Y respectively;
(iii) $y_0 = \vartheta$;
(iv) $M \cap \{u \mid \Pi u \in N\} \neq \emptyset$;

We apply now the "Fenchel type" perturbations, i.e. we set $X = U$, $V = Y$ (cf. the notation of Sec. 2), and

$$\Phi(u, p) = J(u, \Pi u - p) + \delta_M(u) + \delta_N(\Pi u - p) . \tag{3.3}$$

Furthermore, we add still one important assumption, namely

(v) either M is bounded or the functional

$$f_p(u) = J(u, \Pi u - p) + \delta_N(\Pi u - p) \tag{3.4}$$

is coercive on M uniformly on some $\mathcal{O} \in \mathfrak{U}_Y(\vartheta)$, i.e.

$$\lim_{\substack{\|u\|_U \to +\infty \\ u \in M}} f_p(u) = +\infty \text{ uniformly for } p \in \mathcal{O}. \tag{3.5}$$

Assumptions listed above ensure that the infimum of ($\mathscr{P}$ 3) is finite and a solution $\hat{u}$ of ($\mathscr{P}$ 3) exists (cf. Prop. 1.17).

Proposition 3.1. ($\mathscr{P}$ 3) is inf-stable.

Proof. Merely the lower semicontinuity of the extremal-value function h at ϑ remains to be shown. Let us assume, by contradiction, that there exists an $\varepsilon > 0$ and a net $\{\bar{p}_\nu \mid \nu \in \mathscr{A}\}$ converging to ϑ in Y such that for every $\nu \in \mathscr{A}$

$$h(\bar{p}_\nu) < h(\vartheta) - \varepsilon .$$

It is possible to find points $\bar{u}_\nu \in U$ such that

$$\Phi(\bar{u}_\nu, \Pi\bar{u}_\nu - \bar{p}_\nu) < h(\vartheta) - \tfrac{1}{2}\varepsilon .$$

The family $\mathscr{H} = \{\bar{u}_\nu \mid \nu \in \mathscr{A}\}$ is bounded due to assumption (v). Therefore, it is possible to find a weak cluster point of $\mathscr{H}$ denoted by u_0. Because of the weak lower semicontinuity of Φ

$$h(\vartheta) \leqq \Phi(u_0, \Pi u_0) \leqq h(\vartheta) - \tfrac{1}{2}\varepsilon ,$$

which is the desired contradiction. □

Concerning the stability of $(\mathscr{P}\,3)$, we have the following direct consequence of Prop. 2.5:

Proposition 3.2. Both following conditions are sufficient for $(\mathscr{P}\,3)$ to be stable.

(i) $\boldsymbol{Y}$ is a barrelled space and there exists a vector $u_0 \in M$ such that

$$\Pi u_0 \in \operatorname{int} N , \tag{3.6}$$

(ii) $\boldsymbol{Y}$ and $\boldsymbol{Y}^*$ are both Banach spaces (in the designated compatible topologies) and

$$\vartheta \in \operatorname{int} (\Pi M - N) . \tag{3.7}$$

Proof. Condition (i) implies that

$$\Pi u_0 \in \operatorname{int} (\operatorname{dom} \Phi_{u_0}) ,$$

where $\Phi_{u_0} : p \mapsto \Phi(u_0, p)$. As $\boldsymbol{Y}$ is a barrelled space, Φ_{u_0} is continuous over the interior of its effective domain, in particular at $p = \vartheta$. Hence, Cor. 1.2.2 and Prop. 1.2 may be applied.

Concerning (ii), observe just that

$$\operatorname{dom} h = \Pi M - N . \qquad \square$$

Remark. Incl. (3.6) is usually termed the Slater condition for $(\mathscr{P}\,3)$. If neither (3.6) nor (3.7) applies, the dual objective is to be examined with respect to an eventual application of assertion (iii) of Prop. 2.5.

$$\Phi^*(\vartheta, p^*) = \sup_{\substack{u \in \boldsymbol{U} \\ p \in \boldsymbol{Y}}} [\langle p^*, p \rangle_{\boldsymbol{Y}} - J(u, \Pi u - p) - \delta_M(u) - \delta_N(\Pi u - p)] =$$

$$= \sup_{\substack{u \in \boldsymbol{U} \\ v \in \boldsymbol{Y}}} [\langle p^*, \Pi u \rangle_{\boldsymbol{Y}} - \langle p^*, v \rangle_{\boldsymbol{Y}} - J(u, v) - \delta_M(u) - \delta_N(v)] =$$

$$= \varphi^*(\Pi^* p^*, -p^*) ,$$

where $\varphi(u, y) = J(u, y) + \delta_M(u) + \delta_N(y)$. Thus, the dual problem attains the form

$$(\mathscr{D}\,3) \qquad \begin{aligned} & -\varphi^*(\Pi^* p^*, -p^*) \to \sup \\ & \text{subj. to} \\ & p^* \in \boldsymbol{Y}^* . \end{aligned}$$

The problem $(\mathscr{D}\,3)$ seems to be an unconstrained extremal problem over $\boldsymbol{Y}^*$.

Unfortunately, it may be implicitly constrained due to the incidental unboundedness of sets M, N. However, as a solution of $(\mathscr{P}\,3)$ exists, these sets may be substituted by

$$(3.8)\qquad \begin{aligned}\tilde{M} &= \{u \in M \mid \|u\|_U \leqq L_0\}\\ \tilde{N} &= \{v \in N \mid \langle v^*, v\rangle_Y \leqq L_0\ \forall v^* \in \mathcal{O}\}\,,\end{aligned}$$

where $\mathcal{O} \in \mathfrak{U}_{Y^*}(\vartheta)$ and L_0 is a sufficiently large number. In such a way, also implicit dual constraints will be eliminated.

Let $\hat{p}^*$ be a solution of $(\mathscr{D}\,3)$. A natural question arises, whether we can now easily compute the desired optimal control $\hat{u}$ of $(\mathscr{P}\,3)$. The extremality relations (Prop. 2.4) imply that if

$$(3.9)\qquad (\bar{u}, \bar{y}) \in \arg\sup_{\substack{u\in M\\ y\in N}} [\langle \Pi^*\hat{p}^*, p\rangle_U - \langle \hat{p}^*, y\rangle_Y - J(u, y)]\,,$$

and $\Pi\bar{u} = \bar{y}$, then $\bar{u}$ is a solution of the primal problem. Indeed, by Prop. 2.4 (i) the value of the supremum in (3.9) equals $-\,J(\hat{u}, \Pi\hat{u})$ so that

$$(3.10)\qquad \langle \Pi^*\hat{p}^*, \bar{u}\rangle_U - \langle \hat{p}^*, \bar{y}\rangle_Y - J(\bar{u}, \bar{y}) = -J(\bar{u}, \Pi\bar{u}) = -J(\hat{u}, \Pi\hat{u})\,.$$

If, however, $J(\cdot, y)$ is strictly convex over M, there is only one control $\bar{u}$ satisfying (3.9) and this is also the only solution of $(\mathscr{P}\,3)$. For an approximate solution p_a^* of $(\mathscr{D}\,3)$ it may be impossible to find among the vectors u_a and y_a solving the appropriate Incl. (3.9) such a couple satisfying the system equation. In other words $\Pi u_a \notin N$ and/or $J(u_a, \Pi u_a) > J(\hat{u}, \Pi\hat{u})$. In concrete cases, various estimates may be obtained by Prop. 2.8.

The attractivity of $(\mathscr{D}\,3)$ with respect to solution of $(\mathscr{P}\,3)$ further increases if the dual objective $G(p^*) = -\varphi^*(\Pi^*p^*, -p^*)$ happens to be differentiable in some sense on Y^*. Let us now examine this case:

Proposition 3.3. Let Y be a reflexive Banach space and φ be strongly coercive on $U \times Y$, i.e.

$$\lim_{\|u\| + \|y\| \to \infty} \frac{\varphi(u, y)}{\|u\| + \|y\|} = +\infty\,.$$

Furthermore, let J be strictly convex over $M \times N$ and $(\bar{u}, \bar{y})$ be given by (3.9). Then G is Gâteaux differentiable at any $p^* \in Y^*$ with $\bar{p} = \bar{y} - \Pi\bar{u}$ being its gradient.

Proof. The strong coercivity of φ and the strict convexity of J ensure that

$$(3.11)\qquad \sup_{\substack{u\in M\\ y\in N}} [\langle \Pi^*p^*, u\rangle_U - \langle p^*, y\rangle_Y - J(u, y)] = \langle \Pi^*p^*, \bar{u}\rangle_U - \langle p^*, \bar{y}\rangle_Y - J(\bar{u}, \bar{y})$$

is attained at the unique couple $(\bar{u}, \bar{y}) \in M \times N$. Using the same argumentation as in the proof of Prop. 3.1 we conclude that $h \in \Gamma_0(Y)$. We prove, moreover, that h

is strongly coercive on $\boldsymbol{Y}$. Define for $r \in \mathbb{R}_+$ and a fixed $k > \|\Pi\|_{\mathscr{L}[\boldsymbol{U},\boldsymbol{Y}]}$

$$c(r) = \inf\left\{\frac{\varphi(u, y)}{k\|u\| + \|y\|} \,\middle|\, k\|u\| + \|y\| \geqq r\,, \quad u \in \boldsymbol{U},\, y \in \boldsymbol{Y}\right\}.$$

Then c is nondecreasing and $\lim_{r\to\infty} c(r) = +\infty$. Thus,

$$\varphi(u, \Pi u - p) \geqq c(k\|u\| + \|\Pi u - p\|)\,[k\|u\| + \|\Pi u - p\|] \geqq$$
$$\geqq c((k - \|\Pi\|)\,\|u\| + \|p\|)\,[(k - \|\Pi\|)\,\|u\| + \|p\|]\,,$$

hence $h(p) \geqq c(\|p\|)\,\|p\|$ so that G is continuous over $\boldsymbol{Y}^*$.

In view of Prop. 1.14 it remains to be proved that at arbitrary $p^* \in \boldsymbol{Y}^*$ $\partial(-G)(p^*)$ is a singleton. Prop. 1.12 implies that

$$p \in \partial(-G)(p^*) \Leftrightarrow h(p) - G(p^*) = \langle p^*, p\rangle_{\boldsymbol{Y}}\,.$$

The last equality may be rewritten into the form

$$\inf_{u\in \boldsymbol{U}}[J(u, \Pi u - p) + \delta_M(u) + \delta_N(\Pi u - p)] - \langle p^*, p\rangle_{\boldsymbol{Y}} =$$
$$= \inf_{\substack{u\in \boldsymbol{U}\\ y\in \boldsymbol{Y}}}[J(u, y) + \delta_M(u) + \delta_N(y) - \langle p^*, \Pi u - y\rangle_{\boldsymbol{Y}}]\,.$$

The infimum on the right hand side is attained at the single couple $(\bar{u}, \bar{y})$ due to the strict convexity of J. Hence

$$\bar{p} = \Pi\bar{u} - \bar{y} = \nabla^G(-G)(p^*)\,. \qquad \square$$

Remark. The strong coercivity of φ on $\boldsymbol{U} \times \boldsymbol{Y}$ is clearly ensured e.g. by the boundedness of M, N.

Proposition 3.4. Let $\boldsymbol{Y}$ be a reflexive Banach space and $\varphi(u, \Pi u - p)$ be coercive in u on $\boldsymbol{U}$ locally uniformly with respect to $p \in \boldsymbol{Y}$, i.e. for each $p \in \boldsymbol{Y}$ there is $\mathcal{O} \in \mathfrak{U}_{\boldsymbol{Y}}(p)$ such that

$$\lim_{r\to\infty} \inf_{\substack{u\in \boldsymbol{U}\\ q\in \boldsymbol{Y}}} \{\varphi(u, \Pi u - q) \mid \|u\| > r,\, q \in \mathcal{O}\} = +\infty\,.$$

Furthermore, let points $\bar{u} \in M$, $\bar{y} \in N$ be determined by (3.11) and there exists a function $\beta_{\bar{u},\bar{y}}[\mathbb{R}^2_+ \to \mathbb{R}_+]$ such that $\inf_{[\varepsilon,\infty)\times[\delta,\infty)} \beta_{\bar{u},\bar{y}} > 0$ if either ε or $\delta > 0$, and

$$(3.12)\quad \varphi(u', y') \geqq \varphi(\bar{u}, \bar{y}) + D\,\varphi(\bar{u}, \bar{y}; u' - \bar{u}, y' - \bar{y}) + \beta_{\bar{u},\bar{y}}(\|u' - \bar{u}\|, \|y' - \bar{y}\|)$$
$$\forall u' \in \boldsymbol{U},\quad y' \in \boldsymbol{Y}\,.$$

Then G is Fréchet differentiable at p^* with $\bar{y} - \Pi\bar{u}$ being its gradient.

Proof. The assumptions being imposed imply that

$$\inf_{\substack{u\in \boldsymbol{U}\\ y\in \boldsymbol{Y}}}[\varphi(u, y) - \langle \Pi^* p^*, u\rangle_{\boldsymbol{U}} + \langle p^*, y\rangle_{\boldsymbol{Y}}]$$

is attained strongly at $(\bar{u}, \bar{y})$ with respect to the norm topology due to Lemma 1.1. Rearranging the terms as in the proof of Prop. 3.3, we see that

$$\inf_{p \in Y} \left[h(p) - \langle p^*, p \rangle_Y\right]$$

is attained strongly at $\bar{p} = \Pi\bar{u} - \bar{y}$ with respect to the norm topology. Hence, h is norm-rotund at $\bar{p}$ with respect to p^*.

The assertion of Prop. 1.11 completes the proof because h and $-G$ are conjugate to each other due to the local uniform coercivity of φ. □

Remark. It is easy to see that in the differentiable case

$$\nabla^G G(\hat{p}^*) = \vartheta \Leftrightarrow \hat{y} = \Pi \hat{u}\,,$$

where $(\hat{u}, \hat{y}) \in \arg\sup_{\substack{u \in M \\ y \in N}} \left[\langle \Pi^* \hat{p}^*, u \rangle_U - \langle \hat{p}^*, y \rangle_Y - J(u, y)\right]$, and hence $\hat{u}$ is the optimal control of $(\mathscr{P}\,3)$. Thus, if we maximize the concave Gâteaux differentiable function G, we approach the solution of $(\mathscr{P}\,3)$ by the couples (u, y) which are unfeasible with respect to the system equation. Since the system equation is satisfied, the corresponding $\hat{p}^*$ belongs to the solution set of $(\mathscr{D}\,3)$.

There is a certain freedom in the choice of the space Y in $(\mathscr{P}\,3)$. E.g. for Π being given as a system of first-order linear ordinary differential equations we may set $Y = H^1$ and use the Sobolev pairing or we may set Y equal to H^1, C_0, L_∞ or L_2 and use the generalized L_2 pairing. The latter situation is reflected in the following general structure:

Besides assumption (i)–(v) imposed at the beginning of the section we assume that

(vi) there is a pair of reflexive l.c.s. W, W^* such that Y is continuously embedded in W, $\mathrm{cl}_W Y = W$ and the pairing $\langle \cdot, \cdot \rangle_Y$ is the "canonical" generalization of $\langle \cdot, \cdot \rangle_W$;

(vii) there is a convex closed set $N' \subset W$ such that $N = Y \cap N'$, $N' = \mathrm{cl}_W N$;

(viii) J possesses an extension $J' \in \Gamma_0(U \times W)$ which is finite and continuous on $M \times N'$.

Now, we can set $V = W$ and derive another "Fenchel type" perturbed essential objective Φ on $U \times W$ and a new dual problem $(\mathscr{D}\,3)'$. Because of (vi)–(viii) the cost in $(\mathscr{D}\,3)'$ is the restriction of the cost function in $(\mathscr{D}\,3)$ onto W^*. Of course, we may have difficulties with the existence of a solution of $(\mathscr{D}\,3)'$ because neither condition of Prop. 3.2 can be satisfied. On the other hand, if $(\mathscr{P}\,3)$ is inf-stable with respect to the new perturbations, $(\mathscr{D}\,3)'$ provides us with minimizing sequences for $(\mathscr{D}\,3)$ and may be handled substantionally more easily than $(\mathscr{D}\,3)$. This approach is especially advantageous when Y is a Sobolev space and W is an L_2 space.

Let Ω be a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary, $Q = (0, 1) \times \Omega$. Let (a_{ij}) be a symmetric matrix with sufficiently smooth elements and a coefficient of the ellipticity $a_0 > 0$ on cl Ω. Put $\tilde{Y} = \mathring{H}^{1,2}(\Omega) \cap L_2(0, 1; \mathring{H}^1(\Omega))$

with the norm of $\boldsymbol{H}^{1,2}(\Omega)$ ($\boldsymbol{H}^{1,2}(\Omega)$ is the appropriate anisotropic Sobolev space of the Hilbert type), $\boldsymbol{U} = \boldsymbol{L}_2(Q)$. Put $\mathscr{H}_1 := \{y \in \tilde{\boldsymbol{Y}} \mid y(0, \cdot) = 0\}$ and denote by $C : \mathscr{H}_1 \to \mathscr{H}_2 = \mathring{\boldsymbol{H}}^1(\Omega)$ the mapping $y \mapsto y(1, \cdot)$. Put $\boldsymbol{Y} := \{(y, w) \in \mathscr{H}_1 \times \mathscr{H}_2 \mid w = Cy\}$. For $y \in \tilde{\boldsymbol{Y}}$ put $Ay = \sum_{i,j=1}^{n} (\partial/\partial x_i) [a_{ij}(\partial y/\partial x_j)]$. Denote by $\|\cdot\|_{0,Q}$, $\|\cdot\|_{0,\Omega}$ the norms in $\boldsymbol{L}_2(Q), \boldsymbol{L}_2(\Omega)$, respectively. Let $\tilde{y} \in \boldsymbol{L}_2(\Omega)$ (a "desired" terminal state), a be a continuous function on Q such that $\tilde{y} \leqq Ca$, $\vartheta \leqq a(0, \cdot)$ a.e. in Ω. The problem is stated in the following way:

$$J(u, y) \to \inf$$

$$(\mathscr{P}\, 3) \qquad \text{subj. to} \quad \frac{\partial y}{\partial t} = Ay + u \quad \text{a.e. in } Q\,, \quad y(0, \cdot) = \vartheta\,,$$

$$y = 0 \quad \text{a.e. in} \quad (0, 1) \times \partial\Omega\,,$$

$$y \leqq a \quad \text{a.e. in} \quad Q\,,$$

where $J(u, y) = J_1(u) + J_2(y) = \frac{1}{2}\|u\|_{0,Q}^2 + [\frac{1}{2}\varepsilon\|y\|_{0,Q}^2 + \frac{1}{2}\eta\|Cy - \tilde{y}\|_{0,\Omega}^2]$, $\varepsilon, \eta > 0$ are given.

It is well known (cf. [11]) that for $u \in \boldsymbol{U}$ the corresponding state $\Pi_1 u$ belongs to $\mathscr{H}_1$ and Π_1 is an isomorphismus of $\boldsymbol{U}$ and $\mathscr{H}_1$. Hence the operator $\Pi[\boldsymbol{U} \to \boldsymbol{Y}]$, $\Pi = (\Pi_1, C\Pi_1)$, is epimorphic. We have $M = \boldsymbol{U}$, $N' = K \times K_1$, where $K = \{z \in \boldsymbol{L}_2(Q) \mid z \leqq a \text{ a.e. in } Q\}$, $K_1 = \{z \in \boldsymbol{L}_2(\Omega) \mid z \leqq Ca \text{ a.e. in } \Omega\}$. Prop. 3.2 (ii) implies immediately the stability of $(\mathscr{P}\, 3)$ for the "Fenchel type" perturbations with $\boldsymbol{V} = \boldsymbol{Y}$. Taking $\boldsymbol{V} = \boldsymbol{W} := \boldsymbol{L}_2(Q) \times \boldsymbol{L}_2(\Omega)$ we may prove merely the inf-stability of $(\mathscr{P}\, 3)$ by means of Prop. 3.1.

The dual problem $(\mathscr{D}\, 3)$ has the form

$$G(p^*) \equiv -\Phi^*(\theta, p^*) = -J_1^*(\Pi^* p^*) - J_2^*(-p^*) \to \inf\,, \quad p^* \in \boldsymbol{Y}^*\,,$$

where

$$J_1^*(v^*) = \tfrac{1}{2}\|v^*\|_{0,Q}^2,\ v^* \in \boldsymbol{L}_2(Q),\ J_2^*(p_1^*, p_2^*) =$$

$$= \sup_{\substack{p_1 \in \mathscr{H}_1 \\ p_2 \in \mathscr{H}_2 \\ p_2 = Cp_1}} [\langle p_1^*, p_1 \rangle - \tfrac{1}{2\varepsilon}\|p_1\|_{0,Q}^2 - \delta_K(p_1) + \langle p_2^*, p_2 \rangle - \tfrac{1}{2\eta}\|p_2 - \tilde{y}\|_{0,\Omega}^2 - \delta_{K_1}(p_2)]\,.$$

The dual problem $(\mathscr{D}\, 3)'$ has the same form, the supremum can be decoupled (the constraint $p_2 = Cp_1$ vanishes) and both suprema are taken over $\boldsymbol{L}_2(Q)$, $\boldsymbol{L}_2(\Omega)$, respectively. As the conditions (vi)–(viii) are fulfilled, it holds:

Proposition 3.5. There exists a sequence $\{(p_{1,i}^*, p_{2,i}^*)\} \subset \boldsymbol{L}_2(Q) \times \boldsymbol{L}_2(\Omega)$ minimizing $(\mathscr{D}\, 3)$. Every such sequence tends weakly in $\boldsymbol{Y}^*$ to $\hat{p}^*$, the unique solution of $(\mathscr{D}\, 3)$. In particular, $\Pi^*(p_{1,i}^*, p_{2,i}^*)$ tends weakly in $\boldsymbol{L}_2(Q)$ to $\Pi^*\hat{p}^*$.

Proof. Observe first that h is continuous over $\boldsymbol{Y}$ due to the surjectivity of Π in view of (3.7). This implies in particular the Gâteaux differentiability of h at ϑ

because of the strict concavity of G (Props. 1.14, 3.2). Hence, $-G$ is $w(\boldsymbol{Y}^*, \boldsymbol{Y})$-rotund at $\hat{p}^*$ with respect to $\vartheta \in \boldsymbol{Y}$ due to Prop. 1.11. □

Clearly for $p^* = (p_1^*, p_2^*) \in \boldsymbol{W}^*$ $\Pi^* p^*$ is represented by the solution of the problem

$$(3.13) \quad \frac{\partial v^*}{\partial t} + A v^* = -p_1^* \text{ on } Q, \quad v^*(1, \cdot) = p_2^* \text{ on } \Omega, \quad v^*/_{(0,1)\times\Omega} = \vartheta$$

(the same is formally true for $(p_1^*, p_2^*) \in \boldsymbol{Y}^*$). For $p^* \in \boldsymbol{W}^*$

$$(3.14) \quad \Phi^*(\vartheta, p^*) = \tfrac{1}{2}\|v^*\|_{0,Q}^2 + \frac{1}{2\varepsilon}\int_{\{(\tau,\xi)\in Q \mid -p_1^*(\tau,\xi) \leqq \varepsilon a(\tau,\xi)\}} (p_1^*(t, z))^2 \,\mathrm{d}t\,\mathrm{d}z +$$

$$+ \int_{\{(\tau,\xi)\in Q \mid -p_1^*(\tau,\xi) > \varepsilon a(\tau,\xi)\}} \left[-p_1^*(t, z)\, a(t, z) - \frac{\varepsilon}{2}(a(t, z))^2\right] \mathrm{d}t\,\mathrm{d}z +$$

$$+ \int_{\{\xi\in\Omega \mid -p_2^*(\xi) \leqq \eta[a(1,\xi) - \bar{y}(\xi)]\}} \left[-p_2^*(z)\,\bar{y}(z) + \frac{1}{2\eta}(p_2^*(z))^2\right] \mathrm{d}z +$$

$$+ \int_{\{\xi\in\Omega \mid -p_2^*(\xi) > \eta[a(1,\xi) - \bar{y}(\xi)]\}} \left[-p_2^*(z)\, a(1, z) + \frac{\eta}{2}(a(1, z) - \bar{y}(z))^2\right] \mathrm{d}z\,,$$

where v^* is defined in (3.13). Thus $(\mathscr{D}\,3)'$ is the maximization of $-\Phi^*(\vartheta, p^*)$ given by (3.14) under the equality constraint given by the dual system equation (3.13).

Generally, for the numerical solution of $(\mathscr{D}\,3)'$ a subgradient minimization technique must be employed, cf. [13]. However, the specific structure of $(\mathscr{P}\,3)$ enables us to apply Prop. 3.4 to the cost function of $(\mathscr{D}\,3)'$ with $\beta_{\bar{u},\bar{y}}(\|u' - \bar{u}\|, \|y' - \bar{y}\|) = \tfrac{1}{2}\|u' - \bar{u}\|_{0,Q}^2 + \tfrac{1}{2}\varepsilon\|y_1' - \bar{y}_1\|_{0,Q}^2 + \tfrac{1}{2}\eta\|y_2' - \bar{y}_2\|_{0,\Omega}^2$ for any $(\bar{u}, \bar{y}) \equiv (\bar{u}, (\bar{y}_1, \bar{y}_2)) \in \operatorname{dom} \varphi$. Hence, the Fréchet derivative $\nabla_{p^*} G$ is given by

(3.15)

$$\nabla_{p_1^*} G(p^*) = -\Pi_1 v^* + \begin{cases} p_1^*/\varepsilon & \text{on } \{(\tau, \xi) \in Q \mid -p_1^*(\tau, \xi) \leqq \varepsilon\, a(\tau, \xi)\} \\ -a & \text{elsewhere on } Q\,, \end{cases}$$

$$\nabla_{p_2^*} G(p^*) = -C\Pi_1 v^* + \begin{cases} p_2^*/\eta - \bar{y} & \text{on } \{\xi \in \Omega \mid -p_2^*(\xi) \leqq \eta[a(1, \xi) - \bar{y}(\xi)]\} \\ -Ca & \text{elsewhere on } \Omega\,, \end{cases}$$

where v^* is from (3.13). In such a way, effective gradient methods may be applied to the numerical solution of $(\mathscr{D}\,3)'$.

Let $p_a^* \in \boldsymbol{L}_2(Q) \times \boldsymbol{L}_2(\Omega)$ be an approximate solution of $(\mathscr{D}\,3)'$ satisfying the stopping rule condition

$$G(p_a^*) \geqq h(\vartheta) - \varepsilon\,, \quad \varepsilon > 0\,.$$

The approximate optimal solution u_a of $(\mathscr{P}\,3)$ can now be obtained by

$$u_a = v_a^* = \Pi p_a^*\,.$$

Its accuracy may be estimated by

$$\|u_a - \hat{u}\|_{\boldsymbol{L}_2} \leqq \sqrt{(2\varepsilon)}$$

where $\hat{u}$ is the (unique) optimal control of ($\mathscr{P}$ 3). Indeed, by Prop. 2.8

$$(\hat{u}, \vartheta) \in \partial^{\varepsilon} \Phi^*(\vartheta, p_a^*)$$

which implies

$$\tfrac{1}{2}\|\hat{u}\|_{0,Q}^2 + \tfrac{1}{2}\|u_a\|_{0,Q}^2 - \langle \hat{u}, u_a \rangle_{L_2(Q)} \leqq \varepsilon .$$

It is reasonable to solve ($\mathscr{P}$ 3) by way of its Fenchel dual problem ($\mathscr{D}$ 3) or ($\mathscr{D}$ 3)′ if Φ^* is given by a formula so that we do not need to solve auxiliary extremal problems to evaluate a value of the dual objective. This approach is especially recommended if

(i) Y is finite dimensional (optimal control of ordinary differential equations with merely terminal state constraints);

(ii) J is nonsmooth (then we replace a constrained nonsmooth program by an uncostrained one);

(iii) G is Gâteaux or Fréchet differentiable on Y^*.

In [20], [22] and many other papers the reader may find various further applications of the above approach to optimal control problems.

4. Lagrange dualisation

The perturbations studied in the present section are said to be of "Lagrange type" because the extremal relations of Prop. 2.4 are in this case in fact the well-known multiplier rules of Lagrange and Kuhn-Tucker cf. e.g. [16]. We apply here these perturbations to the mathematical programming problem ($\mathscr{P}$ 2).

Throughout this section it will be assumed that $\boldsymbol{X}$ is a reflexive Banach space and $\boldsymbol{Z}$, $\boldsymbol{Z}^*$ are two paired l.c.s. with respect to some pairing $\langle \cdot, \cdot \rangle$. Furthermore, we assume that

(i) $f \in \Gamma_0(\boldsymbol{X})$ and $A \subset \operatorname{dom} f$;

(ii) A is a closed convex subset of $\boldsymbol{X}$;

(iii) D is a closed convex cone in $\boldsymbol{Z}$ with the vertex at the origin;

(iv) $q[\boldsymbol{X} \to \boldsymbol{Z}]$ is a continuous D-convex map, i.e. for any $\alpha \in (0, 1)$ and any $x_1, x_2 \in \boldsymbol{X}$

$$\alpha\, q(x_1) + (1 - \alpha)\, q(x_2) - q(\alpha x_1 + (1 - \alpha)\, x_2) \in D ;$$

(v) $A \cap \{x \mid -q(x) \in D\} \neq \emptyset$;

We set now $\boldsymbol{X} = \boldsymbol{X}$, $\boldsymbol{V} = \boldsymbol{Z}$ (cf. the notation of Sec. 2), and

$$\Phi(x, h) = f(x) + \delta_A(x) + \delta_D(p - q(x)) . \tag{4.1}$$

Furthermore, we impose still one important assumption, namely

(vi) either A is bounded or the functional

$$f_p(x) = f(x) + \delta_D(p - q(x)) \tag{4.2}$$

is coercive on A uniformly on some $\mathcal{O} \in \mathfrak{U}_{\boldsymbol{Z}}(\vartheta)$.

Assumption listed above ensure that the infimum of $(\mathscr{P}\ 2)$ is finite and a solution $\hat{x}$ of $(\mathscr{P}\ 2)$ exists (cf. Prop. 1.17).

Proposition 4.1. $(\mathscr{P}\ 2)$ is inf-stable.

The proof may be performed by means of the same argument as in Prop. 3.1.

Proposition 4.2. Both following conditions are sufficient for $(\mathscr{P}\ 2)$ to be stable:

(i) $\boldsymbol{Z}$ is a barrelled space and there exists a vector $x_0 \in A$ such that

(4.3)
$$-q(x_0) \in \operatorname{int} D\ ;$$

(ii) $\boldsymbol{Z}$ and $\boldsymbol{Z}^*$ are both Banach spaces (in the designated compatible topologies) and for each $p \in \boldsymbol{Z}$ there exist $\varepsilon > 0$ and $x \in A$ such that

(4.4)
$$\varepsilon p - q(x) \in D\ .$$

Proof. Concerning the condition (i) termed again usually the Slater condition, we may apply the same technique as in Prop. 3.2 (i). Condition (ii) is a direct consequence of Prop. 2.5 (ii) with int (dom h) replaced by core (dom h), cf. [32], 8, Example 4. □

Various other stability conditions for $(\mathscr{P}\ 2)$, mostly of the dual type (Prop. 2.5 (iii)) can be found in [25].

We introduce now the Lagrangian $L(x, p^*)$ defined on $\boldsymbol{X} \times \boldsymbol{Z}^*$ according to Eq. (2.7), i.e.

(4.5)
$$\begin{aligned} L(x, p^*) &= -\sup_{p\in \boldsymbol{Z}} \left[\langle p^*, p\rangle - f(x) - \delta_A(x) - \delta_D(p - q(x))\right] = \\ &= f(x) + \delta_A(x) - \langle p^*, q(x)\rangle - \delta_D^*(p^*) = \\ &= f(x) + \delta_A(x) - \langle p^*, q(x)\rangle - \delta_{-D^*}(p^*)\,, \end{aligned}$$

where D^* is the polar cone of D i.e. $D^* = \{p^* \in V^* | \langle p^*, p\rangle \geqq 0\ \forall p \in D\}$. The dual problem attains the form

$(\mathscr{D}\ 2)$
$$\sup_{p^* \in -D^*} \inf_{x \in A} \{f(x) - \langle p^*, q(x)\rangle\}\ .$$

It is an easy exercise to obtain the well-known Kuhn-Tucker optimality conditions (or Lagrange multiplier rule for the case of affine equality constraints) from Props. 2.4 and 2.7. We have to distinguish two principal cases:

(i) $\boldsymbol{Z} = \mathbb{R}^n$;

(ii) $q(x)$ is Gâteaux differentiable over $\boldsymbol{X}$,

and therefore we impose for the purpose of the next assertion the following assumption on the structure of $(\mathscr{P}\ 2)$:

(4.6)
$$\boldsymbol{Z} = \mathbb{R}^n \times \boldsymbol{Z}_1\,,$$
$$D = \mathbb{R}^n_+ \times D_1\,, \quad \text{where} \quad D_1 \subset \boldsymbol{Z}_1\,,$$

$$q(x) = \begin{bmatrix} q^1(x) \\ \vdots \\ q^n(x) \\ Q(x) \end{bmatrix}, \quad \text{with} \quad q^i[X \to \mathbb{R}], \quad i = 1, 2, \ldots, n, \quad \text{and}$$

$Q[X \to Z_1]$ being Gâteaux differentiable on X.

Proposition 4.3. Let $(\mathscr{P}\,2)$ possess the structure given by (4.6) and be inf-dif stable. Then $\hat{x}$ solves $(\mathscr{P}\,2)$ if and only if there exist scalars $\hat{p}^{*i} \leqq 0$, and a vector $\hat{P}^* \in -D_1^*$ such that

$$(4.7) \qquad \vartheta \in \partial f(\hat{x}) - \sum_{i=1}^{n} \hat{p}^{*i}\, \partial q^i(\hat{x}) - \nabla^G Q^*(\hat{x})\, \hat{P}^* + N_A(\hat{x}),$$

$$(4.8) \qquad \sum_{i=1}^{n} \hat{p}^{*i}\, q_i(\hat{x}) + \langle \hat{P}^*, Q(\hat{x}) \rangle_{Z_1} = 0,$$

$$(4.9) \qquad \hat{x} \in A, \quad q^i(\hat{x}) \leqq 0, \quad i = 1, 2, \ldots, n, \quad -Q(\hat{x}) \in D_1.$$

Proof. Let $\hat{x}$ solve $(\mathscr{P}\,2)$. The inf-dif stability implies the existence of a vector $\hat{p}^* = (\hat{p}^{*1}, \hat{p}^{*2}, \ldots, \hat{p}^{*n}, \hat{P}^*) \in \mathbb{R}^n_- \times (-D_1^*)$ which solves $(\mathscr{D}\,2)$ and $\alpha = \beta$. Hence, $(\hat{x}, \hat{p}^*)$ is a saddle point of L given by (4.5) due to Prop. 2.7. Therefore,

$$\hat{x} \in \arg\inf_{x \in X} L(x, \hat{p}^*) \Leftrightarrow \vartheta \in \partial_x L(\hat{x}, \hat{p}^*).$$

Prop. 1.15 implies now Incl. (4.7). Simultaneously, according to Prop. 2.4

$$\Phi(\hat{x}, \vartheta) + \Phi^*(\vartheta, \hat{p}^*) = 0,$$

i.e.

$$f(\hat{x}) - L(\hat{x}, \hat{p}^*) = 0$$

which implies (4.8).

On the other hand, let there exist $\hat{p}^* = (\hat{p}^{*1}, \ldots, \hat{p}^{*n}, \hat{P}^*) \in \mathbb{R}^n_- \times (-D_1^*)$ and $\hat{x}$, satisfying the relations (4.7), (4.8), (4.9). Incl. (4.7) implies that $\vartheta \in \partial_x L(\hat{x}, \hat{p}^*)$ which is equivalent to $\hat{x} \in \arg\inf_{x \in X} L(x, \hat{p}^*)$, and relations (4.8), (4.9) imply after some computation that $\vartheta \in \partial_{p^*}(-L(\hat{x}, p^*))$ which is equivalent to $\hat{p}^* \in \arg\sup_{p^* \in Z^*} L(\hat{x}, p^*)$. Hence, $(\hat{x}, \hat{p}^*)$ is a saddle-point for $L(\hat{x}, \hat{p}^*)$ so that $\hat{x}$ is a solution of $(\mathscr{P}\,2)$ because of Prop. 2.7. □

With respect to the above assertion we will make use of the following

Definition 4.1. The solution $\hat{p}^*$ of $(\mathscr{D}\,2)$ is termed the *Kuhn-Tucker vector* of $(\mathscr{P}\,2)$. Provided $D = \vartheta$, the vector $\hat{p}^*$ is alternatively termed the *Lagrange multiplier* of $(\mathscr{P}\,2)$.

In most cases it is not recommendable to solve $(\mathscr{P}\,2)$ by way of $(\mathscr{D}\,2)$ with the exception of problems with a certain separable structure. An example of this type will be discussed at the end of this section. However, various two-step algorithms which have been developed for the solution of $(\mathscr{D}\,2)$ can be successfully applied (may be in a slightly modified form) to other dual problems and therefore we mention now briefly one representative of this family.

Uzawa's algorithm

Let $\boldsymbol{X}$ and $\boldsymbol{Z}$ be two Hilbert spaces and $(\mathscr{P}\,2)$ be inf-dif stable. The algorithms is based on the construction of two sequences of elements $\{p_k^*\}$, $\{x_k\}$.

k-th step: 1) Given $p_k^* \in -D^*$, determine x_k as the element of $\boldsymbol{X}$ which minimizes $L(\cdot, p_k^*)$.

2) Set

$$p_{k+1}^* = (p_k^* - \varrho_k\, q(x_k))^{-D^*},$$

where $\varrho_k > 0$ is a suitably chosen step-size.

Under various additional requirements (to those which have been imposed at the beginning of this section) it is possible to show that $\{p_k^*\}$ converges to a solution of $(\mathscr{D}\,2)$ and $\{x_k\}$ converges to a solution of $(\mathscr{P}\,2)$ in some sense. One set of such requirements can be found in [8], VII. Simultaneously one derives the upper bound for ϱ_k. Generally, we have to ensure that

(i) the intermediate minimizations $L(\cdot, p_k^*)$ possess solutions x_k;

(ii) $\inf_{x \in X} L(x, p_k^*) \to \beta = \alpha$ and the sequence $\{p_k^*\}$ is bounded which is in fact the problem of a suitable choice of stepsizes $\{\varrho_k\}$.

(iii) the corresponding sequence $\{x_k\}$ is also bounded.

Then it may be proved that the weak limits of some subsequences $\{x_{k'}\}$, $\{p_{k'}^*\}$ form in fact a saddle point of L.

Remark. Observe that for the function $\mathscr{L}(x, p^*) = L(x, p^*) + \delta_{-D^*}(p^*)$ and for any p^*, p_0^* and $x_0 \in \arg\min_{x \in \boldsymbol{X}} \mathscr{L}(x, p_0^*)$

$$\inf_{x \in \boldsymbol{X}} \mathscr{L}(x, p^*) - \inf_{x \in \boldsymbol{X}} \mathscr{L}(x, p_0^*) \leqq \mathscr{L}(x_0, p^*) - \mathscr{L}(x_0, p_0^*) = -\langle p^* - p_0^*, q(x_0)\rangle .$$

Hence,

$$q(x_0) \in \partial(-\inf_{x \in \boldsymbol{X}} \mathscr{L}(x, p_0^*))$$

and we may apply as stepsizes any nonnegative sequence $\{\varrho_k\}$ satisfying

$$\lim_{k \to \infty} \varrho_k = 0\,, \quad \sum_{k=0}^{\infty} \varrho_k = \infty\,. \tag{4.10}$$

Indeed, then the Uzawa algorithm turns out to be a method of Polyak [23], [24] for maximization of a concave nondifferentiable function $\inf_{x \in X} \mathscr{L}(x, p^*)$ over the set $-D^*$. For various different stepsize choices see [12] and [39].

At this place we have to remark, that for $\hat{p}^*$ being the Kuhn-Tucker vector of $(\mathscr{P}\,2)$, $\bar{x} \in \arg\inf_{x \in X} L(x, \hat{p}^*)$ need not be necessarily a solution of $(\mathscr{P}\,2)$ because it need not hold that $-q(\bar{x}) \in D$. Only such $\bar{x}$ for which the last constraint is satisfied are in fact solutions of $(\mathscr{P}\,2)$.

The above mentioned separable structure of $(\mathscr{P}\,2)$ favourable for the solution by way of $(\mathscr{D}\,2)$ may very well be illustrated on the following economic example taken from [8].

Decentralized management of a firm

Let us consider a firm consisting of N factories. Each factory is characterized by its production set $X_i \subset \mathbb{R}^n$, $i = 1, 2, \ldots, N$. To say that $\bar{x}_i \in X_i$ means that the factory is in a position to produce articles $\bar{x}_i^j$, if $\bar{x}_i^j > 0$, while consuming articles $\bar{x}_i^l$, if $\bar{x}_i^l < 0$, j, $l \in \{1, 2, \ldots, n\}$. We assume a nonlinear (concave) growth of profit with production expressed at each factory by a function $g_i[\mathbb{R}^n \to \mathbb{R}]$. We denote by $c = (c^1, \ldots, c^n)$ the initial resources, i.e. the quantities disposable on the market. However, some of our factories can produce article i and supply it to others. Therefore,

$$\sum_{i=1}^{N} x_i^l + c^l \geqq 0 \quad \text{for} \quad l = 1, 2, \ldots, n. \tag{4.11}$$

The firm seeks to maximize its total profit subj. to given constraints, i.e. solves the problem

$$-\sum_{i=1}^{N} g_i(x_i) \to \inf \tag{4.12}$$

subj. to

$$x_i \in X_i, \quad i = 1, 2, \ldots, N,$$

$$\sum_{i=1}^{N} x_i^l + c^l \geqq 0, \quad l = 1, 2, \ldots, n.$$

It is assumed that

(i) $-g_i$ are proper convex, l.s.c. functions coercive on X_i with $\operatorname{dom} g_i \supset X_i$, $i = 1, 2, \ldots, N$;

(ii) X_i are closed convex sets containing $\mathbb{R}^n_-$ (any factory is capable to destroy any quantity of goods);

(iii) there exists an n-tuple of vectors $[x_1, x_2, \ldots, x_N] \in \bigtimes_{i=1}^{N} X_i$ feasible with respect to Ineq. (4.11).

Under these supposition we obtain a problem falling completely into the scope of this section. The stability condition (4.3) attains the form

$$[\sum_{i=1}^{N} X_i + c] \cap \operatorname{int} \mathbb{R}^n_+ \neq \emptyset \,, \tag{4.13}$$

and it is trivially satisfied if, e.g. $c \in \operatorname{int} \mathbb{R}^n_+$ due to assumption (ii).

(iv) The dual problem attains the form

$$\sup_{p^* \leqq \vartheta} \inf_{x_i \in X_i} [- \sum_{i=1}^{N} g_i(x_i) + \langle p^*, q \rangle_n] \,,$$

where $q^l = \sum_{i=1}^{N} x_i^l + c^l$, $l = 1, 2, \dots, n$. We rewrite it into a more convenient form

$$- \inf_{v^* \geqq \vartheta} \sup_{x_i \in X_i} [\sum_{i=1}^{N} g_i(x_i) + \sum_{l=1}^{n} (\sum_{i=1}^{N} x_i^l + c^l) v^{*l}] \,. \tag{4.14}$$

The constant $\langle v^*, c \rangle$ may be omitted, summation may be interchanged, and we obtain due to the separable structure of the cost and nonfunctional constraints X_i a much simpler problem

$$- \inf_{v^* \geqq \vartheta} \sum_{i=1}^{N} \sup_{x_i \in X_i} [g_i(x_i) + \langle v^*, x_i \rangle] \,. \tag{4.15}$$

Thus, since a Kuhn-Tucker vector $- \hat{v}^*$ is computed, the optimal productions $\hat{x}_i \in X_i$ for single factories, $i = 1, 2, \dots, N$ can be determined by evaluating convex conjugate functions $\Psi_i^*(v^*)$, where

$$\Psi_i(x_i) = -g_i(x_i) + \delta_{X_i}(x_i) \,. \tag{4.16}$$

Using this notation, problem (4.15) can be written in a very compact form

$$- \sum_{i=1}^{N} \Psi_i^*(v^*) \to \sup \tag{4.17}$$

subj. to

$$v^* \geqq \vartheta \,.$$

The object of a firm is now to estimate a certain approximation of $\hat{v}^*$, meanwhile the object of the ith factory is to compute $\Psi_i^*(v^*)$, or, in other words to solve the subproblem

$$g_i(x_i) + \langle \hat{v}^*, x_i \rangle \to \sup \tag{4.18}$$

subj. to

$$x_i \in X_i$$

of dimension n. If $\hat{x}_i$ is a solution of (4.18), then $g_i(\hat{x}_i)$ is in fact the real profit. The value $\langle \hat{v}^*, x_i \rangle$ is termed the *shadow profit* and the vector $\hat{v}^*$ is called the vector of ***shadow prices***. It is well justified, because $\hat{v}^*$ belongs to the subdifferential of the extremal-value function of (4.12) (taken in the concave sense) at $p = \vartheta$, cf. (1.5). Hence, $\hat{v}^{*i}$ would give us an estimate of the change of the whole optimal profit $\sum_{i=1}^{N} g_i(\hat{x}_i)$, if c^i increases by a unit.

Doubtless the solution of (4.12) by way of (4.15) is in many cases much easier than a direct attacking of (4.12). The Uzawa's algorithm represents in the economic theory a dialogue between the centre (the firm) and the periphery (the factories) as a result of which we obtain a sequence of shadow prices vectors $v_1^*, v_2^*, \ldots$ converging in an appropriate sense to the required system of shadow prices.

5. Shifted penalties

The Lagrange dualisation possesses three principal disadvantages:

(i) the dual $(\mathscr{D}2)$ is a *constrained* maximization of a generally *nondifferentiable* function $\mathscr{L}(x, .)$;

(ii) there is hardly any chance for a nonconvex problem $(\mathscr{P}2)$ to be inf-stable with respect to these perturbations;

(iii) for $\hat{p}^*$ solving $(\mathscr{D}2)$, $\bar{x} \in \arg\inf_{x \in X} \mathscr{L}(x, \hat{p}^*)$ need not be a solution of $(\mathscr{P}2)$.

Therefore, it has been a considerable effort to construct a better dualisation for $(\mathscr{P}2)$ started in 1969 by works of Hestenes [10] and Powell [26], in which a shifted or also generalized quadratic penalty has been proposed for problems with nonlinear equality constraints. Since that time many papers and books have appeared, where the method has been gradually generalized so that now we have a rather complete theory. Simultaneously, this effort has pointed out further research directions which will be discussed in the next part of the paper.

In this section we confine ourselves to the completely convex problem $(\mathscr{P}2)$ as in Sec. 4. Further possibilities of this approach for nonconvex problems (in which even the extremal-value function need not be convex) will be studied in the second part of the paper. We will assume that X is a reflexive Banach space and $\boldsymbol{Z} = \boldsymbol{H}$ (a Hilbert space). Furthermore, we impose assumptions (i)...(vi) of the preceding section. Under these assumptions the infimum of $(\mathscr{P}2)$ is finite and $(\mathscr{P}2)$ possesses solutions.

We set now $\boldsymbol{X} = X$, $\boldsymbol{V} = \boldsymbol{Z} = \boldsymbol{H}$ (cf. the notation of Sec. 2), and

$$\Phi(x, p) = f(x) + \delta_A(x) + \delta_D(p - q(x)) + r\|p\|^2 , \tag{5.1}$$

where $r > 0$ is so called *penalty parameter*. Note that for $r = 0$ we obtain the Lagrange perturbations. On denoting h_0 the extremal-value function for (4.1) and h_r the extremal-value function for (5.1), we conclude that

$$h_r(p) = h_0(p) + r\|p\|^2 \tag{5.2}$$

so that the stability of $(\mathscr{P}2)$ with respect to perturbations (5.1) is exactly the same as with respect to perturbations (4.1). Therefore, Props. 4.1, 4.2 remain true even here ($\boldsymbol{H}$ is of course a Banach space, hence also barrelled).

In the computation of the Lagrangian, we utilize the following lemma proved e.g. in [38]:

Lemma 5.1. For any $p \in \boldsymbol{H}$ the following holds:

$$p = p^D + p^{-D^*}, \quad (-p)^{D^*} = -p^{-D^*}. \tag{5.3}$$

By definition

$$L(x, p^*) = -\sup_{p \in \boldsymbol{H}} [\langle p^*, p \rangle - f(x) - \delta_A(x) - \delta_D(p - q(x)) - r\|p\|^2] = \tag{5.4}$$

$$= f(x) + \delta_A(x) - \frac{1}{4r}\|p^*\|^2 + r \inf_{v \in D} \left\| v - \left(\frac{p^*}{2r} - q(x)\right)\right\|^2.$$

The last expression implies first (using Lemma 5.1) that

$$L(x, p^*) = f(x) + \delta_A(x) + r\left\|\left(q - \frac{p^*}{2r}\right)^{D^*}\right\|^2 - \frac{1}{4r}\|p^*\|^2 = \tag{5.5}$$

$$= f(x) + \delta_A(x) - \left\langle \left(\frac{p^*}{2r} - q(x)\right)^D + q(x), p^* \right\rangle + r\left\|\left(\frac{p^*}{2r} - q(x)\right)^D + q(x)\right\|^2.$$

Secondly, we may apply Props. 1.7, 1.8 ($u = v$, $z = p^*/(2r) - q(x)$) and conclude that

$$\nabla_{p^*} L(x, p^*) = -\left(\frac{p^*}{2r} - q(x)\right)^D - q(x). \tag{5.6}$$

In what follows we will denote the Lagrangian (5.5) by $L_r(x, p^*)$ to emphasize the role of the penalty parameter r and term it the *augmented Lagrangian* of $(\mathscr{P}2)$.

We immediately see that for r tending to zero $L_r(x, p^*)$ converges to the classical Lagrangian of $(\mathscr{P}2)$ given by (4.5) and for $r > 0$ dom $L_r = \{(x, p^*) \in \boldsymbol{X} \times \boldsymbol{H} \mid L_r(x, p^*)$ is finite$\} = A \times \boldsymbol{H}$; hence the dual problems

$$(\mathscr{D}2_r) \qquad \begin{aligned} & G_r(p^*) = \inf_{x \in \boldsymbol{X}} L_r(x, p^*) \to \sup \\ \text{subj. to} \quad & \\ & p^* \in \boldsymbol{H} \end{aligned}$$

are in fact unconstrained optimizations over a Hilbert space $\boldsymbol{H}$. We denote $G_0(p^*)$ the dual objective of the preceding section (equals the cost in $(\mathscr{D}2_r)$ above for $r = 0$) and state the following important result of Rockafellar [31]:

Proposition 5.1. For every $r > 0$

$$G_r(p^*) = \max_{a^* \in \boldsymbol{H}} \left[G_0(a^*) - \frac{1}{4r}\|p^* - a^*\|^2 \right]. \tag{5.7}$$

Thus the dual problems $(\mathscr{D}2_r)$ all have the same optimal solutions as the dual $(\mathscr{D}2)$ of the preceding section. Moreover, G_r is continuously Fréchet differentiable on $\boldsymbol{H}$.

Specifically, if for a given p^* $\bar{x} \in \arg\inf_{x\in X} L_r(x, p^*)$, then

$$\nabla G_r(p^*) = \nabla_{p^*} L_r(\bar{x}, p^*) . \tag{5.8}$$

Proof. Clearly, from Eq. (5.2)

$$h_r^*(p^*) = (h_0 + r\|\cdot\|^2)^* (p^*) = h_0^*(p^*) \,\square\, \frac{1}{4r}\|p^*\|^2 =$$
$$= \min_{v^*\in H}\left[h_0^*(v^*) + \frac{1}{4r}\|p^* - v^*\|^2\right]$$

due to the coercivity of the minimized functional. As $G_r = -h_r^*$, $G_0 = -h_0^*$, we have obtained Eq. (5.7).

Concerning the second assertion of Prop. 5.2, we may rewrite (5.7) into the form

$$G_r(p^*) = -\frac{1}{2r}\min_{a^*\in H}\left[-2rG_0(a^*) + \tfrac{1}{2}\|p^* - a^*\|^2\right]$$

and apply Props. 1.7, 1.8. They imply that G_r is Fréchet differentiable for all $p^* \in H$, and

$$\nabla G_r(p^*) = -\frac{1}{2r}\operatorname{prox}_{(-2rG_0)^*} p^* = -\frac{1}{2r}(p^* - \operatorname{prox}_{-2rG_0} p^*) = -\frac{1}{2r}(p^* - \bar{a}^*) ,$$

where $\bar{a}^*$ is the (unique) solution of the maximization problem in (5.7). Since L_r is concave and Fréchet differentiable for all $x \in A$ with respect to the second variable, with $\nabla_{p^*} L_r(x, p^*)$ given by (5.6), we have for all $y^* \in H$ and for $\bar{x} \in \arg\inf_{x\in X} L_r(x, p^*)$

$$G_r(y^*) \leqq L_r(\bar{x}, y^*) \leqq L_r(\bar{x}, p^*) + \langle y^* - p^*, \nabla_{p^*} L_r(\bar{x}, p^*)\rangle =$$
$$= G_r(p^*) + \langle y^* - p^*, \nabla_{p^*} L_r(\bar{x}, p^*)\rangle .$$

Hence, $\nabla_{p^*} L_r(\bar{x}, p^*) \in \partial G_r(p^*) = \nabla G_r(p^*)$ and we are done. □

Remark. Under any assumption guaranteeing that h is l.s.c., the differentiability of G_r is implied by Prop. 1.11. Indeed, h_r and $-G_r^*$ are then proper convex functions conjugate to each other and the infimum of $h_r(\cdot) - \langle p^*, \cdot\rangle$ is attained for all $p^* \in H$ at some $p \in H$ strongly with respect to the norm-topology, whenever $r > 0$.

For the solution of $(\mathcal{D}2_r)$ the following numerical scheme has been proposed in [30]:

Given $r > 0$, $p_0^* \in H$ and a nonnegative sequence $\{\alpha_k\}$ converging to zero with

$$\sum_{k=1}^{\infty} \sqrt{\alpha_k} < +\infty . \tag{5.9}$$

kth step: 1. Given $p_k^* \in H$, determine $x_k \in A$ such that

$$L_r(x_k, p_k^*) \leqq \inf_{x\in X} L_r(x, p_k^*) + \alpha_k . \tag{5.10}$$

2. Set

(5.11) $$p^*_{k+1} = p^*_k + 2r\,\nabla_{p^*} L_r(x_k, p^*_k)\,.$$

When studying the convergence properties of this numerical scheme, we will make use of the

Definition 5.1. A sequence $\{x_k\}$ will be termed *asymptotically minimizing* (ASMS) for ($\mathscr{P}2$) if
(i) it is a *minimizing sequence*, i.e.

(5.12) $$\lim_{k\to\infty} f(x_k) = \alpha\,;$$

(ii) it is an *asymptotically feasible sequence*, i.e.

(5.13) $$\lim_{k\to\infty} \operatorname{dist}\{x_k, A\} = 0\,,\qquad \lim_{k\to\infty} \operatorname{dist}\{-q(x_k), D\} = 0\,.$$

Proposition 5.2. Let the sequence $\{p^*_k\}$ generated by (5.10), (5.11) be bounded. Then the corresponding sequence $\{x_k\}$ is assymptotically minimizing for ($\mathscr{P}2$) and all of its weak cluster points are solutions of ($\mathscr{P}2$). Simultaneously, every weak cluster point of $\{p^*_k)$ is a Kuhn-Tucker vector of ($\mathscr{P}2$).

In the proof we involve following lemmas:

Lemma 5.2. The dual objective G_r satisfies for all y^* and p^* from $\boldsymbol{H}$ the inequality

(5.14) $$G_r(y^*) \geqq G_r(p^*) + \langle y^* - p^*, \nabla G_r(p^*)\rangle - \frac{1}{4r}\|y^* - p^*\|^2\,.$$

Proof. Eq. (5.7) implies, for any given p^*, the existence of a quadratic function of the form

$$b(y^*) = G_0(z^*) - \frac{1}{4r}\|y^* - z^*\|^2$$

satisfying $b(p^*) = G_r(p^*)$, while $b(y^*) \leqq G_r(y^*)$ for all $y^* \in \boldsymbol{H}$. The two properties imply $\nabla b(p^*) = \nabla G_r(p^*)$. But since b is quadratic, we have

$$b(y^*) = b(p^*) + \langle y^* - p^*, \nabla b(p^*)\rangle - \frac{1}{4r}\|y^* - p^*\|^2\,.$$

Thus $b(y^*)$ equals the expression on the right side of (5.14). □

Lemma 5.3. It holds

(5.15) $$r\,\|\nabla_{p^*} L_r(x_k, p^*_k) - \nabla G_r(p^*_k)\|^2 \leqq \alpha_k\,.$$

Proof. Using the Ineq. (5.14) and the concavity of $L_r(x_k, \cdot)$, we have for every $y^* \in \boldsymbol{H}$

$$L_r(x_k, p_k^*) + \langle y^* - p_k^*, \nabla_{p^*} L_r(x_k, p_k^*)\rangle \geqq L_r(x_k, y^*) \geqq G_r(y^*) \geqq$$

$$\geqq G_r(p_k^*) + \langle y^* - p^*, \nabla G_r(p_k^*)\rangle - \frac{1}{4r} \|y^* - p_k^*\|^2 ,$$

and hence

$$L_r(x_k, p_k^*) - G_r(p_k^*) \geqq \sup_{u \in \boldsymbol{H}} \left[\langle u, \nabla G_r(p_k^*) - \nabla_{p^*} L_r(x_k, p_k^*)\rangle - \frac{1}{4r} \|u\|^2 \right] =$$

$$= r\|\nabla G_r(p_k^*) - \nabla_{p^*} L_r(x_k, p_k^*)\|^2 . \qquad \square$$

Lemma 5.4. For a bounded sequence $\{p_k^*\}$ of elements of $\boldsymbol{H}$ generated by (5.10), (5.11)

$$\lim_{k\to\infty} [G_r(p_{k+1}^*) - G_r(\operatorname*{prox}_{-2rG_0} p_k^*)] = 0 .$$

Proof. As G_r is a concave function continuous over $\boldsymbol{H}$, it is locally Lipschitz (cf. Cor. 1.2.1). Hence, there exists a constant L such that

$$\lim_{k\to\infty} [G_r(p_{k+1}^*) - G_r(\operatorname*{prox}_{-2rG_0} p_k^*)] \leqq \lim_{k\to\infty} L\|p_{k+1}^* - \operatorname*{prox}_{-2rG_0} p_k^*\| .$$

We already know (Proof of Prop. 5.1) that

$$\operatorname*{prox}_{-2rG_0} p_k^* = p_k^* + 2r\, \nabla G_r(p_k^*) .$$

Thus, Ineq. (5.15) and the *up-date rule* (5.11) imply

$$\|p_{k+1}^* - \operatorname*{prox}_{-2rG_0} p_k^*\|^2 \leqq 4r\alpha_k$$

which completes the proof. $\square$

Lemma 5.5. Let $\{p_k^*\}$ be a bounded maximizing sequence for $(\mathscr{D}2_r)$ and a sequence $\{x_k\}$ be given by (5.10). Then $\{x_k\}$ is ASMS for $(\mathscr{P}2)$, possesses at least one cluster point $\hat{x}$ and this vector is an actual solution of $(\mathscr{P}2)$.

Proof. To show that $\{x_k\}$ is ASMS for $(\mathscr{P}2)$, we have merely to verify Eqs. (5.12), (5.13), because all $x_k \in A$. By definition

$$L_r(x_k, p_k^*) = \min_{p \in \boldsymbol{H}} [\Phi(x_k, p) - \langle p_k^*, p\rangle] ,$$

and the minimum is attained at the unique point

$$\bar{p}_k = -\nabla_{p^*} L(x_k, p_k^*) .$$

Therefore, $\bar{p}_k \to \vartheta$ by Lemma 5.3. Since by hypothesis

$$\lim_{k\to\infty} L_r(x_k, p_k^*) = \lim_{k\to\infty} G_r(p_k^*) = \alpha ,$$

while the sequence $\{p_k^*\}$ is bounded, we see that

$$\lim_{k\to\infty} \Phi(x_k, \bar{p}_k) = \lim_{k\to\infty} [L_r(x_k, p_k^*) + \langle p_k^*, \bar{p}_k\rangle] = \alpha$$

which gives immediately Eqs. (5.12), (5.13).

Indeed, the sequence $\{x_k\}$ is bounded either due to the boundedness of A or due to the uniform coercivity of f_p (given by (4.2)) on some neighborhood of $\vartheta \in \boldsymbol{H}$. Hence, it possesses weak cluster points. Let x_0 by one of them. The assumptions being imposed imply that both functions $f(\cdot)$ and dist $(-q(\cdot), D)$ are weakly l.s.c. Thus, for $x_k \xrightarrow{w} x_0$

$$\alpha \leqq f(x_0) \leqq \lim_{k\to\infty} f(x_k) = \alpha$$

$$0 \leqq \text{dist}\,(-q(x_0), D) \leqq \lim_{k\to\infty} \text{dist}\,(-q(x_k), D) = 0\,. \qquad \square$$

P r o o f o f P r o p. 5.2. With respect to the assertion of the preceding lemma we show first that the sequence $\{p_k^*\}$ generated by (5.10, 5.11) is maximizing for $(\mathscr{D}2_r)$, i.e.

$$\lim_{k\to\infty} G_r(p_k^*) = \beta = \alpha\,.$$

Eq. (5.7) implies that

(5.16)

$$G_r(p_k^*) = G_0(\operatorname{prox}_{-2rG_0} p_k^*) - \frac{1}{4r}\,\|\operatorname{prox}_{-2rG_0} p_k^* - p_k^*\|^2 = G_0(\operatorname{prox}_{-2rG_0} p_k^*) - r\|\nabla G_r(p_k^*)\|^2\,,$$

but simultaneously also

(5.17)
$$G_r(\operatorname{prox}_{-2rG_0} p_k^*) \geqq G_0(\operatorname{prox}_{-2rG_0} p_k^*)\,.$$

Combining (5.16) with (5.17), we have

$$G_r(\operatorname{prox}_{-2rG_0} p_k^*) \geqq G_r(p_k^*) + r\|\nabla G_r(p_k^*)\|^2\,.$$

We involve now the assertion of Lemma 5.4 and conclude that

$$\lim_{k\to\infty} \|\nabla G_r(p_k^*)\| = 0\,,$$

because G_r is bounded from above. Consequently, $\{p_k^*\}$ is a maximizing sequence for $(\mathscr{D}2_r)$. If $\hat{p}^*$ is one of its weak cluster points (their existence is ensured by the boundedness of $\{p_k^*\}$, then actually

$$\nabla G_r(\hat{p}^*) = \vartheta\,,$$

and $\hat{p}^*$ is a Kuhn-Tucker vector for $(\mathscr{P}2)$. $\square$

Remark. Any condition guaranteeing that $\vartheta \in \text{int}\,(\text{dom}\,h)$ ensures also the boundedness of all sequences $\{p_k^*\}$ generated by (5.10), (5.11). Indeed, according to Prop. 1.9, in such a case the level sets

$$\{p^* \in \boldsymbol{H} \mid g_r^*(p^*) \geqq \beta\},\ \beta \in \mathbb{R}$$

are bounded.

Remark. Lemma 5.5 implies in particular that if $\hat{p}^*$ is a Kuhn-Tucker vector of $(\mathscr{P}2)$ then any $\hat{x} \in \arg\min L_r(x, \hat{p}^*)$ is an actual solution of $(\mathscr{P}2)$. In theory of convex--concave functions this property is termed *stability in x* and it was not generally present in the Lagrange duality. In 1976 G. Maistrovskii [18] proved in a finite--dimensional setting the convergence of a substantionally simplified version of the Uzawa type algorithm (5.9)–(5.11) provided f and q are continuously differentiable:

Given $r > 0$, $x_0 \in A$, $p_0^* \in \boldsymbol{H}$ and a nonnegative sequence $\{\alpha_k\}$ such that

$$\lim_{k\to\infty} \alpha_k = 0\,, \quad \lim_{k\to\infty} \sum_{i=0}^{k} \alpha_i = \infty\,.$$

kth step: Given $x_k \in A$, $p_k^* \in \boldsymbol{H}$ compute

$$x_{k+1} = (x_k - \alpha_k \nabla_x L(x_k, p_k^*))^A,\ p_{k+1}^* = p_k^* + \alpha_k \nabla_{p^*} L(x_k, p_k^*)\,.$$

Such "diagonal" algorithms have been further investigated by Golshtein in [9] for a larger class of augmented Lagrangians generated by perturbations

$$\Phi(x, p) = f(x) + \delta_A(x) + \delta_D(p - q(x)) + \alpha(p)\,,$$

where α is a convex function, $\alpha(\vartheta) = 0$, $\nabla\alpha(\vartheta) = \vartheta$.

We illustrate now the shifted penalty approach on the simple convex optimal control problem

$$\varphi(x(T)) + \int_0^T \psi(u(t))\,\mathrm{d}t \to \inf$$

subj. to

$$(5.18)\qquad \begin{aligned} &\dot{x}(t) = Sx(t) + Bu(t) \quad \text{a.e. on} \quad [O, T]\,,\\ &x(0) = \vartheta\,,\\ &Fx(t) + d_0 \leqq \vartheta \quad \text{on} \quad [O, T]\,,\\ &u \in L_2[O, T; \mathbb{R}^l]\,, \end{aligned}$$

where S, B, F are constant $[n \times n]$, $[n \times l]$, $[m \times n]$ matrices, respectively, $\varphi[\mathbb{R}^n \to \mathbb{R}_{cl}]$, $\psi[\mathbb{R}^l \to \mathbb{R}_{cl}]$, $d_0 \in W_2^1[0, T; \mathbb{R}^m]$ and T is the fixed terminal time. Problem (5.18) can be easily converted into the form $(\mathscr{P}2)$ as follows: We denote Π the linear operator which assigns to each control vector u the corresponding trajectory, i.e.

$$(\Pi u)(t) = x(t) = \int_0^t \exp(S(t - \tau))\,Bu(\tau)\,\mathrm{d}\tau,\ t \in [0, T]\,.$$

As Π is defined on $L_2[0, T, \mathbb{R}^l]$, it is natural to set $\boldsymbol{X} = L_2[0, T; \mathbb{R}^n]$. The constraint map q is given by

$$q(u) = F \circ \Pi u + d_0$$

so that we may choose for V any Hilbert space containing $\mathscr{R}(q) = W_2^1[0, T, \mathbb{R}^m]$. D is then the cone of nonnegative functions in V. Finally, $A = X$ and the cost is given by

$$f(u) = \varphi((\Pi u)(T)) + \int_0^T \psi(u(t))\,dt\,. \tag{5.19}$$

The choice of V is a crucial question. In a standard approach one takes $L_2[0, T; \mathbb{R}^m]$ with a very easy computation of projections $(\cdot)^D$. However, the interior of D in $L_2[0, T; \mathbb{R}^m]$ is empty and therefore the Slater condition does not apply. Unfortunately, also the other stability conditions do not help us and hence the case of an unbounded sequence $\{p_k^*\}$ generated by (5.10, 5.11) cannot be avoided. So, we set here $V = W_2^1[0, T; \mathbb{R}^m]$ which is Hilbert with the inner product e.g.

$$\langle v, \mu \rangle_{W_2^1} = \langle v(0), \mu(0) \rangle_m + \int_0^T \langle \dot{v}(t), \dot{\mu}(t) \rangle_m\,dt\,.$$

In $W_2^1[0, T; \mathbb{R}^m]$ the interior of D is nonempty and therefore the Slater condition may be imposed. However, the projections $(\cdot)^D$ are now rather complicated. This projection problem has been studied in [21], where also an effective numerical procedure has been obtained for the discretized version of an appropriate dual problem to (5.18). But, unfortunately, for Sobolev spaces of functions of two or more variables (which need not be continuous) the computation of $(\cdot)^D$ cannot be performed in such a direct way as in [21] and generally a certain variational inequality must be solved. We impose the following assumptions, guaranteeing that the requirements (i)...(vi) of Sec. 4 are satisfied:

(i) $\varphi \in \Gamma_0(\mathbb{R}^n)$, $\psi \in \Gamma_0(\mathbb{R}^l)$, dom $\varphi = \mathbb{R}^n$, dom $\psi = \mathbb{R}^l$;

(ii) there exist constants $M > 0$ and L such that $|\psi(z)| \leqq M|z|_l^2 + L$;

(iii) f (given by (5.19)) is coercive on $L_2[0, T, \mathbb{R}^l]$;

(iv) there exists a control u_0 such that the corresponding trajectory x_0 satisfies the generalized Slater condition $-Fx_0 - d_0 \in \text{int } D$.

To simplify the expression for the augmented Lagrangian, we denote

$$c^* = \left(\frac{p^*}{2r} - Fx - d_0\right)^D$$

and introduce instead of L_r a new function $\Lambda_r[X \times H \times H \to \mathbb{R}_{cl}]$ related with L_r by

$$\Lambda_r(u, x, p^*) = L_r(u, p^*) \quad \text{if} \quad x = \Pi u\,.$$

Then,

$$\Lambda_r(u, x, p^*) = \varphi(x(T)) + \int_0^T \psi(u(t))\,dt - \langle c^*(0) + Fx(0) + d_0(0), p^*(0) \rangle_m -$$

$$- \int_0^T \langle \dot{c}^*(t) + F\dot{x}(t) + \dot{d}_0(t), \dot{p}^*(t) \rangle_m \, dt + r|c(0) + Fx(0) + d_0(0)|_m^2 +$$
$$+ r \int_0^T |\dot{c}^*(t) + F\dot{x}(t) + \dot{d}_0(t)|_m^2 \, dt \,,$$

and the dual problem of (5.18) attains the form

$$\min_{\substack{x = \Pi u \\ u \in L_2[0,T,\mathbb{R}^l]}} \Lambda_r(u, x, p^*) \to \max \tag{5.20}$$

subj. to

$$p^* \in W_2^1[0, T; \mathbb{R}^m] \,.$$

We are entitled to write min and max instead of inf and sup due to assumptions being imposed. For each fixed p^* the intermediate minimization

$$\Lambda_r(u, x, p^*) \to \min \tag{5.21}$$

subj. to

$$x = \Pi u$$
$$u \in L_2[0, T, \mathbb{R}^l]$$

is again an optimal control problem with the same system, but without the state-space constraint which has been augmented to the cost. Problem (5.21) may be solved by a gradient technique in the case described by the assertion below:

Proposition 5.3. Let φ, ψ be continuously differentiable, S be regular and Θ_{p^*} $[L_2[0, T; \mathbb{R}^m] \to \mathbb{R}]$ be the functional given by

$$\Theta_{p^*}(u) = L_r(u, p^*) \quad \text{for} \quad u \in L_2[0, T; \mathbb{R}^l] \,.$$

Let a function $\varkappa$ be the solution of the adjoint equation

$$-\dot{\varkappa}(t) = S^*\varkappa(t) - 2rF^*(F\dot{x}(t) + \dot{d}_0(t) - \frac{\dot{p}^*(t)}{2r} + \dot{c}(t)) \quad \text{a.e.}$$

backwards on $[0, T]$ from the terminal condition

$$\varkappa(T) = -(S^*)^{-1} \nabla\varphi(x(T)) \,.$$

Then, the functional Θ_{p^*} is Fréchet differentiable over $L_2[0, T; \mathbb{R}^l]$ and $\nabla\Theta_{p^*}(u) = B^*\dot{\varkappa} + \nabla\psi(u)$.

Proof. Cf. [16] and [21].

Remark. If some simple (affine) control constraints are imposed, it is not recommendable to augment them to the objective, but generally it is more effective to handle them directly in (5.21) within the minimization routine. Therefore, also in the whole theory of Secs. 4, 5 we do not assign a certain structure to the set A and do not augment this constraint to the objective.

The method just described is now frequently used for the solution of various extremal problems not only in the convex case or in the case with a convex extremal-value function but also for general nonconvex but differentiable problems (all functionals and operators are differentiable in an appropriate sense). In this extension we meet some new problems and therefore we return to the shifted penalties once more in the second part of the sequel.

6. Miscellaneous Items

The first part of this section is devoted to dualisations of two significant mathematical programming problems. In the second part we describe a class of nonconvex problems, the extremal-value function of which is convex and thus the theory of Sec. 2 is applicable.

We start with the fractional programming. Optimization problems of this kind occur if rates of economical and/or technical terms define the objective function cf. [34]. The problems attain the form

$$\frac{f(x)}{g(x)} \to \sup$$

(6.1) subj. to

$$x \in S_0 \subset \mathbb{R}^n ,$$

where $f[S_0 \to \mathbb{R}]$ is concave and nonnegative on S_0, $g[S_0 \to \mathbb{R}]$ is convex and positive on S_0 and S_0 is convex compact. We apply now the Charnes-Cooper variable transformation, cf. [5]:

$$y = \frac{1}{g(x)}\, x\,, \quad \tau = \frac{1}{g(x)}$$

for $x \in S_0$, and obtain an equivalent problem

$$\tau f(y/\tau) \to \sup$$

(6.2) subj. to

$$\tau > 0$$

$$y/\tau \in S_0$$

$$\tau g(y/\tau) = 1 .$$

Lemma 6.1. Let q be an arbitrary convex (concave) function on S_0. Then the function $(y, \tau) \to \tau q(y/\tau)$ is convex (concave) on

$$S_0^* = \left\{(y, \tau) \in \mathbb{R}^{n+1} \mid \tau > 0\,, \frac{y}{\tau} \in S_0\right\}.$$

The proof is a straightforward consequence of the definitions, cf. [35].

However, problem (6.2) is not a convex program yet because of its equality constraint. Following [33] we relax this constraint to the form

$$\tau g(y/\tau) \leqq 1 ,$$

thereby arriving at a new problem which will be denoted by (6.3). The following assertion holds:

Proposition 6.1. We assume that there exists a point feasible with respect to constraints of (6.1) and such that $f(\bar{x}) > 0$. Then, problem (6.3) possesses a solution iff problem (6.1) possesses a solution. These solutions are connected by Charnes--Cooper transformation and the appropriate objective values are equal.

Proof. Let (y, τ) satisfy the constraints of (6.3) with $0 < \tau g(y/\tau) = \alpha < 1$ and $\tau f(y/\tau) > 0$. For $\lambda = 1/\alpha$ $(\lambda y, \lambda\tau)$ is an admissible point for (6.3). Clearly, for $\lambda > 1$ $\lambda\tau f(\lambda y/\lambda\tau) > \tau f(y/\tau)$ cf. also [33]. □

In fact, we have obtained under the imposed assumptions for a nonconvex fractional program (6.1) an equivalent convex program of the form ($\mathscr{P}2$). For nondifferentiable functions f and/or g it is recommended in [34] to perform the Lagrange dualisation (Sec. 4). We obtain the dual program

$$\inf_{p^* \leqq 0} \sup_{(y,\tau)\in S_0^*} \{\tau f(y/\tau) + p^*(\tau g(y/\tau) - 1)\} . \tag{6.4}$$

If there exists a couple $(y_0, \tau_0) \in S_0^*$ such that $\tau_0 g(y_0/\tau_0) < 1$ which is the stability condition, an eventual duality gap disappears, infimum in (6.4) may be replaced by minimum, and e.g. the Uzawa algorithm may be applied to its solution. Provided both f and g are continuously differentiable, other dual concepts are more suitable, cf. [34].

The same idea, namely the convexification by the construction of an equivalent convex program and the subsequent suitable dualisation leads to an easily tractable dual program also in the case of the geometric programming. We will consider the problem

$$\begin{aligned} &\sum_{k=1}^{T_0} c_0^k \prod_{j=1}^{n} (t^j)^{a_{0jk}} \to \inf \\ \text{subj. to} \quad &\sum_{k=1}^{T_i} c_i^k \prod_{j=1}^{n} (t^j)^{a_{ijk}} \leqq 1 \quad \text{for} \quad i = 1, 2, \ldots, m , \\ &t^j > 0 , \quad j = 1, 2, \ldots, n , \end{aligned} \tag{6.5}$$

where $c_i^k \geqq 0$, $i = 0, 1, \ldots, m$, $k = 1, 2, \ldots, T_i$. Problems of this type appear frequently in engineering design cf. [6]. (6.5) is a nonconvex program, but if we perform the simple transformation

$$y^j = \ln t^j \quad \text{for} \quad j = 1, 2, \ldots, n , \tag{6.6}$$

and take logarithms of the objective and of the constraint functions, we obtain the convex program

$$\ln \sum_{k=1}^{T_0} c_0^k \exp\left(\sum_{j=1}^{n} a_{0jk} y^j\right) \to \inf$$

(6.7) subj. to

$$\ln \sum_{k=1}^{T_i} c_i^k \exp\left(\sum_{j=1}^{n} a_{ijk} y^j\right) \leqq 0 \quad \text{for} \quad i = 1, 2, \ldots, m\,.$$

The convexity of the above program may be proved by the Hölder inequality. The optimal cost values μ of (6.7) and ν of (6.5) are related by

$$\mu = \ln \nu\,.$$

The problem (6.7) can be embedded into the following class of mathematical programs:

$$f_0(y) \to \inf$$

(6.8) subj. to

$$f_i(y) \leqq 0\,, \quad i = 1, 2, \ldots, m\,,$$

where each f_i may be expressed in the form

$$f_i(y) = h_i(A_i y + \alpha_i) + \langle b_i, y \rangle_{n_i} + d^i \tag{6.9}$$

with $h_i[\mathbb{R}^{n_i} \to \mathbb{R}]$, $y \in \mathbb{R}^n$, $\alpha_i \in \mathbb{R}^{n_i}$, $b_i \in \mathbb{R}^n$, $d^i \in \mathbb{R}$ and A_i being a constant matrix $[n_i \times n]$. We will assume that all h_i are convex and continuous and involve the perturbations $p = (u, v_0, v_1, \ldots v_m) \in \mathbb{R}^m \times \mathbb{R}^{n_0} \times \ldots \times \mathbb{R}^{n_m}$ by

$$\begin{aligned} \Phi(y, p) = {} & h_0(A_0 y + \alpha_0 - v_0) + \langle b_0, y \rangle_{n_0} + d^0 + \\ & + \sum_{i=1}^{m} \delta_{\mathbb{R}_-}\left(h_i(A_i y + \alpha_i - v_i) + \langle b_i, y \rangle_{n_i} + d^0 - u^0\right). \end{aligned} \tag{6.10}$$

As the corresponding extremal-value function is convex, the theory of Sec. 2 may be applied, and we obtain the dual problem

$$d^0 + \langle \alpha_0, v_0^* \rangle_{n_0} - h_0^*(v_0^*) + \sum_{i=1}^{m} \left[d^i u^{*i} + \langle \alpha_i, v_i^* \rangle_{n_i} - u^{*i} h_i^*(v_i^*/u^{*i})\right] \to \sup$$

(6.11) subj. to

$$b_0 + \sum_{i=1}^{m} u^{*i} b_i + \sum_{i=1}^{m} A_i^* v_i^* = \vartheta\,,$$

$$u^* \in \mathbb{R}_+^m\,.$$

Under an appropriate stability condition, a solution of (6.11) exists, and we may replace the solution of (6.8) by an eventually easier solution of the above dual problem. However, we must not forget the implicit constraints

$$v_0^* \in \operatorname{dom} h_0^*\,, \quad v_i^* \in u^{*i} \operatorname{dom} h_i^*\,, \quad i = 1, 2, \ldots, m$$

which may be of a rather complicated nature. In [29] a very elegant way is proposed for the characterization of these explicit constraints provided all h_i are "faithfully" convex (in particular convex analytic).

In (6.7) we have

$$h_i(z_i) = \ln \sum_{k=1}^{T_i} c_i^k \exp(z_i^k) \quad i = 0, 1, \ldots, m\,,$$

$$z_i^k = \sum_{j=1}^{n} a_{ijk} y^j\,, \quad j = 1, 2, \ldots, n$$

so that α_i, b_i, d^i are zero vectors and scalars, respectively, $i = 0, 1, \ldots, m$. Thus, by a direct computation, we obtain

$$h_i^*(z_i^*) = \sum_{k=1}^{T_i} z_i^{*k} \ln \frac{z_i^{*k}}{c_i^k} \quad (\text{with } 0 \ln 0 = 0)\,,$$

and

$$\operatorname{dom} h_i^* = \{z_i^* \in \mathbb{R}^{T_i} \mid z_i^{*k} \geqq 0, k = 1, 2, \ldots, T_i, \sum_{k=1}^{T_i} z_i^{*k} = 1\}$$

so that the dual problem generated by perturbations (6.10) attains the form

$$\sum_{i=0}^{m} \sum_{k=1}^{T_i} v_i^{*k} \ln \frac{c_i^k}{v_i^{*k}} + \sum_{i=1}^{m} u^{*i} \ln u^{*i} \to \sup$$

subj. to

(6.12) $$v_i^{*k} \geqq 0 \quad \text{for} \quad i = 0, 1, \ldots, m\,,\ k = 1, 2, \ldots, T_i\,,$$

(6.13) $$\sum_{k=1}^{T_0} v_0^{*k} = 1\,,$$

$$\sum_{k=1}^{T_i} v_i^{*k} = u^{*i} \quad \text{for} \quad i = 1, 2, \ldots, m\,,$$

(6.14) $$\sum_{i=0}^{m} \sum_{k=1}^{T_i} a_{ijk} v_i^{*k} = 0 \quad \text{for} \quad j = 1, 2, \ldots, n\,.$$

Inequalities (6.12) are termed the *positivity condition*, and equalities (6.13) and (6.14) are termed *normality* and *orthogonality* conditions, respectively, ([6]). Taking the exp function of the dual cost above creates the function

(6.15) $$\prod_{i=0}^{m} \prod_{k=1}^{T_i} \left(\frac{c_i^k}{v_i^{*k}}\right)^{v_i^{*k}} \left(\prod_{i=1}^{m} (u^{*i})^{u^{*i}}\right)$$

which is the standard dual cost in geometric programming.

Assertion (ii) of Prop. 2.5 implies that if there exists a vector $\bar{t}$ with positive components such that

(6.16) $$\sum_{k=1}^{T_i} c_i^k \prod_{j=1}^{n} (\bar{t}^j)^{a_{ijk}} < 1 \quad \text{for} \quad i = 1, 2, \ldots, \mathrm{m}\,,$$

then a solution of the above dual problem exists and the corresponding value of (6.15) equals the optimal cost value in (6.5). Geometric programming problems satisfying the stability condition (6.16) are said to be *superconsistent*. The solution $\hat{t}$ of the primal problem (6.5) and the solution $(\hat{u}^*, \hat{v}_0^*, \hat{v}_1^* \dots \hat{v}_m^*)$ of the dual problem (6.15) are related by

$$\hat{v}_0^{*k} = c_0^k \prod_{j=1}^n \frac{(\hat{t}_j)^{a_{0jk}}}{v}, \quad k = 1, 2, \dots, T_0,$$

$$\hat{v}_i^{*k} = \hat{u}^{*i} c_i^k \prod_{j=1}^n \frac{(\hat{t}_j)^{a_{ijk}}}{v}, \quad k = 1, 2, \dots, T_i, \; i = 1, 2, \dots, m,$$

see e.g. [6], [17].

Perturbations (6.10) may very well be used also in other mathematical programming problems like l^p-programs or quasiseparable programs, cf. [29].

We conclude now intentionally the first part of the paper with an example demonstrating how a nonconvex problem can be "convexly" perturbed to obtain a convex extremal-value function. We take the optimal control problem $(\mathscr{P}3)$ in which we require neither the operator Λ to be affine nor the set M to be convex. On the other hand we suppose that

(i) the objective depends only on the state variable y;

(ii) the reachable set ΛM is convex and $w(Y, Y^*)$-compact.

The remaining requirements are of the same kind as those of Sec. 3. $\boldsymbol{Y}$ and $\boldsymbol{Y}^*$ are two paired l.c.s. with respect to some pairing $\langle \cdot, \cdot \rangle_Y$ and assumptions (iii), (iv), (v) below correspond to assumptions (i), (ii), (iv) there, respectively:

(iii) $J \in \Gamma_0(Y)$ with $N \subset \operatorname{dom} J$;

(iv) N is a closed convex subset of $\boldsymbol{Y}$;

(v) $\Lambda M \cap N \neq \emptyset$.

The above assumptions guarantee the existence of a solution of $(\mathscr{P}3)$ and the finiteness of the corresponding cost value. We apply now the "Fenchel type" perturbations in the same way as in Sec. 3, i.e. we set

$$(6.17) \qquad \Phi(u, p) = J(\Lambda u - p) + \delta_M(u) + \delta_N(\Lambda u - p) = \tilde{J}(\Lambda u - p) + \delta_M(u),$$

where $\tilde{J} = J + \delta_N$.

Lemma 6.2. The extremal-value function corresponding to perturbations (6.17) is convex.

Proof. Let $p, q \in \boldsymbol{Y}$ and $\lambda \in (0, 1)$. Let $h(p)$ and $h(q)$ be finite – otherwise the assertion is obvious. For every $a > h(p)$, $b > h(q)$ there are $v, w \in M$ such that

$$h(p) \leqq \tilde{J}(\Lambda v - p) \leqq a, \quad h(q) \leqq \tilde{J}(\Lambda w - q) \leqq b.$$

Then:

$$h(\lambda p + (1 - \lambda) q) \leqq \inf_{u \in M} \tilde{J}(\Lambda u - \lambda p - (1 - \lambda) q) \leqq$$

$$\leqq \tilde{J}(\lambda \Lambda v + (1 - \lambda) \Lambda w - \lambda p - (1 - \lambda) q) \leqq$$
$$\leqq \lambda \tilde{J}(\Lambda v - p) + (1 - \lambda) \tilde{J}(\Lambda w - q) \leqq \lambda a + (1 - \lambda) b$$

because of the convexity of ΛM and $\tilde{J}$. If we let now a decrease towards $h(p)$ and b decrease towards $h(q)$ we obtain the desired inequality. □

Concerning the stability, Prop. 3.2 remains valid. If an appropriate condition is satisfied, the concave dual problem

$$(6.18) \qquad \begin{aligned} & -\tilde{J}^*(-p^*) - \delta^*_{\Lambda M}(p^*) \to \sup \\ & \text{subj. to} \quad p^* \in Y^* \end{aligned}$$

possesses a solution and an eventual duality gap vanishes. Of course, assumption (ii) is very stringent and hence very rarely satisfied. However, for Λ being a nonlinear ordinary differential equation we have some results dealing with this problem.

Proposition 6.2. Let the operator $\Lambda[\boldsymbol{U} \to \boldsymbol{Y}]$ be given by

$$(6.19) \qquad \begin{aligned} & \dot{y}(t) = f(y(t), u(t)) \quad \text{a.e. on} \quad [0, T], \\ & y(0) = a, \end{aligned}$$

where $f[\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n]$, $\boldsymbol{U} = \boldsymbol{L}_\infty[0, T; \mathbb{R}^m]$, $\boldsymbol{Y} = \boldsymbol{C}_0[0, T; \mathbb{R}^n]$. Let $M = \{u \in \boldsymbol{L}_\infty[0, T; \mathbb{R}^m] \mid u(t) \in K \text{ a.e. on } [0, T]\}$, where K is a subset of $\mathbb{R}^m$. We suppose that

(i) the "set of admissible speeds" $f(y, K)$ is a convex compact subset of $\mathbb{R}^n$ for all $y \in \mathbb{R}^n$;

(ii) the function
$$c(y, e) = \delta^*_{F(y)}(e)$$
is concave in y for $F(y) = f(y, K)$ and an arbitrary unit vector e.

Then the set of all trajectories of the system (6.19) generated by feasible controls $u \in M$ is convex.

For the proof see [3]. The closedness and boundedness of ΛM is implied by the following statement proved in [8]:

Proposition 6.3. Let K be compact and f be continuous from $\mathbb{R}^n \times K$ into $\mathbb{R}^n$. Moreover, we assume that

(i) for all $\varrho \geqq 0$ there exists $k \geqq 0$ such that for all $v \in K$
$$|f(x, v) - f(x', v)|_n \leqq k|x - x'|_n \;\; \forall x, x' \in B^\varrho_n(\vartheta);$$

(ii) there exists a constant $l \geqq 0$ such that
$$\forall x, v \in \mathbb{R}^n \times K \; |\langle x, f(x, v)\rangle| \leqq l(1 + |x|^2_n).$$

Finally, let assumption (i) of Prop. 6.2 be satisfied. Then:

A. For any control $u \in M$ there exists a unique trajectory of the system (6.19);
B. the reachable set ΛM is compact in $C_0[0, T; \mathbb{R}^n]$.
The proof may be found in [8].

We show now on a concrete "minimum energy" problem with a nonlinear system equation how one proceeds. Let us solve the problem

$$
\begin{aligned}
&\int_0^1 \langle u(t), u(t)\rangle \to \inf \\
(6.20) \qquad &\text{subj. to} \\
&\dot{y}(t) = \lambda(y(t))\, u(t) \text{ a.e. on } [0, 1] \\
&y(0) = a \\
&u \in M = \{u \in L_\infty[0, 1; \mathbb{R}^n] \mid u(t) \in K \text{ a.e. on } [0, 1]\} \\
&y(1) \in N = B_n^\varepsilon(b),
\end{aligned}
$$

where $\lambda[\mathbb{R}^n \to \mathbb{R}]$ is a concave continuously differentiable function, $K \subset \mathbb{R}^n$ is compact and $a, b \in \mathbb{R}^n$. To meet the assumption (i) in the introduction of this part we rewrite (6.20) into the form (arguments t are omitted for the simplicity)

$$
\begin{aligned}
&x^{n+1}(1) \to \inf \\
(6.21) \qquad &\text{subj. to} \\
&\dot{x}^i = \lambda(x^1, x^2 \ldots x^n)\, u^i, \quad i = 1, 2, \ldots, n \\
&\dot{x}^{n+1} = \langle u(t), u(t)\rangle_n \\
&x^i(0) = a^i, \quad i = 1, 2, \ldots, n \\
&x^{n+1}(0) = 0 \\
&u \in M \\
&(x^1(1), \ldots, x^n(1)) \in N.
\end{aligned}
$$

In this problem $\Lambda[L_\infty[0, 1; \mathbb{R}^n] \to \mathbb{R}^{n+1}]$ assigns to each control u the "extended" terminal state $x(1)$. If λ is such that assumption (ii) of Prop. 6.3 is satisfied, then ΛM is convex compact. If also the "controllability" assumption (v) in the present specification of the optimal control problem ($\mathscr{P}3$) is met, we may apply the above theory and obtain the dual problem

$$
\begin{aligned}
&\langle p^*, b\rangle_n - |p^*|_n\, \varepsilon - \delta^*_{\Lambda M}(p^*) \to \sup \\
(6.22) \qquad &\text{subj. to} \\
&p^* \in \mathbb{R}^n,
\end{aligned}
$$

where the value $\delta^*_{\Lambda M}(p^*)$ is the optimal cost value of the problem

$$\langle p^*, y(1)\rangle_n - \int_0^1 \langle u(t), u(t)\rangle_n \, dt \to \sup$$

(6.23) subj. to

$$\dot{y} = \lambda(y(t))\, u(t) \quad \text{a.e.} \quad \text{on} \quad [0, 1],$$

$$y(0) = a,$$

$$u \in M.$$

The above problem does not possess any terminal constraint and may be solved e.g. by the method of Krylov, cf. [4].

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PART II: NONCONVEX PROBLEMS

Introduction

All duality results of the first part of the supplement depend heavily on the fact that in a l.c.s. any closed convex set with a nonempty interior may be supported by a closed hyperplane at every boundary point. (It is a consequence of the Hahn-Banach theorem, cf. [8], 1, Cor. 1.3). In particular, at stable problems, epi h may be supported at $(\vartheta, h(\vartheta))$ and the solution of the dual problem amounts in fact the seeking for such a supporting hyperplane.

Of course, for h being nonconvex, such a hyperplane can exist only exceptionally. This having been one of the reasons, there has been a considerable effort in recent years to generalize the notion of convexity or the concept of separation. We see various directions in this research, e.g.

(i) introducing generalized convex functions as a supremum of a family of (generally) nonaffine functionals and generalized convex sets as level sets of such generalized convex functions cf. e.g. [14], [15], [43], [53], [54], [55], [74].

(ii) separation of sets by graphs or level sets of convex functionals cf. e.g. [67], [68];

(iii) extension of the Jensen inequqlity

$$f(\alpha x_1 + (1 - \alpha) x_2) \leqq \alpha f(x_1) + (1 - \alpha) f(x_2) \ \forall \alpha \in [0, 1], x_1, x_2 \in X$$

characterizing convex functions $f[X \to \mathbb{R}_{cl}]$ by various ways cf. e.g. [44], [48], [69];

(iv) detailed investigations concerning pseudoconvexity and quasiconvexity, cf. e.g. [49], [50], [51].

Especially, item (i) has lead to several very general duality theories. We note also a close relationship between the references given in (i) and (ii).

In the sequel we will not mutually compare single duality theories of the first approach, we just choose the nonconvex perturbational theory of duality due to Lindberg ([14], [15]) as the theoretical basis for various nonconvex dualisations appearing in the second and the third part of the supplement. This theory is explained in Sec. 8 and it is the nonconvex counterpart of the perturbational theory of duality presented in Sec. 2. It does not require any differentiability assumptions directly – however, the success of single dualisations may be substantionally influenced by the nature of the primal problem from this point of view. The dualisations presented in Secs. 9 and 10 are particularly suited for smooth problems; they represent the numerous family of augmented Lagrangian methods for the solution of mathematical programming and optimal control problems. The dualisations of Secs. 12 and 13 lead to nonsmooth dual problems and therefore we have included them into the third part of the supplement devoted to nonsmooth optimization.

Besides the general methods of items (i), (ii) and (iii) there is a considerable number of other duality theories (with or without the perturbational character) with various purposes and applicability fields. Two of such theories, requiring certain special properties to remove the duality gap but still having a considerable applicability area are briefly described in Sec. 7. It is the duality theory of Toland ([36], [37]) and the relaxation duality theory of Ekeland and Temam ([8], [78]). At this place it is interesting to note that Toland's results have been republished in [62] and [76] without any reference to the author. We start the "nonconvex" part of the sequel by these two theories because they are essentially based merely upon the results of the convex analysis and the convex perturbational theory of duality.

Let us still mention at least here, in the introduction, some other duality theories which could not be discussed in the sequel because of the extent limitations. In [57] the author proposes a general nonconvex duality theory for smooth extremal problems via the Legendre transformation. As the Legendre transform of a smooth nonconvex function need not be a function, he broadens the framework from smooth functions to Lagrangian manifolds.

In [56] a new duality concept termed the H-duality has been developed originating in the results of Singer [77] in convex optimization. In this duality scheme one replaces nonlinear (or even nonfunctional) constraints by a single affine equality constraint. Solving the dual problem amounts seeking for the optimal linear functional specifying this affine constraint.

There are various duality approaches in quasiconvex optimization cf. [42], [49], [60].

Furthemore, besides these theories with a still considerable level of generality we have a number of quite special duality schemes, where a specific nature of the problem is utilized. An example of this kind can be found in [58].

We shall employ the notation of the first part. Additionally, $(x)^+ = \max\{0, x\}$ for $x \in \mathbb{R}$, $\nabla^2 f(x)$ is the Hessian of f at x, sp A is the span of a set A, χ_A is the characteristic function of a set A and the sum over an empty index set equals zero.

7. Nonconvex dualisations derived by the convex perturbational theory

If we dualise a nonconvex extremal problem according to the pattern of Sec. 2, we cannot generally avoid the presence of a duality gap which makes the obtained dual problem useless. However, sometimes a special structure of the original problem together with some suitably chosen perturbations make the duality gap disappear. This section contains two examples of this kind. Let us start with the duality theory of Toland.

Let X, X^*, and V, V^* be two pairs of paired l.c.s. and we have to solve the problem $(\mathscr{P}1)$. Differently from the first part of the paper, we make no convexity assumptions concerning f; we assume merely that a "suitable" perturbed essential objective $\Phi[X \times V \to \mathbb{R}_{cl}]$, $\Phi(X, \vartheta) = f(x)$ can be found. The meaning of the adjective "suitable" will become clear in the course of our considerations.

We introduce the Lagrangian function $L[X \times V^* \to \mathbb{R}_{cl}]$ by

$$L(x, p^*) = -\sup_{p \in V} [\langle p^*, p \rangle + \Phi(x, p)] = -(-\Phi_x)^* (p^*) , \tag{7.1}$$

i.e. with the opposite sign at Φ in comparison with (2.7). Let us examine the dual problem

$$(\mathscr{D}1) \qquad \begin{aligned} & G(p^*) = -\sup_{x \in X} L(x, p^*) \to \inf \\ & \text{subj. to} \\ & p^* \in V^* . \end{aligned}$$

As previously we denote by α and β the optimal cost values of ($\mathscr{P}1$) and ($\mathscr{D}1$), respectively, and state the following assertions:

Proposition 7.1.

$$\alpha \leqq \beta .$$

Proof. By definition

$$\alpha \leqq \Phi(x, \vartheta) \leqq -L(x, p^*) \ \forall x \in X, p^* \in V^* .$$

Hence also

$$\alpha \leqq -\sup_{p^* \in V^*} \sup_{x \in X} L(x, p^*) = \inf_{p^* \in V^*} G(p^*) = \beta . \qquad \square$$

Proposition 7.2. If for all $x \in X$

$$(-\Phi_x)^{**}(\vartheta) = (-\Phi_x)(\vartheta) , \tag{7.2}$$

then $\alpha = \beta$.

Proof.

$$\alpha = \inf_{x \in X} \Phi_x(\vartheta) = -\sup_{x \in X} (-\Phi_x(\vartheta)) = -\sup_{x \in X} (-\Phi_x)^{**}(\vartheta) =$$
$$= -\sup_{x \in X} \sup_{p^* \in V^*} L(x, p^*) = \inf_{p^* \in V^*} G(p^*) = \beta . \qquad \square$$

Condition (7.2) is satisfied, if either $-\Phi_x \in \Gamma_0(V)$ or $\partial(-\Phi_x)(\vartheta) \neq \emptyset$ for each $x \in X$. Under this condition we could solve ($\mathscr{P}1$) by way of ($\mathscr{D}1$); however, ($\mathscr{D}1$) is a minimization of a generally nonconvex functional G over V^* so that it is not so favourable as the duals of Secs. 2–6. Extremality relations reflect the nonconvexity of both problems:

Proposition 7.3. If $\hat{x}$ solves ($\mathscr{P}1$) and $\hat{p}^* \in \partial(-\Phi_{\hat{x}})(\vartheta)$, then $\hat{p}^*$ solves ($\mathscr{D}1$). Furthermore,

$$\begin{aligned} & \Phi(\hat{x}, \vartheta) + L(\hat{x}, \hat{p}^*) = 0 , \\ & G(\hat{p}^*) + L(\hat{x}, \hat{p}^*) = 0 . \end{aligned} \tag{7.3}$$

Proof.

$$-\Phi(\hat{x}, p) = -\Phi_{\hat{x}}(p) \geqq -\Phi_{\hat{x}}(\vartheta) + \langle \hat{p}^*, p \rangle = -\alpha + \langle \hat{p}^*, p \rangle \ \forall p \in V .$$

Therefore,

$$\alpha \geqq -L(\hat{x}, \hat{p}^*)$$

which implies further that $-G(\hat{p}^*) = \sup_{x \in X} L(x, \hat{p}^*) \geqq -\alpha$ so that $\hat{p}^*$ solves $(\mathscr{D}I)$. Since $\hat{p}^* \in \partial(-\Phi_{\hat{x}})(\vartheta)$, we have

$$-\Phi_{\hat{x}}(\vartheta) + (-\Phi_{\hat{x}})^* (\hat{p}^*) = 0$$

due to Prop. 1.12. It proves relations (7.3). □

In [36] one can find various other important assertions concerning the existence of solutions $\hat{x}$, $\hat{p}^*$ of $(\mathscr{P}I)$, $(\mathscr{D}I)$, respectively, minimizing sequences etc. Here, we illustrate rather briefly these developments on an example which has in fact motivated this research. So, let

$$f = H - F\,, \tag{7.4}$$

where $F \in \Gamma_0(X)$. We may set $V = X$,

$$\Phi(x, p) = H(x) - F(x + p)$$

and observe that condition (7.2) is satisfied. Therefore, the strong duality result of Prop. 7.2 holds.

$$\begin{aligned} L(x, p^*) &= -\sup_{p \in X} [\langle p^*, p\rangle - F(x + p)] - H(x) = \\ &= -\sup_{v \in X} [\langle p^*, v\rangle - F(v)] + \langle p^*, x\rangle - H(x) = \\ &= -F^*(p^*) - H(x) + \langle p^*, x\rangle\,. \end{aligned}$$

Thus, $(\mathscr{D}I)$ attains the form

$$\begin{aligned} & F^*(p^*) - H^*(p^*) \to \inf \\ \text{subj. to} \quad & \\ & p^* \in X^*\,. \end{aligned} \tag{7.5}$$

The extremality relations (7.3) are in this case particularly simple: If $\hat{x}$ solves $(\mathscr{P}I)$ with f given by (7.4) and $\hat{p}^* \in \partial F(\hat{x})$, then $\hat{p}^*$ solves $(\mathscr{D}I)$. Moreover,

$$H(\hat{x}) + H^*(\hat{p}^*) = \langle \hat{p}^*, \hat{x}\rangle\,, \quad F(\hat{x}) + F^*(\hat{p}^*) = \langle \hat{p}^*, \hat{x}\rangle\,. \tag{7.6}$$

It implies in particular that if $\partial F(x) \neq \emptyset$ for all $x \in X$ and a solution $\hat{x}$ of the primal problem with α finite exists, then a solution of (7.5) exists. We have also some converse existence results:

Proposition 7.4. Let $\hat{p}^*$ be a solution of (7.5) and $\{x_n\} \subset X$ be such a sequence that

$$\lim_{n \to \infty} [\langle \hat{p}^*, x_n\rangle - H(x_n)] = H^*(\hat{p}^*)\,, \tag{7.7}$$

i.e. $\{x_n\}$ is a maximizing sequence for the functional $\langle \hat{p}^*, \cdot \rangle - H$. Then,

$$\lim_{n\to\infty} [\langle \hat{p}^*, x_n \rangle - F(x_n)] = F^*(\hat{p}^*), \tag{7.8}$$

and $\{x_n\}$ is a minimizing sequence in the primal problem.

Proof. Since $F \in \Gamma_0(X)$,

$$\alpha = F^*(\hat{p}^*) - \lim_{n\to\infty} [\langle \hat{p}^*, x_n \rangle - H(x_n)] .$$

By definition, $H(x_n) - F(x_n) \geqq \alpha$ for all n, and so

$$\varliminf_{n\to\infty} [\langle \hat{p}^*, x_n \rangle - F(x_n)] \geqq F^*(\hat{p}^*) .$$

Hence, Eq. (7.8) holds, and $H(x_n) - F(x_n) \to \alpha$. □

Corollary 7.4.1. Let H be weakly l.s.c., and $\hat{p}^*$ be a solution of (7.5). If there is a sequence $\{x_n\} \subset X$ satisfying (7.7) and $w(X, X^*)$-convergent to some point $\bar{x}$, then $\bar{x}$ is a solution of the primal problem.

Proof. Since $x_n \xrightarrow{w} \bar{x}$, $\lim_{n\to\infty} \langle \hat{p}^*, x_n \rangle \to \langle \hat{p}^*, \bar{x} \rangle$; thus the weak lower semicontinuity of H implies that

$$\lim_{n\to\infty} H(x_n) = -H^*(\hat{p}^*) + \langle \hat{p}^*, \bar{x} \rangle \leqq H(\bar{x}) \leqq \varliminf_{n\to\infty} H(x_n) .$$

Hence, $H(x_n) \to H(\bar{x})$. As also (see (7.8)) $F(x_n) \to F(\bar{x})$, $\bar{x}$ is a solution of the primal problem due the assertion of Prop. 7.4. □

Corollary 7.4.2. Suppose that X is a reflexive Banach space, H is weakly l.s.c. and strongly coercive, and $\hat{p}^*$ is a solution of (7.5). Then there exists a solution $\hat{x}$ of the primal problem.

Proof. Let $\{x_n\} \subset X$ be a maximizing sequence for the functional $\langle \hat{p}^*, \cdot \rangle - H$, i.e. a minimizing sequence for the primal problem by Prop. 7.4. By the assumptions $\{x_n\}$ possesses a weakly convergent subsequence $\{x_{n'}\}$ with a limit point $\hat{x}$. Thus, Cor. 7.4.1 may be applied. □

Props. 7.3, 7.4 enable to replace under some conditions the existence problem in $(\mathscr{P}1)$ with f given by (7.4) by the existence problem in (7.5) and vice versa. The extremality relations provide us with valuable necessary optimality conditions especially if any lack of differentiability occurs. In variational problems possessing this structure they imply a weak form of the appropriate Euler-Lagrange equation. Moreover, with the help of them it was possible to generalize the notion of a stationary point of a nonconvex nondifferentiable functional f having the prescribed structure. On the other hand, Eqs. (7.6) do not guarantee that $\hat{x}$ and $\hat{p}^*$ are indeed the desired minimizers unlike the extremal relations in convex optimization.

Toland's duality theory may well be applied also in optimal control problems or variational problems of the type ($\mathscr{P}3$), if $\Lambda \in \mathscr{L}[U, Y]$, Λ and Λ^* are bijective and $-J(\cdot, y) \in \Gamma(U)$ for all $y \in Y$, cf. [36]. The case of $J(u, y) = H \circ \Lambda u - F(u)$ has been successfully used to explain the connection between the differential equations describing the motion of a heavy rotating chain, cf. [37].

The summation exhibits remarkable "convexifying" properties; it has been utilized in a very fruitful relaxation technique described in [8] in detail. For a more general concept of the relaxation (from a different point of view) cf. [82]. For further applications see e.g. [78]. The rest of this section is devoted to a very brief description of some elements of this theory.

The method of relaxation consists in replacing a cost functional f (in the framework of problem ($\mathscr{P}$ 1)) by an approximate cost functional $\tilde{f}$ possessing the following properties:

(i) $\inf_{x \in A} f(x) = \min_{x \in A} \tilde{f}(x)$;

(ii) for each $x \in \operatorname{argmin}(\tilde{f} + \delta_A)$ there is a sequence $\{x_n\}$ such that $f(x_n) \to \inf_{x \in A} f(x)$ and $x_n \to x$ in a certain sense;

(iii) every cluster point of an arbitrary minimizing sequence in ($\mathscr{P}$ 1) belongs to $\operatorname{argmin}(\tilde{f} + \delta_A)$.

The method is particularly useful in such extremal problems, where a solution either fails to exist or its existence is hardly to be proved. It may be applied e.g. to the extremal problems of the type ($\mathscr{P}$ 3), where U and Y are Banach spaces (mostly reflexive ones) of functions mapping a measurable set $\Omega \subset \mathbb{R}^n$ (frequently a domain with a sufficiently smooth boundary) into $\mathbb{R}^m$, $\mathbb{R}^p$, respectively, $\Lambda[U \to Y]$ possesses mostly a differential or integral character,

$$J(u, y) = \int_\Omega \omega(x, u(x), y(x))\, \mathrm{d}x\,,$$

$\omega[\Omega \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}_{cl}]$ and the sections $\omega(x, \cdot, \cdot)$ are (generally) nonconvex. We assume futhermore that M, N are convex and closed and U, U^*, Y, Y^* are paired with respect to the appropriate canonical pairings. We denote here such problems also ($\mathscr{P}$ 3) for the sake of simplicity.

Denoting $\Theta(u) = \int_\Omega \omega(x, u(x), \Lambda\, u(x))\, \mathrm{d}x$, we will study simultaneously the bidual problem

$$(\mathscr{B}\ 3) \qquad \tilde{\Theta}(u) = \int_\Omega \omega^{**}(x, u(x), \Lambda\, u(x))\, \mathrm{d}x \to \inf$$

$$\text{subj. to} \quad x \in M \subset U\,,\ \Lambda u \in N \subset Y\,.$$

As in Sec. 1 conjugations of integrands are taken merely over those variables in which the integrands are not intrgrated. Under some conditions $\tilde{\Theta} = \Theta^{**}$ and some other conditions ensure that $\tilde{\Theta} = \bar{\Theta}^w$, the largest weakly l.s.c. minorant. Then the infima in ($\mathscr{P}$ 3) and ($\mathscr{B}$ 3) are equal. The existence of a solution of ($\mathscr{B}$ 3) can be ensured e.g. by a certain coercivity condition in connection with the reflexivity.

The "dualisation" phase of the method consists in solving ($\mathscr{B}$ 3) by way of a suitable dual problem which provides us simultaneously with a minimizing sequence for the original nonconvex problem. With respect to the limited extend of the sequel we confine ourselves to the problem

$$
\begin{aligned}
&J(u, y) = F(u) + G(y) = \int_\Omega f(x, u(x))\,\mathrm{d}x + \int_\Omega g(x, y(x))\,\mathrm{d}x \to \inf \\
(7.9)\qquad &\text{subj. to} \\
&y = \Lambda u\,.
\end{aligned}
$$

The appropriate assumptions will be imposed later, To prove the results concerning such type of problems, the following auxiliary theory is necessary:

Definition 7.1. Let B be a Borel subset of $\mathbb{R}^m$. A mapping $f[\Omega \times B \to \mathbb{R}_{cl}]$ is termed a *normal integrand* on $\Omega \times B$ if

(i) for almost all $x \in \Omega$ $f(x,\cdot)$ is l.s.c. on B;

(ii) there exists a Borel function $\bar{f}[\Omega \times B \to \mathbb{R}_{cl}]$ such that $\bar{f}(x, \cdot) = f(x, \cdot)$ for a.e. $x \in \Omega$.

Remark. A function $f[\mathbb{R}^p \to \mathbb{R}^q]$ is called *Borel* if for each open subset $\mathcal{O} \subset \mathbb{R}^q$ $f^{-1}(\mathcal{O})$ is a Borel set.

Clearly, for every measurable function $u[\Omega \to B]$ $f(\cdot, u(\cdot))$ is measurable on Ω provided f is a normal integrand on $\Omega \times B$. Def. 7.1. implies that for f and g arbitrary normal integrands on $\Omega \times B$ and $\lambda \geqq 0$ $\lambda(f + g)$, $\inf(f, g)$ are normal integrands there and for a sequence $\{f_n\}$ of normal integrands on $\Omega \times B$ $\sup_{n \in \mathbb{N}} f_n$ is a normal integrand there. The following Lusin type theorem holds:

Proposition 7.5. Let B be a Borel subset of $\mathbb{R}^m$. Then for a function $f[\Omega \times B \to \mathbb{R}_{cl}]$ the following assertions are equivalent to each other:

(i) f is a normal integrand on $\Omega \times B$;

(ii) for every compact set $K \subset \Omega$ and all $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset K$ such that meas $(K \setminus K_\varepsilon) \leqq \varepsilon$ and $f|_{K_\varepsilon \times B}$ is l.s.c.

Proof. Suppose $f(\Omega \times B) \subset [0, 1]$ (if necessary, an isomorphism of the arctg type can be used). The implication (ii) $\Rightarrow$ (i) is obvious. For the proof of the opposite implication we denote by $\mathfrak{Y}$ a denumerable basis of open subsets on B and set

$$\mathcal{B} := \{k\chi_Q \mid Q \in \mathfrak{Y}, k \in \mathbb{Q} \cap [0, 1]\}\,.$$

$\mathcal{B} \equiv \{\varphi_n\}_{n=1}^{+\infty}$ is denumerable and for a.e. $x \in \Omega$

$$f(x, \cdot) = \sup\{\varphi \in \mathcal{B} \mid \varphi \leqq f(x, \cdot) \text{ on } B\}\,.$$

The set

$$E_n := \{x \in \Omega \mid f(x, \cdot) \geqq \varphi_n\}$$

is measurable as the complement of the projection of $\{(x, u) \in \Omega \times B \mid f(x, u) < \varphi_n(u)\}$ on Ω which is implied e.g. by the Choquet capacity theorem. Thus, by virtue of Lusin's theorem we can find a compact set $K_\varepsilon \subset K$ with meas $(K \setminus K_\varepsilon) \leqq \varepsilon$

such that for each $n \in \mathbb{N}$ $\chi_{E_n}/_{K_\varepsilon}$ is continuous on K_ε. Then functions $\psi_n = \varphi_n \chi_{E_n}$ are l.s.c. on $K_\varepsilon \times B$ and f is the least upper bound of $\{\psi_n\}_{n=1}^{+\infty}$ there. □

Definition 7.2. Let B be a Borel subset of $\mathbb{R}^m$. A function $f[\Omega \times B \to \mathbb{R}_{cl}]$ is said to be a *Carathéodory function* if

(i) for a.e. $x \in \Omega$ $f(x, \cdot)$ is continuous on B;

(ii) for all $v \in B$ $f(\cdot, v)$ is measurable on Ω.

Proposition 7.6. Every Carathéodory function on $\Omega \times B$ is a normal integrand there.

Proof. Suppose $f(x, \cdot)$ is continuous for every $x \in \Omega$ and $f(\Omega \times B) \subset [0, 1]$. Taking $\mathfrak{Y}, \mathscr{B} = \{\varphi_n\}_{n=1}^{+\infty}$ as above and $\mathscr{A}$ a dense denumerable set in B, we set

$$E_{n,a} := \{x \in \Omega \mid f(x, a) \geqq \varphi_n(a)\} \quad \text{for} \quad a \in \mathscr{A} \quad \text{and} \quad n \in \mathbb{N}.$$

Since $f(\cdot, a)$ is measurable, $E_n = \bigcap_{a \in \mathscr{A}} E_{n,a}$ is measurable. From the continuity of the sections $f(x, \cdot)$, $E_n = \{x \in \Omega \mid f(x, \cdot) \geqq \varphi_n\}$. For each $n \in \mathbb{N}$ there is a Borel subset $C_n \subset \Omega$ such that $\chi_{C_n} = \chi_{E_n}$ a.e. in Ω. Setting $\bar{f}(x, v) = \sup\{\varphi_n(v)\, \chi_{C_n}(x) \mid n \in \mathbb{N}\}$ which is Borel, we easily see that $\bar{f}(x, \cdot) = f(x, \cdot)$ for a.e. $x \in \Omega$. □

Corollary 7.6.1. (Scorza, Dragoni). A function $f[\Omega \times B \to \mathbb{R}_{cl}]$ is Carathéodory iff for all compact sets $K \subset \Omega$ and all $\varepsilon > 0$ there are compact sets $K_\varepsilon \subset K$ having meas $(K \setminus K_\varepsilon) \leqq \varepsilon$ and such that $f/_{K_\varepsilon \times B}$ is continuous.

Proof. The sufficiency is clear. The necessity follows from Props. 7.5. and 7.6. and the fact that $-f$ is Carathéodory too. □

To prove that f^* is a normal integrand, we need the following

Lemma 7.1. Let B be a compact subset of $\mathbb{R}^m$ and d be a normal integrand on $\Omega \times B$ Let

$$d_0(x, v) = \begin{cases} 0 & \text{if} \quad d(x, v) = \min_{b \in B} d(x, b) \\ +\infty & \text{if} \quad d(x, v) > \min_{b \in B} d(x, b). \end{cases}$$

Then d_0 is a normal integrand on $\Omega \times B$. Moreover, there exists a measurable mapping $\bar{u}[\Omega \to B]$ such that $d_0(x, \bar{u}(x)) = 0$ a.e. in Ω.

Proof. Suppose $d(\Omega \times B) \subset [0, 1]$. For all $\varepsilon > 0$ and all compacts $K \subset \Omega$ we can find a compact set K_ε with meas $(K \setminus K_\varepsilon) \leqq \varepsilon$ and such that $d/_{K_\varepsilon \times B}$ is l.s.c. Using the compactness of $K_\varepsilon \times B$ it can easily be proved that $\varphi(x) = \min\{d(x, b) \mid b \in B\}$ is l.s.c. on K_ε. The set

$$C = \{(x, v) \in K_\varepsilon \times B \mid d(x, v) = \varphi(x)\}$$

is Borel. Hence $\delta_C = d_0/_{K_\varepsilon \times B}$ is Borel and $C_x = \{v \in B \mid (x, v) \in C\}$ is closed for a.e. $x \in \Omega$.

Since $d_0/_{K_\varepsilon \times B}$ is a normal integrand, we find a compact set $K_{2\varepsilon} \subset K_\varepsilon \subset K$ with meas$(K \setminus K_{2\varepsilon}) \leqq 2\varepsilon$ such that $d_0/_{K_{2\varepsilon} \times B}$ is l.s.c. due to Prop. 7.5. It shows that d_0 is in fact a normal integrand on $\Omega \times B$ (Prop. 7.5).

Let $\{b_n\}_{n=1}^{+\infty}$ be a dense sequence in B. We define by induction for $n \in \mathbb{N}$

$$e_n(x, v) = d_{n-1}(x, v) + |v - b_n|_m ,$$

$$d_n(x, v) = \begin{cases} 0 & \text{if } \; e_n(x, v) = \inf_{b \in B} e_n(x, b) , \\ +\infty & \text{elsewhere on } \; \Omega \times B. \end{cases}$$

Using the preceding assertion successively, we can see that every d_n, $n = 0,1 \ldots$, is a normal integrand on $\Omega \times B$. Hence, $\bar{d} = \sup\{d_n \mid n \in \mathbb{N}\}$ is also a normal integrand for which dom $\bar{d}(x, \cdot)$ consists of a single point for a.e. $x \in \Omega$. The function $\bar{u} : x \mapsto \operatorname{dom} \bar{d}(x, \cdot)$ possesses the required property. □

Proposition 7.7. If f is a normal integrand on $\Omega \times \mathbb{R}^m$, then $f^* : (x, v^*) \mapsto$ $\mapsto \sup\{\langle v^*, v\rangle - f(x, v) \mid v \in \mathbb{R}^m\}$ is a normal integrand on $\Omega \times \mathbb{R}^m$ as well.

Proof. As $f^*(x, \cdot)$ is a convex l.s.c. function, it remains to verify the requirement (ii) of Def. 7.1. Clearly,

$$f^* = \sup_k f_k^* \quad \text{for} \quad f_k(x, \xi) = f(x, \xi) + \delta_{B_n{}^k(\vartheta)}(\xi),\ k \in \mathbb{N} .$$

As

$$-f_k^*(x, \xi^*) = \inf_{|\xi|_m \leqq k} [f_k(x, \xi) - \langle \xi^*, \xi\rangle_m] \quad \forall x \in \Omega , \quad k \in \mathbb{N} ,$$

$f_k^*(x, \cdot)$ is either a finite convex function or identically equal to $+\infty$. By Lemma 7.1. there exists a measurable function $u_{k,\xi^*}[\Omega \to B_n^k(\vartheta)]$ such that

$$-f_k^*(x, \xi^*) = -\langle \xi^*, u_{k,\xi^*}(x)\rangle_m + f_k(x, u_{k,\xi^*}(x)) \quad \text{a.e. in } \Omega .$$

Hence f_k^* is a Carathéodory function. By Prop. 7.6. f^* is a normal integrand. □

Proposition 7.8. Let f be a normal integrand on $\Omega \times \mathbb{R}^n$ satisfying the condition

$$(7.10) \quad f(x, \cdot) \geqq h_0(x) \quad \text{for a.e.} \quad x \in \Omega \quad \text{and a suitable function} \quad h_0 \in L_1(\Omega) .$$

Let $\{u_k\}_{k=1}^{+\infty}$ be a sequence of measurable functions from Ω into $\mathbb{R}^m$ converging to some $\bar{u}[\Omega \to \mathbb{R}^m]$ almost everywhere. Then,

$$\int_\Omega f(x, \bar{u}(x))\, dx \leqq \varliminf_{k \to \infty} \int_\Omega f(x, u_k(x))\, dx .$$

Proof. Apply Fatou's lemma. □

In the rest of the section we shall denote by $\|\cdot\|_{L_v{}^c}$ the norm in $L_v[\Omega, \mathbb{R}^c]$ for $c \in \boldsymbol{N}$ and $v \in [1, +\infty]$. We shall omit c if $c = 1$. We shall suppose that $\Omega \subset \mathbb{R}^n$ is bounded and open.

Proposition 7.9. Let f be a normal integrand on $\Omega \times \mathbb{R}^m$ and F given as in (7.9) map $L_\nu[\Omega, \mathbb{R}^m]$ into $\mathbb{R}_{cl}$ for a certain $\nu \in [0, +\infty]$. Let there is a function $u_0 \in L_\infty[\Omega, \mathbb{R}^m]$ such that $F(u_0) < +\infty$. Then for all $u^* \in L_{\nu^*}[\Omega, \mathbb{R}^m]$ $F^*(u^*) = \int_\Omega f^*(x, u^*(x))\,dx$.

We remark that for $\nu \in (1, +\infty)$ $\nu^* = \nu/(\nu - 1)$, for $\nu = 1$ $\nu^* = +\infty$ and for $\nu = +\infty$ $\nu^* = 1$. F maps $L_\nu[\Omega, \mathbb{R}^m]$ into $\mathbb{R}_{cl}$ for every $\nu \in [1, \infty]$ e.g. provided the condition (7.10) is satisfied.

Proof of Prop. 7.9. We fix some $u^* \in L_{\nu^*}[\Omega, \mathbb{R}^m]$ and use the notation

$$\varphi(x) = f^*(x, u^*(x)),$$

$$\varphi_k(x) = f_k^*(x, u^*(x)),\ k = 1, 2, \dots\ (f_k^* \text{ introduced in the proof of Prop. 7.7}).$$

As $\{q_k\}$ tends monotonicaly to φ and the function $x \mapsto \langle u^*(x), u_0(x)\rangle_m - f(x, u_0(x))$ is an integrable minorant of φ_k for $k \geqq \|u_0\|_{L_\infty m}$, we have

$$\lim_{k\to+\infty} \int_\Omega \varphi_k(x)\,dx = \int_\Omega \varphi(x)\,dx.$$

By Lemma 7.1 we find $u_k \in L_\infty[\Omega, \mathbb{R}^m]$, $\|u_k\|_{L_\infty m} \leqq k$ such that

$$\varphi_k(x) = \langle u^*(x), u_k(x)\rangle_m - f(x, u_k(x)) \quad \text{a.e. in } \Omega.$$

Hence $\int_\Omega \varphi(x)\,dx \leqq F^*(u^*)$. The converse inequality is obvious. □

Corollary 7.9.1. Let f be a normal integrand on $\Omega \times \mathbb{R}^m$ minorized by a function $h_0 \in L_1(\Omega)$ as in (7.10). If there are $u_0, u_0^* \in L_\infty[\Omega, \mathbb{R}^m]$ such that $F(u_0) < +\infty$, $F^*(u_0^*) < +\infty$, then

$$F^{**}(u) = \int_\Omega f^{**}(x, u(x))\,dx \quad \text{for each} \quad u \in L_\nu[\Omega, \mathbb{R}^m],\ \nu \in [1, +\infty].$$

The assertion is a straightforward consequence of Prop. 7.9 and its proof.

To prove the equality $F^{**} = \bar{F}^w$, we carry out the following construction. We denote

$$\mathfrak{R}_i := \{\prod_{j=1}^{n} [k_j 2^{-i}, (k_j + 1) 2^{-i}) \mid k_1, k_2, \dots, k_n \in \mathbb{Z}\}$$

$$\mathscr{K}_i := \{K \in \mathfrak{R}_i \mid K \subset \Omega\}, \quad i = 1, 2, \dots.$$

Let $\beta_1, \beta_2, \dots, \beta_s$ be arbitrary positive numbers such that $\sum_{r=1}^{s} \beta_r = 1$. For $K = [k_1 2^{-i}, (k_1 + 1) 2^{-i}) \times \prod_{j=2}^{n} [k_j 2^{-i}, (k_j + 1) 2^{-i}) \in \mathscr{K}_i$ we set

$$K^{\beta_r} := [(k_1 + \sum_{l=1}^{r-1} \beta_l) 2^{-i}, (k_1 + \sum_{l=1}^{r} \beta_l) 2^{-i}) \times \prod_{j=2}^{n} [k_j 2^{-i}, (k_j + 1) 2^{-i}),\ r = 1, \dots, s.$$

Let for $i = 1, 2, \dots$, $\tilde{B}_i^{\beta_r} = \cup \{K^{\beta_r} \mid K \in \mathscr{K}_i\}$. For an arbitrary s-tuple of measurable functions $f = (f_1, \ldots, f_s)$ we define the measurable function

$$(7.11) \qquad (T_i f)(\mathrm{q}) = \begin{cases} f_r(x) & \text{for } x \in \tilde{B}_i^{\beta_r}, \quad r = 1, \ldots, s, \\ f_1(x) & \text{elsewhere on } \Omega. \end{cases}$$

Furthermore, we define

$$C_i = \mathrm{sp}\{\chi_K \mid K \in \mathscr{K}_{ij}\}, \quad C = \bigcup_{i=1}^{+\infty} C_i.$$

The following assertions hold for Ω being an open bounded subset of $\mathbb{R}^n$:

Lemma 7.2. For all $i \in \mathbb{N}$ the mapping T_i is linear and continuous from $\boldsymbol{L}_1[\Omega, \mathbb{R}^s]$ into $\boldsymbol{L}_1(\Omega)$. Moreover, for each $f = (f_1, \ldots, f_s) \in \boldsymbol{L}_1[\Omega, \mathbb{R}^s]$

$$(7.12) \qquad \lim_{i \to \infty} \int_\Omega T_i f \,\mathrm{d}x = \sum_{r=1}^{s} \beta_r \int_\Omega f_r \,\mathrm{d}x,$$

Proof. From (7.11) it follows immediately that T_i are linear and

$$\|T_i f\|_{L_1} \leqq \sum_{r=1}^{s} \|f_r\|_{L_1}.$$

T_i are thus equicontinuous and the linear functionals $f \mapsto \int_\Omega T_i f \,\mathrm{d}x$ are equicontinuous over $\boldsymbol{L}_1[\Omega, \mathbb{R}^s]$ as well. By Ascoli's theorem it is sufficient to prove the convergence (7.12) for all s-tuples of the functions from C. Considering the definition of C and the fact that $\lim_{i \to +\infty} \mathrm{meas}\,(\Omega \setminus \cup \mathscr{K}_i) = 0$, we arrive easily at the assertion. □

For an s-tuple $(f_1, \ldots, f_s) \in \boldsymbol{L}_\nu[\Omega, \mathbb{R}^{ms}]$ we define $T_i f \in \boldsymbol{L}_\nu[\Omega, \mathbb{R}^m]$ as $(T_i f^j)_{j=1}^m$, where $f^j = (f_1^j, \ldots, f_s^j)$.

Lemma 7.3. The mappings T_i are linear and continuous from $\boldsymbol{L}_\nu[\Omega, \mathbb{R}^{ms}]$ into $\boldsymbol{L}_\nu[\Omega, \mathbb{R}^m]$ for $\nu \in [1, +\infty]$ and $m \in \mathbb{N}$. For an arbitrary s-tuple $f = (f_1, \ldots, f_s) \in \boldsymbol{L}_\nu[\Omega, \mathbb{R}^{ms}]$ $T_i f$ tends weakly (for $\nu = +\infty$ weakly *) to $\sum_{r=1}^{s} \beta_r f_r \in \boldsymbol{L}_\nu[\Omega, \mathbb{R}^m]$.

Proof. Clearly, $\|T_i f\|_{L_\nu^m} \leqq \sum_{r=1}^{s} \|f_r\|_{L_\nu^m}$. To obtain the convergence result, let $h \in \boldsymbol{L}_{\nu^*}[\Omega, \mathbb{R}^m]$ be arbitrary and $(h \,.\, f) : x \mapsto (\langle h(x), f_1(x)\rangle_m, \ldots, \langle h(x), f_s(x)\rangle_m)$. Now $T_i(h \,.\, f) = \sum_{j=1}^{m} h^j (T_i f^j)$ and Eq. (7.12) implies the required result. □

Lemma 7.4. Let $u = (u_1, \ldots, u_s)$ be an s-tuple of functions from Ω into some arbitrary set S and let $f[\Omega \times S \to \mathbb{R}_{cl}]$ be such that all functions $x \mapsto f(x, u_r(x))$, $r = 1, 2, \ldots, s$, belong to $\boldsymbol{L}_1(\Omega)$. We define for $(\beta_1, \ldots, \beta_s) \in \mathbb{R}_+^s$, $\sum_{r=1}^{s} \beta_r = 1$:

$$T_i u(x) = \begin{cases} u_r(x) & \text{if } x \in \tilde{B}_i^{\beta_r}, \quad r = 1, \ldots, s, \\ u_1(x) & \text{elsewhere on } \Omega. \end{cases}$$

Then,

$$(7.13) \qquad \lim_{i \to \infty} \int_\Omega f(x, T_i u(x)) \,\mathrm{d}x = \sum_{r=1}^{s} \beta_r \int_\Omega f(x, u_r(x)) \,\mathrm{d}x.$$

Proof. Denote $f_r = f(\cdot, u_r(\cdot))$, $r = 1, \ldots, s$. Then $f = (f_1, \ldots, f_s) \in L_1[\Omega, \mathbb{R}^s]$ and we may apply the assertion of Lemma 7.2. □

Proposition 7.10. Let f be a normal integrand satisfying the condition (7.10) for a suitable $h_0 \in L_1(\Omega)$. Then $F^{**} = \bar{F}^w$ (the weak- and for $v = +\infty$ the weak * – l.s.c. regularization of F) on $L_v[\Omega, R^n]$, $v \in [1, +\infty]$.

Proof. Of course, $F^{**} \leqq \bar{F}^w$. Conversely, if $(v, a) \in \text{epi } F^{**} = \overline{\text{co}}\,(\text{epi } F)$, $a \in \mathbb{R}$, then for arbitary $\mathcal{O} \in \mathfrak{U}_{L_v^m, w}(\vartheta)$ (w denotes the $w(L_v[\Omega, \mathbb{R}^m], L_{v^*}[\Omega, \mathbb{R}^m])$ topology) and for arbitary $\varepsilon > 0$ there exists a suitable $p \in \mathbb{N}$, an s-tuple $(u_r, a_r)_{r=1}^s \in (\text{epi } F)^s$ and nonnegative scalars $\beta_1, \beta_2, \ldots, \beta_s$, $\sum_{r=1}^{s} \beta_r = 1$, such that $v - \sum_{r=1}^{s} \beta_r u_r \in \mathcal{O}$ and $|a - \sum_{r=1}^{s} \beta_r F(u_r)| \leqq \varepsilon$. As $F(u_r) < +\infty$, $r = 1, 2, \ldots, s$, we can use Lemmas 7.3, 7.4 to find a sufficiently large $i_0 \in \mathbb{N}$ such that for $u = (u_1, u_2, \ldots, u_s)$

$$T_{i_0} u - \sum_{r=1}^{s} \beta_r u_r \in \mathcal{O}$$

and

$$\left| \int_\Omega f(x, (T_{i_0} u)(x))\,\mathrm{d}x - \sum_{r=1}^{s} \beta_r \int_\Omega f(x, u_r(x))\,\mathrm{d}x \right| \leqq \varepsilon\,.$$

Hence, $T_{i_0} u - v \in 2\mathcal{O}$ and $(T_{i_0} u, a + 2\varepsilon) \in \text{epi } F$ so that $(v, a) \in \text{epi } \bar{F}^w$. □

Proposition 7.11. Let U be a Banach space paired with its dual U^*. We assume that $F[U \to \mathbb{R}_{cl}]$ satisfies $F^{**} = \bar{F}^w$ and $\mathcal{G} \in \Gamma_0(U)$ is such that $\mathcal{G}/_{\text{lev}_\lambda F}$ is $w(U, U^*)$ – continuous for every λ. Then $(\mathcal{G} + F)^{**} = \mathcal{G} + F^{**}$.

Proof. For $u^* \in U^*$ $(\mathcal{G} + F)^{**} = (\mathcal{G} - u^* + F)^{**} + u^*$. As $\overline{(\mathcal{G} + F)}^w \geqq (\mathcal{G} + F)^{**} \geqq \mathcal{G} + F^{**}$ and there is $u_0^* \in U^*$ minorizing $\mathcal{G}$ on U, it suffices to prove that $\overline{(\mathcal{G} + F)}^w = \mathcal{G} + \bar{F}^w$ for $\mathcal{G} \geqq 0$. For $u \in U$ and $\lambda > \varliminf_{v \xrightarrow{w} u} [\mathcal{G}(v) + F(v)] \geqq \varliminf_{v \xrightarrow{w} u} F(v)$, we clearly have

$$\varliminf_{v \xrightarrow{w} u} [\mathcal{G}(v) + F(v)] = \varliminf_{\substack{v \in \text{lev}_\lambda F \\ v \xrightarrow{w} u}} [\mathcal{G}(v) + F(v)] = \mathcal{G}(u) + \varliminf_{v \xrightarrow{w} u} F(v)\,.$$ □

Let us now return to the problem (7.9) with the following set of requirements:

(i) $\Omega \subset \mathbb{R}^n$ is an open bounded set,

(ii) f is a normal integrand on $\Omega \times \mathbb{R}^m$ satisfying the condition (7.10);

(iii) g is a Carathéodory function on $\Omega \times \mathbb{R}^\mu$ convex in y for a.e. $x \in \Omega$ and satisfying for some $c_0 > 0$, $\gamma \in [1, +\infty)$ and $h_1 \in L_1[\Omega]$ the growth condition

$$|g(x, y)| \leqq h_1(x) + c_0 |y|_\mu^\gamma\,; \tag{7.14}$$

(iv) the functional $\mathcal{G} = G \circ \Lambda$ is $w(L_v[\Omega, \mathbb{R}^m], L_{v^*}[\Omega, \mathbb{R}^m])$-continuous;

(v) $\Lambda \in \mathcal{L}[U, Y]$.

We set $U = L_\nu[\Omega, \mathbb{R}^m]$, $Y = L_\gamma[\Omega, \mathbb{R}^\mu]$ and introduce the perturbed essential objective

$$\Phi(u, p) = F(u) + G(\Lambda u - p)\,. \tag{7.15}$$

The dual problem attains the form

$$-F^*(\Lambda^* p^*) - G^*(-p^*) \to \sup \tag{7.16}$$

subj. to

$$p^* \in L_{\gamma^*}[\Omega, \mathbb{R}^\mu]\,,$$

and the bidual (relaxed) problem

$$F^{**}(u) + G(y) \to \inf \tag{7.17}$$

subj. to

$$y = \Lambda u\,.$$

Control constraints of a certain type may be included to F and hence treated within the formulation (7.9).

We impose now still one additional assumption, namely that
(vi) there exist functions $u_0, u_0^* \in L_\infty[\Omega, \mathbb{R}^m]$ such that

$$F(u_0) < +\infty, F^*(u_0^*) < +\infty\,.$$

Then we can express F^*, F^{**} by means of f^*, f^{**} respectively. Props. 7.10, 7.11 imply that the infima in (7.9) and (7.17) are equal and for each solution $\hat{u}$ of (7.17) there is a sequence $\{u_n\}$ such that $u_n \xrightarrow{w} \hat{u}$ and $F(u_n) \to F^{**}(\hat{u})$ (for $\nu = +\infty$ in the w^* topology). If $\inf(7.9) \in \mathbb{R}$, we may solve (7.17) by way of (7.16) (the dualisation phase of the method) because the bidual problem is stable ($\operatorname{dom} G = Y$) and hence problem (7.16) possesses a solution $\hat{p}^*$. The extremality relations (Prop. 2.4) provide us with the following trivial statement:

Proposition 7.12. Let $\hat{p}^*$ be a solution of (7.16). Suppose that

$$u_a \in \partial_{\varepsilon/2} F^*(\Lambda^* \hat{p}^*)\,,$$
$$\Lambda u_a \in \partial_{\varepsilon/2}\, G^*(-\hat{p}^*)\,.$$

Then

$$F(u_a) + G(\Lambda u_a) \leqq -F^*(\Lambda^* \hat{p}^*) - G(-\hat{p}^*) + \varepsilon\,. \tag{7.18}$$

Conversely, if Ineq. (7.18) holds, then

$$u_a \in \partial_\varepsilon F^*(\Lambda^* \hat{p}^*)$$
$$\Lambda u_a \in \partial_\varepsilon G^*(-\hat{p}^*)\,.$$

Remark. There are various possibilities to apply the above relaxation dualisation method to more complicated problems than (7.9). Moreover, the above theory completes the results concerning normal convex integrands in Sec. 1 and the relaxation represents an important tool even if we have to solve bidual problems directly (e.g. with the help of the Carathéodory theorem, cf. [8]). In such a case, we do not need to assume strictly the linearity of Λ in problem (7.9).

8. Nonconvex perturbational duality theory of Lindberg

This section is in fact a natural extension of the results of Sec. 2 to the case, where the extremal-value function h need not be convex. This extension was achieved by a generalization of the Γ-regularization and hence also the conjugation introduced in Sec. 1.

Let X and Y be arbitrary spaces (sets) and $\varphi[X \times Y \to \mathbb{R}]$ be a given function. For functions defined on X we will use minorants of the form $\mu - \varphi(\cdot, y)$, called *φ-minorants*.

Each φ-minorant is given by a couple $(y, \mu) \in Y \times \mathbb{R}$. To a function $f[X \to \mathbb{R}_{cl}]$ we associate its *minorant set* $m(f)$,

$$m(f) = \{(y, \mu) \in Y \times \mathbb{R} \mid \mu - \varphi(x, y) \leqq f(x) \ \forall x \in X\}.$$

Clearly,

(i) $$(y, \bar{\mu}) \in m(f) \Rightarrow (y, \mu) \in m(f) \quad \text{for} \quad \mu \leqq \bar{\mu};$$

(ii) $$(y, \mu_r) \in m(f) \quad \text{and} \quad \mu_r \to \mu \Rightarrow (y, \mu) \in m(f)$$

so that $m(f)$ is the hypograph of some function defined on Y. This function will be termed the *lower φ-conjugate* of f and denoted $f^\frown$.

(8.1) $$f^\frown(y) = \sup\{\mu \mid (y, \mu) \in m(f)\} = \inf_{x \in X}[f(x) + \varphi(x, y)].$$

Similarly, we may also try to recover a function f from its majorants of the form $\mu + \varphi(\cdot, y)$, called *φ-majorants*. As above, we define the *majorant set* $M(f)$,

$$M(f) = \{(y, \mu) \in Y \times \mathbb{R} \mid \mu + \varphi(x, y) \geqq f(x) \ \forall x \in X\}$$

and find that it is the epigraph of a function $f^\smile[Y \to \mathbb{R}_{cl}]$ termed *the upper φ-conjugate of f*.

(8.2) $$f^\smile(y) = \sup_{x \in X}[f(x) - \varphi(x, y)].$$

Evidently,

(8.3) $$(-f)^\frown = -(f^\smile).$$

Proposition 8.1. If $\varphi(x, \cdot)$ is concave for each fixed x, then for any function $f[X \to \mathbb{R}_{cl}]$, $f^\frown$ and $f^\smile$ are concave and convex, respectively.

Proof. Since $\varphi(x, \cdot)$ is concave, (8.1) implies that $f^\frown$ is the pointwise infimum of a family of concave functions and hence also concave. For $f^\smile$ the result follows from (8.2). □

Remark. The conjugation (1.1) would rather correspond to minorants of the form $\varphi(\cdot, y) - \mu$. However, Lindberg provides us with sufficient reasons for this choice of signs in [14].

Definition 8.1. A function f is said to be *lower (upper) φ-regular* at x if $f(x)$ is the supremum (infimum) of the values at x of the φ-minorants (φ-majorants) of f. If f is lower (upper) φ-regular at each point, then it is called *lower (upper) φ-regular.*

The following assertion is the generalization of the Fenchel-Moreau theorem (Prop. 1.4).

Proposition 8.2. $(f^{\frown})^{\smile}(x) \leqq f(x)$ with equality iff f is lower φ-regular at x.

Proof.

$$(f^{\frown})^{\smile}(x) = \sup_{y \in Y} [f^{\frown}(y) - \varphi(x, y)] = \sup \{\mu - \varphi(x, y) \mid (y, \mu) \in \boldsymbol{Y} \times \mathbb{R}, \mu \leqq f^{\frown}(y)\} =$$
$$= \sup \{\mu - \varphi(x, y) \mid (y, \mu) \in m(f)\} \leqq f(x) .$$

The inequality is by Definition 8.1 an equality iff f is lower φ-regular at x. □

Analogously, we have

Proposition 8.3. $(f^{\smile})^{\frown}(x) \geqq f(x)$ with equality iff f is upper φ-regular at x.

One may anticipate that the lower (upper) φ-regularity plays a crucial role in the developed duality theory and we have to provide meaningful criteria for its testing. Supposing $\boldsymbol{X}$ equipped with a topology, we introduce

Definition 8.2. φ is said to be *sharp* at $\bar{x} \in \boldsymbol{X}$ if for all $\bar{y} \in \boldsymbol{Y}$, all $\mathcal{O} \in \mathfrak{U}_{\boldsymbol{X}}(\bar{x})$, all $\mu \in \mathbb{R}$ and all $\varepsilon > 0$ there is a $\hat{y} \in \boldsymbol{Y}$ and a neighborhood $\hat{\mathcal{O}} \in \mathfrak{U}_{\boldsymbol{X}}(\bar{x})$, $\hat{\mathcal{O}} \subset \mathcal{O}$ such that

$$\varphi(x, \hat{y}) - \varphi(\bar{x}, \hat{y}) \geqq \varphi(x. \bar{y}) + \mu \quad \forall x \in \boldsymbol{X} \setminus \hat{\mathcal{O}} \tag{8.4}$$

and

$$\varphi(x, \hat{y}) \geqq \varphi(\bar{x}, \hat{y}) - \varepsilon \quad \forall x \in \hat{\mathcal{O}} . \tag{8.5}$$

If φ is sharp at all $x \in \boldsymbol{X}$ we shall say that φ is *sharp*.

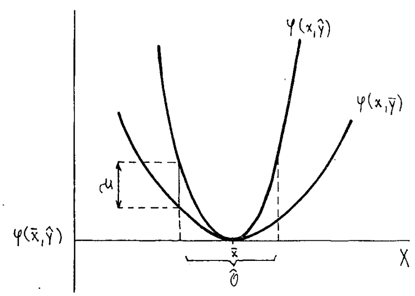

Fig 1. $\varphi(x, y) = y|x - \bar{x}|^2$ is sharp at $\bar{x}$.

Proposition 8.4. Suppose that φ is sharp at $\bar{x} \in \boldsymbol{X}$. Then every function $f[\boldsymbol{X} \to \mathbb{R}_{cl}]$ which is l.s.c. (u.s.c.) at $\bar{x}$ and has some φ-minorant (φ-majorant) is lower (upper) φ-regular at $\bar{x}$.

Proof. We will discuss only the case of f l.s.c. at $\bar{x}$. Suppose that $f(\bar{x})$ is finite. Then for any $\varepsilon > 0$ there exists a neighbourhood $\mathcal{O} \in \mathfrak{U}_{\boldsymbol{X}}(\bar{x})$ such that

$$f(x) \geqq f(\bar{x}) - \varepsilon \quad \forall x \in \mathcal{O}\,.$$

By the assumption

$$f(x) \geqq \bar{\mu} - \varphi(x, \bar{y}) \quad \forall x \in \boldsymbol{X} \text{ and some fixed couple } (\bar{y}, \bar{\mu}).$$

The sharpness of φ at $\bar{x}$ implies the existence of a $\hat{y}$ and $\hat{\mathcal{O}} \in \mathfrak{U}_{\boldsymbol{X}}(\bar{x})$, $\hat{\mathcal{O}} \subset \mathcal{O}$ such that

$$\varphi(x, \hat{y}) - \varphi(\bar{x}, \hat{y}) \geqq \varphi(x, \bar{y}) - \bar{\mu} + f(\bar{x}) - 2\varepsilon \quad \forall x \in \boldsymbol{X} \setminus \hat{\mathcal{O}}$$

and

$$\varphi(x, \hat{y}) \geqq \varphi(\bar{x}, \hat{y}) - \varepsilon \quad \forall x \in \hat{\mathcal{O}}\,.$$

Hence, the function

$$\psi_\varepsilon(x) = f(\bar{x}) - 2\varepsilon + \varphi(\bar{x}, \hat{y}) - \varphi(x, \hat{y})$$

satisfies

$$\psi_\varepsilon(x) \leqq f(\bar{x}) - \varepsilon \leqq f(x) \quad \forall x \in \hat{\mathcal{O}}$$

and

$$\psi_\varepsilon(x) \leqq \bar{\mu} - \varphi(x, \bar{y}) \leqq f(x) \quad \forall x \in \boldsymbol{X} \setminus \hat{\mathcal{O}}\,.$$

Thus, functions ψ_ε are φ-minorants and $\lim_{\varepsilon \to 0_+} \psi_\varepsilon(\bar{x}) = f(\bar{x})$. The proof for $f(\bar{x}) = +\infty$ may be performed along the same lines. □

We postpone the investigation of the sharpeness of concrete function φ to concrete dualisations and introduce a natural generalization of the subgradient (Def. 1.7).

Definition 8.3. $\bar{y}$ is a *φ-subgradient* (*φ-supergradient*) of f at $\bar{x}$ if

$$f(x) + \varphi(x, \bar{y}) \geqq f(\bar{x}) + \varphi(\bar{x}, \bar{y}), \; (f(x) - \varphi(x, \bar{y}) \leqq f(\bar{x}) - \varphi(\bar{x}, \bar{y})) \quad \text{for all} \quad x \in \boldsymbol{X}\,.$$

The (possibly empty) set of all φ-subgradients of f at $\bar{x}$ is termed *φ-subdifferential* of f at $\bar{x}$ and denoted $\partial^\frown f(\bar{x})$. Similarly, $\partial^\smile f(\bar{x})$ denotes the *φ-superdifferential* of f at $\bar{x}$, the set of all φ-supergradients of f at $\bar{x}$.

Note that if φ is linear in x, then the φ-subgradients correspond to subgradients in the usual sense. We have also an analogue of Prop. 1.12.

Proposition 8.5. $f^\frown(\bar{y}) \leqq f(\bar{x}) + \varphi(\bar{x}, \bar{y})$ with equality iff $\bar{y} \in \partial^\frown f(\bar{x})$.

Proof. Clear from Eq. (8.1) and Def. 8.3. □

Analogously, we have

Proposition 8.6. $f^\smile(\bar{y}) \geqq f(\bar{y}) - \varphi(\bar{x}, \bar{y})$ with equality iff $\bar{y} \in \partial^\smile f(\bar{x})$.

The apparatus of lower (upper) φ-conjugate functions and φ-subgradients (φ-supergradients) enables us to derive a nonconvex perturbational theory of duality along the same lines as it was done in Sec. 2. We take the general minimization problem ($\mathscr{P}$ 1) and assume that besides the l.c.s. $\boldsymbol{X}$ we are given a l.c.s. $\boldsymbol{V}$ and two arbitrary sets $\boldsymbol{S}$, $\boldsymbol{W}$. Let $\boldsymbol{S}$ contain an element denoted by ϑ (usually the origin for a subset of a l.c.s. containing it). Moreover, let $\varphi[\boldsymbol{X} \times \boldsymbol{S} \to \mathbb{R}]$ and $\psi[\boldsymbol{V} \times \boldsymbol{W} \to \mathbb{R}]$ be such functions that

$$\varphi(\cdot, \vartheta) = 0 \quad \text{and} \quad \psi(\vartheta, \cdot) = 0 \,. \tag{8.6}$$

As in Sec. 2 we consider a suitable essential objective given by (2.1) and examine the family of perturbed primal problems ($\mathscr{P}$ 1_p) by means of the extremal-value function h (2.2). One obtains

$$\alpha = h(\vartheta) \geqq (h_\psi^\frown)_\psi^\smile(\vartheta) = \sup_{w \in \boldsymbol{W}} [h_\psi^\frown(w) - \psi(\vartheta, w)] = \sup_{w \in \boldsymbol{W}} h_\psi^\frown(w) = \beta$$

and we may try to solve the maximization problem $\sup_{w \in \boldsymbol{W}} h_\psi^\frown(w)$ instead of ($\mathscr{P}$ 1) provided $\alpha = \beta$. The subscripts φ, ψ will tell us with respect to which function the conjugates are taken. By definition

$$h_\psi^\frown(w) = \inf_{p \in \boldsymbol{V}} [h(p) + \psi(p, w)] = \inf_{\substack{x \in \boldsymbol{X} \\ p \in \boldsymbol{V}}} [\Phi(x, p) + \varphi(x, \vartheta) + \psi(p, w)] = $$
$$= \Phi^\frown(\vartheta, w) \,,$$

where $\Phi^\frown(s, w) = \inf_{\substack{x \in \boldsymbol{X} \\ p \in \boldsymbol{V}}} [\Phi(x, p) + \varphi(x, s) + \psi(p, w)]$.

Thus, the dual problem ($\mathscr{D}$ 1) with respect to Φ and ψ attains the form

($\mathscr{D}$ 1) $\qquad \Phi^\frown(\vartheta, w) \to \sup$

subj. to

$w \in \boldsymbol{W}$.

The technique used to form the dual of ($\mathscr{P}$ 1) can be easily applied also to ($\mathscr{D}$ 1): We introduce the perturbed dual problem

($\mathscr{D}$ 1_s) $\qquad \Phi^\frown(s, w) \to \sup$

subj. to

$w \in \boldsymbol{W}$,

the dual extremal-value function $g[\boldsymbol{S} \to \mathbb{R}_{cl}]$

$$g(s) = \sup_{w \in \boldsymbol{W}} \Phi^\frown(s, w) \tag{8.7}$$

and note that $\beta = g(\vartheta)$. In such a way

$$\beta = g(\vartheta) \leqq (g_\varphi^\smile)_\varphi^\frown(\vartheta) = \inf_{x \in \boldsymbol{X}} [g_\varphi^\smile(x) + \varphi(x, \vartheta)] = \inf_{x \in \boldsymbol{X}} g_\varphi^\smile(x) = \tag{8.8}$$
$$= \inf_{x \in \boldsymbol{X}} \{ \sup_{s \in \boldsymbol{S}} \sup_{w \in \boldsymbol{W}} [\Phi^\frown(s, w) - \varphi(x, s) - \psi(\vartheta, w)]\} = \inf_{x \in \boldsymbol{X}} (\Phi^\frown)^\smile(x, \vartheta) \,.$$

We have obtained the dual problem of $(\mathscr{D}\ 1)$ (bidual problem of $(\mathscr{P}\ 1)$) with respect to $\Phi^{\frown}$ and φ. If Φ is lower (φ, ψ)-regular, it is the primal problem $(\mathscr{P}\ 1)$ due to Prop. 8.2. In this case there is a complete symmetry between $(\mathscr{P}\ 1)$ and $(\mathscr{D}\ 1)$ as in Sec. 2 for $\Phi \in \Gamma_0(X \times V)$.

A slightly more complex situation present in this nonconvex setting forces us to modify Def. 2.1 as follows:

Definition 8.4. 1) The problem $(\mathscr{P}\ 1)$ is ψ-*normal* (with respect to Φ) if the duality gap $\alpha - \beta$ equals zero.

2) The problem $(\mathscr{P}\ 1)$ is ψ-*stable* (with respect to Φ) if there exists a ψ-subgradient of h at ϑ (h is ψ-subdifferentiable at ϑ).

The assertions below are generalizations of Props. 2.1, 2.2.

Proposition 8.7. Let $(\mathscr{P}\ 1)$ be inf-stable (Def. 2.1) and h possess a ψ-minorant. Let ψ be sharp at $p = \vartheta$. Then, $(\mathscr{P}\ 1)$ is ψ-normal (with respect to Φ).

Proof. It is evident that

$$\alpha = \beta \Leftrightarrow h(\vartheta) = (h_{\psi}^{\frown})_{\psi}^{\smile}(\vartheta)\,.$$

Therefore, we need the lower ψ-regularity of h at ϑ (Prop. 8.2) which is implied by the assumptions due to Prop. 8.4. □

Remark. Observe that the nonconvexity of h causes a substantial strengthening of assumptions guaranteeing that $\alpha = \beta$.

Proposition 8.8. If Φ is lower (φ, ψ)-regular at (x, ϑ) for all $x \in X$ and $g(\vartheta) = (g_{\psi}^{\smile})_{\psi}^{\frown}(\vartheta)$, then $(\mathscr{P}\ 1)$ is ψ-normal (with respect to Φ).

The proof is contained in (8.8).

Proposition 8.9. The following two conditions are equivalent to each other:

(i) $(\mathscr{P}\ 1)$ is ψ-stable with $\hat{w} \in \partial_{\psi}^{\frown} h(\vartheta)$;

(ii) $\hat{w}$ solves $(\mathscr{D}\ 1)$ and $\alpha = \beta$.

Proof. Follows from Prop. 8.5 and the following sequence of equalities and inequalities:

$$h^{\frown}(\hat{w}) = \Phi^{\frown}(\vartheta, \hat{w}) \leqq \sup_{w \in W} \Phi^{\frown}(\vartheta, w) = g(\vartheta) \leqq h(\vartheta) = h(\vartheta) + \psi(\vartheta, \hat{w})\,. \qquad \square$$

Unfortunately, there are no ψ-stability criteria having the generality of Prop. 2.5. We return to this question in concrete dualisations.

As in the convex case usually it is not possible to evaluate $\Phi^{\frown}(\vartheta, w)$ explicitly. Therefore, analogously to (2.7) we introduce the *Lagrangian* function $L[X \times W \to \to \mathbb{R}_{cl}]$ by halfway conjugation:

$$L(x, w) = \Phi_{\psi}^{\frown}(x, v) = \inf_{p \in V} [\Phi(x, p) + \psi(p, w)] \tag{8.9}$$

so that $(\mathscr{D}\ 1)$ attains the form

$$\sup_{w \in W} \inf_{x \in X} L(x, w) .$$

By a simple computation,

$$\Phi(x, p) \geqq (\Phi_\psi^\frown)_\psi^\smile (x, p) = L_\psi^\smile(x, p) = \sup_{w \in W} [L(x, w) - \psi(p, w)] .$$

Hence, $\Phi(x, \vartheta) \geqq \sup_{w \in W} L(x, w)$ with equality iff $\Phi(x, \cdot)$ is lower ψ-regular at ϑ. If this lower ψ-regularity holds for each $x \in X$, $(\mathscr{P}\ 1)$ can alternatively be written in the form

$$\inf_{x \in X} \sup_{w \in W} L(x, w) ,$$

and we have the usual relation between the Lagrangian and the primal and dual problems.

If $\psi(p, \cdot)$ is concave for each fixed p, then $L(x, \cdot)$ is also concave due to Prop. 8.1.

The relations between saddle-points and saddle-values of L and solutions of $(\mathscr{P}\ 1)$ and $(\mathscr{D}\ 1)$ are examined in the following assertions:

Proposition 8.10. Let $(\hat{x}, \hat{w})$ be a saddle point of L and $\Phi(\hat{x}, \cdot)$ be lower ψ-regular at ϑ. Then $\hat{w}$ is a solution of $(\mathscr{D}\ 1)$, $\hat{x}$ is a solution of $(\mathscr{P}\ 1)$ and $\alpha = \beta$.

Proof. Generally,

$$(8.10) \quad \Phi(\hat{x}, \vartheta) \geqq (\Phi_\psi^\frown)_\psi^\smile (\hat{x}, \vartheta) = \sup_{w \in W} L(\hat{x}, w) \geqq L(\hat{x}, \hat{w}) \geqq \inf_{x \in X} L(x, \hat{w}) = \Phi^\frown(\vartheta, \hat{w}) .$$

The assumptions ensure that each of the inequalities in (8.10) is in fact an equality. □

Proposition 8.11. The following two conditions are equivalent to each other:
(i) $\alpha = \beta$ and $\hat{w}$ solves $(\mathscr{D}\ 1)$;
(ii) $\inf_{x \in X} \Phi(x, \vartheta) = \inf_{x \in X} L(x, \hat{w})$.

The proof is evident. This assertion is essential if we solve $(\mathscr{P}\ 1)$ by way of $(\mathscr{D}\ 1)$. Of course, every minimizer of $L(\cdot, \hat{w})$ need not be a solution of $(\mathscr{P}\ 1)$ as in the Lagrange duality (Sec. 4).

Props. 8.10, 8.11 concern global minima. However, if $\bar{x}$ is locally optimal in some neighborhood $\mathcal{O} \in \mathfrak{U}_X(\bar{x})$ then the above technique could still be used by adding the constraint $x \in \mathcal{O}$ to $(\mathscr{P}\ 1)$.

The important question of differentiability of the cost in $(\mathscr{D}\ 1)$ will also be postponed to further sections.

9. Shifted penalties in nonconvex problems

There are various differences between the convex and nonconvex perturbational theory of duality. From the user's point of view it is important that in the nonconvex case one has to choose not only the perturbations (Φ), but also the generalized pairing (ψ).

Let V be a normed linear space, V^* its topological dual, $W = \mathbb{R}_+ \times V^*$ (with a typical point (r, v^*) and $b[V \to \mathbb{R}_{cl}]$ be convex. We will examine the generalized pairings

$$\psi(p, w) = \langle v^*, p\rangle + rb(p), \tag{9.1}$$

and

$$\bar{\psi}(p, w) = \langle v^*, p\rangle + r\bar{b}(\|p\|). \tag{9.2}$$

In this investigation the concept of local uniform convexity will be utilized:

Definition 9.1. Let X be a normed space. A function $f[X \to \mathbb{R}_{cl}]$ is said to be *uniformly convex at* $\bar{x}$ if for all $\varepsilon > 0$

$$\delta_{\bar{x}}(\varepsilon) = \inf\{f(x) + f(\bar{x}) - 2f(\tfrac{1}{2}(x + \bar{x})) \mid \|x - \bar{x}\| \geqq \varepsilon\} > 0. \tag{9.3}$$

f is called *locally uniformly convex* if it is so at each point.

Lemma 9.1. Let $f[X \to \mathbb{R}_{cl}]$ be uniformly convex at $\bar{x}$ and let $\bar{y} \in \partial f(\bar{x})$. Then for all $\varepsilon > 0$ there is $\delta_{\bar{x}}(\varepsilon) > 0$ such that

$$f(x) \geqq f(\bar{x}) + \langle \bar{y}, x - \bar{x}\rangle + \delta_{\bar{x}}(\varepsilon) \quad \text{for} \quad \|x - \bar{x}\| > \varepsilon. \tag{9.4}$$

Proof. Ineq. (9.3) implies that

$$f(x) + f(\bar{x}) - 2f(\tfrac{1}{2}(x + \bar{x})) \geqq \delta_{\bar{x}}(\varepsilon) \quad \text{if} \quad \|x - \bar{x}\| \geqq \varepsilon. \tag{9.5}$$

Since $\bar{y} \in \partial f(\bar{x})$

$$f(\tfrac{1}{2}(x + \bar{x})) \geqq f(\bar{x}) + \tfrac{1}{2}\langle \bar{y}, x - \bar{x}\rangle. \tag{9.6}$$

Combining (9.5) and (9.6), we get (9.4). □

Proposition 9.1. Suppose b is convex, finite, continuous and uniformly convex at some point $\bar{p} \in V$. Then the generalized pairing ψ given by (9.1) is sharp at $\bar{p}$.

Proof. Let $\bar{p}, (\bar{r}, \bar{v}^*)$ correspond to $\bar{x}$, $\bar{y}$ in Def. 8.2, respectively, $\mathcal{O}$, μ and ε be like in Def. 8.2. Let

$$\varkappa : p \mapsto \langle \bar{v}^*, p\rangle + \bar{r}b(p).$$

By the continuity of b there is an $\varepsilon' > 0$ such that $\hat{\mathcal{O}} = B_V^{\varepsilon'}(\bar{p}) \subset \mathcal{O}$ and $\varkappa(p) \geqq \varkappa(\bar{p}) - \varepsilon$ on $\hat{\mathcal{O}}$. Prop. 1.13 implies the existence of a subgradient $\bar{y}^* \in \partial b(\bar{p})$ and for any $\varepsilon' > 0$ there is $\delta_{\bar{p}}(\varepsilon') > 0$ such that

$$b(p) \geqq b(\bar{p}) + \langle \bar{y}^*, p - \bar{p}\rangle + \delta_{\bar{p}}(\varepsilon') \quad \text{if} \quad \|p - \bar{p}\| \geqq \varepsilon'.$$

due to Lemma 9.1. Thus if we set

$$\tilde{\varkappa} : p \mapsto \langle -\bar{y}^*, p\rangle + b(p),$$

we have

$$\tilde{\varkappa}(p) \geqq \tilde{\varkappa}(\bar{p}) \quad \text{if} \quad p \in \hat{\mathcal{O}} \text{ (because } \bar{y}^* \in \partial b(\bar{p})), \tag{9.7}$$

and

$$\tilde{\varkappa}(p) \geqq \tilde{\varkappa}(\bar{p}) + \delta_{\bar{p}}(\varepsilon') \quad \text{if} \quad p \notin \hat{\mathcal{O}}. \tag{9.8}$$

We introduce now

$$\hat{\varkappa} = \varkappa + \frac{\beta_0}{\delta_{\bar{p}}(\varepsilon')} \tilde{\varkappa}, \quad \text{where} \quad \beta_0 = (\mu + \varkappa(\bar{p}))^+ . \tag{9.9}$$

Then by (9.7) and a suitable choice of ε'

$$\hat{\varkappa}(p) \geqq \varkappa(\bar{p}) - \varepsilon + \frac{\beta_0}{\delta_{\bar{p}}(\varepsilon')} \tilde{\varkappa}(\bar{p}) = \hat{\varkappa}(\bar{p}) - \varepsilon \quad \text{for} \quad p \in \hat{\mathcal{O}} . \tag{9.10}$$

For $p \in V \setminus \hat{\mathcal{O}}$ we have by (9.8)

$$\begin{aligned} \hat{\varkappa}(p) &\geqq \varkappa(p) + \frac{\beta_0}{\delta_{\bar{p}}(\varepsilon')} (\tilde{\varkappa}(\bar{p}) + \delta_{\bar{p}}(\varepsilon')) \geqq \varkappa(p) + \mu + \varkappa(\bar{p}) + \\ &+ \frac{\beta_0}{\delta_{\bar{p}}(\varepsilon')} \tilde{\varkappa}(\bar{p}) = \varkappa(p) + \mu + \hat{\varkappa}(\bar{p}) . \end{aligned} \tag{9.11}$$

Thus, $(\hat{r}, \hat{v}^*)$ corresponding to $\hat{y}$ in Def. 8.2 may be composed according to (9.9) and Ineq. (9.10), (9.11) show that $\psi(p, w)$ is sharp at $\bar{p}$. □

Henceforth we will assume that V is a *locally uniformly convex* Banach space i.e. at each point $\bar{x} \in V$, $\|\bar{x}\| = 1$ and for all $\varepsilon > 0$ there is $\tilde{\delta}_{\bar{x}}(\varepsilon) > 0$ such that

$$\inf \{\tfrac{1}{2}\|x + \bar{x}\| \mid \|x\| \leqq 1 , \ \|x - \bar{x}\| \geqq \varepsilon\} \leqq 1 - \tilde{\delta}_{\bar{x}}(\varepsilon) .$$

This property will be utilized in the following assertion concerning the generalized pairing (9.2):

Proposition 9.2. Suppose that $\bar{b}$ is convex and nondecreasing on $\mathbb{R}_+$ and locally uniformly convex at $\lambda \in \mathbb{R}_+$. Then $b = \bar{b}(\|\cdot\|)$ is locally uniformly convex at $\lambda\bar{p}$ for any unit vector $\bar{p}$.

Proof. For an arbitrary $\eta > 0$ we take $\delta_\lambda(\eta)$ ensured by Lemma 9.1 for $\bar{b}$ and $\tilde{\delta}_{\bar{p}}(\eta)$ from the local uniform convexity of V. We fix an arbitrary $\varepsilon > 0$ and set

$$h(p) \equiv b(\|p\|) + \bar{b}(\|\lambda\bar{p}\|) - 2\bar{b}(\tfrac{1}{2}\|p + \lambda\bar{p}\|) ,$$
$$C_{\lambda\varepsilon} = V \setminus B_V^{\lambda\varepsilon}(\lambda\bar{p}) .$$

For some $\gamma \in (0, \min(\varepsilon, 1))$ and $p \in C_{\lambda\varepsilon}$ with $\|p\| \in [(1 - \gamma)\lambda, \lambda]$ we have $\frac{1}{2}\|p + \lambda\bar{p}\| \leqq \lambda(1 - \tilde{\delta}_{\bar{p}}(\varepsilon))$. If $p \in C_{\lambda\varepsilon}$ and $\|p\| = q\lambda$ for some $q \in [1, 1 + \gamma]$, then $\|p/q - \lambda\bar{p}\| \geqq (\varepsilon - \gamma)\lambda$ and $\frac{1}{2}\|p + \lambda\bar{p}\| \leqq \frac{1}{2}\|p/q + \lambda\bar{p}\| + \lambda\gamma/2 \leqq \lambda(1 - \tilde{\delta}_{\bar{p}}(\varepsilon - \gamma) + \gamma/2)$. Hence for every $p \in C_{\lambda\varepsilon}$ such that $|\|p\| - \lambda| \leqq \gamma\lambda$ $h(p) \geqq h_\varepsilon(\gamma)$, where

$$\begin{aligned} h_\varepsilon(\gamma) &= \bar{b}((1 - \gamma)\lambda) + \\ &+ \bar{b}(\lambda) - 2 \sup \{\bar{b}((1 - \tilde{\delta}_{\bar{p}}(\varepsilon - \gamma) + \gamma/2)\lambda), \bar{b}(\lambda(1 - \tilde{\delta}_{\bar{p}}(\varepsilon)))\} . \end{aligned}$$

As $\bar{b}$ is strictly increasing (cf. Lemma 9.1), we find sufficiently small γ such that $h_\varepsilon(\gamma) > 0$. If $p \in C_{\lambda\varepsilon}$ and $|\|p\| - \lambda| > \gamma\lambda$, we have $h(p) \geqq \bar{b}(\lambda) + \bar{b}(\|p\|) - 2\bar{b}(\frac{1}{2}\lambda + \frac{1}{2}\|p\|) \geqq \delta_\lambda(\gamma\lambda)$. Thus, $\inf\{h(p) \mid p \in C_{\lambda\varepsilon}\} > 0$. □

Corollary 9.2.1. Let $\bar{b}$ be finite convex and nondecreasing on $\mathbb{R}_+$ and locally uniformly convex at $\lambda \in \mathbb{R}_+$. Then, the generalized pairing (9.2) is sharp at $\lambda \bar{p}$ for any unit vector $\bar{p}$.

Proof. Due to the assumptions being imposed, $b = \bar{b}(\|\cdot\|)$ is convex and continuous. Therefore, assertions of Props. 9.1, 9.2 can be applied. □

Noting that a function on $\mathbb{R}$ is locally uniformly convex iff it is strictly convex, we have

Corollary 9.2.2. Let $\bar{b}$ be finite, strictly convex and nondecreasing on $\mathbb{R}_+$. Then, (9.2) is sharp.

By the above corollary, functions $\|\cdot\|^\alpha$, $\exp\|\cdot\|^\alpha$ and $\cosh\|\cdot\|^\alpha$ generate sharp functionals for $\alpha > 1$. In what follows we confine ourselves to the special case: $\bar{b} = (\cdot)^2$ and V is a Hilbert space $\boldsymbol{H}$ (hence clearly locally uniformly convex). We shall denote

$$\psi_1(p, w) = \langle v^*, p \rangle + r\|p\|^2$$

and apply the class of minorants generated by ψ_1 to the mathematical programming problem ($\mathscr{P}$ 2). It is assumed that

(i) $\boldsymbol{X}$ is a Banach space and $\boldsymbol{Z} = \boldsymbol{V}$ is the Hilbert space $\boldsymbol{H}$;
(ii) f is finite and l.s.c. on A;
(iii) A is a closed subset of $\boldsymbol{X}$.
(iv) D is a closed convex cone in $\boldsymbol{H}$ with the vertex at the origin.

We set

$$\Phi(x, p) = f(x) + \delta_A(x) + \delta_D(p - q(x)) \tag{9.12}$$

so that the Lagrangian attains the form

$$L(x, w) = L(x, r, v^*) = \inf_{p \in \boldsymbol{H}} [f(x) + \delta_A(x) + \delta_D(p - q(x)) + \langle v^*, p \rangle + r\|p\|^2] =$$
$$= f(x) + \delta_A(x) + r \left\| \left(q(x) + \frac{v^*}{2r} \right)^{D^*} \right\|^2 - \frac{1}{4r} \|v^*\|^2$$

be exactly the same reasoning as in Eq. (5.5). However, with respect to the consistency with the notation of Sec. 5, we replace L by a Lagrangian $\tilde{L}$ given by

$$\tilde{L}(x, r, v^*) = L(x, r, -v^*), \tag{9.13}$$

and $\tilde{L}$ has the same form as L_r in Sec. 5. Nevertheless, there is a fundamental difference: meanwhile r was a fixed parameter in the convex case, now r is a part of the dual couple (r, v^*) (with respect to the pairing ψ_1). Therefore we will include it to the arguments of $\tilde{L}$. Concerning a derivative of $\tilde{L}$ with respect to v^*, we cannot apply Props. 1.7, 1.8. Fortunately, the following assertion has been proved in [80].

Lemma 9.2. Let $\beta(p) = \frac{1}{2}\|p^{D^*}\|^2$, $p \in \boldsymbol{H}$. Then β is Fréchet differentiable over $\boldsymbol{H}$

with

$$\nabla\beta(p) = (p)^{D^*}.$$

In our case

$$\text{(9.14)} \quad \nabla_{v^*}\tilde{L}(x, r, v^*) = -\left(\left(q(x) - \frac{v^*}{2r}\right)^{D^*} - \frac{v^*}{2r} = -\left(\frac{v^*}{2r} - q(x)\right)^{D} - q(x)\right),$$

and we have the same result as in the convex case. Therefore, the *augmented Lagrangian* $\tilde{L}$ is Fréchet differentiable with respect to v^* for any $x \in A$, $r \in \mathbb{R}_+$. $\tilde{L}$ is defined over $A \times \mathbb{R}_+ \times \boldsymbol{H}$ and the dual problem has the form

$$G(r, v^*) = \inf_{x \in \boldsymbol{X}} \tilde{L}(x, r, v^*) \to \sup$$

subj. to

$$(r, v^*) \in \mathbb{R}_+ \times \boldsymbol{H}.$$

The "nonconvex" variant of Prop. 5.1 reads as follows:

Proposition 9.3. The functionals $(r, v^*) \mapsto \tilde{L}(x, r, v^*)$ and $G(r, v^*)$ are concave and u.s.c. in $\mathbb{R}_+ \times \boldsymbol{H}$, nondecreasing in $r \in R_+$ and G is nowhere $+\infty$. Furthermore, for $r_1 > r_0 \geqq 0$, $\bar{v}^* \in \boldsymbol{H}$

$$\text{(9.15)} \qquad G(r_1, \bar{v}^*) \geqq \max_{v^* \in \boldsymbol{H}} \left[G(r_0, v^*) - \frac{\|\bar{v}^* - v^*\|^2}{4(r_1 - r_0)} \right].$$

Proof. Clearly,

$$\text{(9.16)} \qquad G(r, v^*) = \inf_{p \in \boldsymbol{H}} \left[h(p) - \langle v^*, p\rangle + r\|p\|^2\right].$$

The first assertion is implied by the definition of $\tilde{L}$ and (9.16), since the pointwise infimum of a collection of affine functions of (r, v^*) which are nondecreasing in r is an u.s.c. concave function which is nondecreasing in r. For any $(r_1, \bar{v}^*)$ and (r_0, v^*) satisfying $r_1 > r_0 \geqq 0$ we have from (9.16) that

$$G(r_1, \bar{v}^*) = \inf_{p \in \boldsymbol{H}} \left[h(p) - \langle v^*, p\rangle + r_0\|p\|^2 + \langle v^* - \bar{v}^*, p\rangle + (r_1 - r_0)\|p\|^2\right] \geqq$$

$$\geqq G(r_0, v^*) - \frac{\|\bar{v}^* - v^*\|^2}{4(r_1 - r_0)}.$$

The maximum in (9.15) is attained due to Prop. 1.17. □

Corollary 9.3.1. For every $v^* \in \boldsymbol{H}$ one has

$$\text{(9.17)} \qquad \lim_{r \to +\infty} G(r, v^*) = \beta.$$

Proof. Given any $\varepsilon > 0$ and $(r_0, \bar{v}^*) \in \mathbb{R}_+ \times \boldsymbol{H}$ such that $G(r_0, v^*) \geqq \beta - \varepsilon$, one has

$$G(r, v^*) \geqq G(r_0, \bar{v}^*) - \varepsilon$$

for all r sufficiently large by (9.15). □

Remark. Eq. (9.17) shows the connection between the shifted penalty and the classic exterior quadratic penalty. Indeed, by Lemma 5.1.

$$\tilde{L}(x, r, \vartheta) = f(x) + \delta_A(x) + r\|q(x) - (q(x))^{-D}\|^2,$$

and consequently

$$G(r, \vartheta) = \inf_{x \in A} [f(x) + r\|q(x) - (q(x))^{-D}\|^2].$$

However, from well-known numerical reasons we are interested rather in such numerical methods, where we need not force r to tend to $+\infty$, cf. [70].

The theory developed in Sec. 8 enables us to investigate the ψ_1-normality and the ψ_1-stability of $(\mathscr{P}\,2)$ with respect to Φ given by (9.12).

Definition 9.2. We say that $(\mathscr{P}\,2)$ satisfies the *quadratic growth condition*, if there is an $r \geqq 0$ such that $\tilde{L}(x, r, \vartheta)$ is bounded below as a function of x.

Clearly,

$$\inf_{x \in X} \tilde{L}(x, r, \vartheta) = \inf_{p \in H} [h(p) + r\|p\|^2]$$

so that the quadratic growth condition holds, iff there exist real numbers $\bar{r}_0 \geqq 0$ and $\bar{q}_0$ such that

$$h(p) \geqq \bar{q}_0 - \bar{r}_0\|p\|^2.$$

If the quadratic growth condition does not hold, the ψ_1-minorant set of h is empty and hence $G \equiv -\infty$.

Proposition 9.4. Suppose that $(\mathscr{P}\,2)$ is inf-stable and satisfies the quadratic growth condition. Then $(\mathscr{P}\,2)$ is ψ_1-normal.

Proof. Apply Prop. 8.7 to $(\mathscr{P}\,2)$ with Φ given by (9.12) and $\psi = \psi_1$. □

However, it is generally very difficult to verify that $(\mathscr{P}\,2)$ is inf-stable For $X = \mathbb{R}^n$, $H = \mathbb{R}^m$, $D = \mathbb{R}^m_+$ and $q(x) = (q^1(x), \ldots, q^m(x))$ one may use the following basic condition, cf. [72]:

(9.18) A is closed, the functions $f, q^1, q^2, \ldots, q^m$ are l.s.c. and for some $p \in \operatorname{int} D$ and $\alpha' > \alpha$ the set $\{x \in A \mid f_0(x) \leqq \alpha', q(x) \leqq p\}$ is compact.

(The inequality is supposed to be valid for all coordinates).

In [38] the authors have tried to prove some infinite-dimensional versions of (9.18). This is one of them:

Proposition 9.5. Let X be reflexive, A be weakly closed, $f[X \to \mathbb{R} \cup \{+\infty\}]$ be weakly l.s.c. and ψ be continuous in the weak topologies of X and H. Moreover, let one of the following conditions be satisfied:

(i) Int $D \neq \emptyset$; there exist numbers η and δ_0 and an element $\bar{p}_0 \in \operatorname{int} D$ such that the sets

(9.19) $$C_\delta = \{x \in A \mid f(x) \leqq \eta, \delta \bar{p}_0 - q(x) \in D\}$$

are bounded and non-empty for each $\delta \in [0, \delta_0]$;

(ii) there exist numbers η and $\delta_0 > 0$ such that the sets

(9.20) $$\tilde{C}_\delta = \{x \in A \mid f(x) \leqq \eta, \|q(x) - (q(x))^{-D}\| < \delta\}$$

are bounded and non-empty for each $\delta \in [0, \delta_0]$.

Then, problem $(\mathscr{P}\ 2)$ is inf-stable.

Proof. The imposed assumptions imply the weak lower semicontinuity of Φ. Suppose the validity of (i). As for $0 \leqq \delta_1 \leqq \delta_2$ $C_{\delta_1} \subset C_{\delta_2}$, we may use the same arguments as in the proof of Prop. 3.1. to prove that the function $\delta \mapsto h(\delta \bar{p}_0)$ is l.s.c. at ϑ. Let $\{p_n\}$ be an arbitrary sequence such that $p_n \to \vartheta$. Since $\bar{p}_0 \in \operatorname{int} D$, there are $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n\} \subset [0, \delta_0]$, $\delta_n \to 0$ such that $\delta_n \bar{p}_0 - p_n \in D$ for $n \geqq n_0$. If $p_n - q(x) \in D$, then $\delta_n \bar{p}_0 - q(x) \in D$, thus $h(p_n) \geqq h(\delta_n \bar{p}_0)$.

If (ii) holds, the proof, based on the idea from the proof of Prop. 3.1. again, is easy. □

Remark. The requirement of the weak continuity of q is very stringent. One possible application of Prop. 9.5 can be found in [61], where $\boldsymbol{X} = \mathbb{R}^n$ and $\boldsymbol{H} = L_2[0, T, \mathbb{R}^n]$.

Proposition 9.6. Let there exist a couple $\hat{w} = (\hat{r}, \hat{v}^*) \in \mathbb{R}_+ \times \boldsymbol{H}$ such that

$$h(p) \geqq h(\vartheta) + \langle \hat{v}^*, p \rangle - \hat{r}\|p\|^2 \quad \forall p \in \boldsymbol{H}.$$

Then,

$$\alpha = \beta = G(\hat{r}, \hat{v}^*),$$

i.e. $\hat{w}$ is a solution of $(\mathscr{D}\ 2)$.

Proof. It is a consequence of Prop. 8.9. □

It is natural to ask about the relationship between $\hat{v}^*$ and an appropriately defined Kuhn-Tucker vector of $(\mathscr{P}\ 2)$. For f and q Gâteaux differentiable over $\boldsymbol{X}$ the following definition has been used in [63].

Definition 9.3. A point $\hat{v}^* \in \boldsymbol{H}$ is called the *Kuhn-Tucker vector* of $(\mathscr{P}\ 2)$ at a point $\hat{x}$ solving $(\mathscr{P}\ 2)$ if

(9.21) $$\hat{v}^* \in -D^*,$$

(9.22) $$\langle \hat{v}^*, q(\hat{x}) \rangle = 0,$$

(9.23) $$\langle \nabla^G f(\hat{x}) - [\nabla^G q(\hat{x})]^* \hat{v}^*, x - \hat{x} \rangle \geqq 0 \quad \forall x \in A.$$

Before we answer the above question, we state an important assertion, known as the Everett theorem for shifted penalties.

Proposition 9.7. Suppose $\bar{x}$ minimizes $\tilde{L}(\cdot, \bar{r}, \bar{v}^*)$ over X. Then $\bar{x}$ is a solution of the problem

$$(9.24) \qquad \begin{aligned} & f(x) \to \inf \\ & \text{subj. to} \quad x \in A \cap \mathscr{B}_{\bar{p}}\,, \end{aligned}$$

where $\mathscr{B}_{\bar{p}} = \{x \in X \mid \bar{p} - q(x) \in D\}$ and $\bar{p} = (q(\bar{x}) - \bar{v}^*/2r)^{D^*} + \bar{v}^*/2r$. Moreover, if f and q are Gâteaux differentiable over X, $\hat{v}^* = 2r(\bar{v}^*/2r - q(\bar{x}))^{-D^*}$ is a Kuhn-Tucker vector of (9.24) at $\bar{x}$.

Proof. By Lemma 5.1

$$\bar{p} - q(\bar{x}) = (\bar{v}^*/2r - q(\bar{x}))^D \in D$$

so that $\bar{x}$ is a feasible point for (9.24). Furthermore, for $x \in \mathscr{B}_{\bar{p}}$

$$\begin{aligned} \|(q(x) - \bar{v}^*/2r)^{D^*}\|^2 &= \|(\bar{v}^*/2r - q(x))^D - (\bar{v}^*/2r - q(x))\|^2 \leqq \\ &\leqq \|\bar{p} - \bar{v}^*/2r\|^2 = \|(q(\bar{x}) - \bar{v}^*/2r)^{D^*}\|^2 \end{aligned}$$

again by Lemma 5.1 and the definition of projection. Since $\tilde{L}(\bar{x}, \bar{r}, \bar{v}^*) \leqq \tilde{L}(x, \bar{r}, \bar{v}^*)$ for all $x \in X$, then $f(\bar{x}) \leqq f(x)$ for each $x \in A \cap \mathscr{B}_{\bar{p}}$.

Clearly $\hat{v}^* \in -D^*$.

$$\langle \hat{v}^*, q(\bar{x}) - \bar{p} \rangle = \langle 2r(\bar{v}^*/2r - q(\bar{x}))^{-D^*}, -(\bar{v}^*/2r - q(\bar{x}))^D \rangle = 0\,,$$

because in the decomposition (5.3) the components p^D, p^{-D^*} are orthogonal. Finally, realize that

$$\nabla_x^G[\tilde{L}(x, r, v^*) - \delta_A(x)] = \nabla^G f(x) + 2r[\nabla^G q(x)]^* (q(x) - v^*/2r)^{D^*}$$

by Lemma 9.2 so that

$$\langle \nabla^G f(\bar{x}) + 2r[\nabla^G q(\bar{x})]^* (q(\bar{x}) - \bar{v}^*/2r)^{D^*}, x - \bar{x} \rangle \geqq 0 \quad \forall x \in A\,.$$

As

$$\hat{v}^* = 2r(\bar{v}^*/2r - q(\bar{x}))^{-D^*} = -2r(q(\bar{x}) - \bar{v}^*/2r)^{D^*}\,,$$

the assertion has been proved. □

Corollary 9.7.1. Let $(\mathscr{P}\,2)$ be ψ_1-stable with $(\hat{r}, \bar{v}^*)$ being a solution of $(\mathscr{D}\,2)$. Let $\bar{x}$ minimize $\tilde{L}(\cdot, \hat{r} + \delta, \bar{v}^*)$ for some $\delta > 0$. Then $\bar{x}$ is a solution of $(\mathscr{P}\,2)$ and $\bar{v}^*$ is a Kuhn-Tucker vector of $(\mathscr{P}\,2)$ at $\bar{x}$ (in differentiable case).

Proof. It is an easy consequence of Prop. 9.3 that if $(\hat{r}, \bar{v}^*)$ solves $(\mathscr{D}\,2)$ that $(r, \bar{v}^*)$ also solves $(\mathscr{D}\,2)$ for every $r > \hat{r}$. By the assumptions $\inf_{p \in H}[h(p) - \langle \bar{v}^*, p \rangle + \hat{r}\|p\|^2]$ is attained at $p = \vartheta$. Hence for all $r = \hat{r} + \delta$ $\hat{p} = \vartheta$ is the only minimizer of the functional $\zeta_r : p \mapsto h(p) - \langle \bar{v}^*, p \rangle + r\|p\|^2$ and $\zeta_r(\vartheta) = G(r, \bar{v}^*)$.

Indeed,

$$\begin{aligned} \inf_{p \in H} \zeta_r(p) = \inf_{x \in X} \{f(x) + \delta_A(x) + \inf_{p \in H} [\delta_D(p - q(x)) - \langle \bar{v}^*, p \rangle + \\ + r\|p\|^2]\} = \inf_{x \in X} \tilde{L}(x, r, \bar{v}^*)\,. \end{aligned}$$

As $h(\vartheta) = \tilde{L}(\bar{x}, r, \bar{v}^*)$ and the infimum over p in the second part of the above development for $\bar{x}$ is attained at ϑ, $-q(\bar{x}) \in D$ and $f(\bar{x}) + \delta_A(\bar{x}) = \alpha$. Thus, $\bar{x}$ solves $(\mathscr{P}\,2)$ and by Prop. 9.7.

$$(q(\bar{x}) - \bar{v}^*/2r)^{D^*} + \bar{v}^*/2r = \vartheta\,.$$

Provided f and q are Gâteaux differentiable over X, the Kuhn-Tucker vector of $(\mathscr{P}\,2)$ at $\bar{x}$ is given by

$$\bar{v}^* = 2r(\bar{v}^*/2r - q(\bar{x}))^{-D^*} = \hat{v}^*$$

again by Prop. 9.7. □

Prop. 9.7 enlightens the shifted penalty approach for the solution of $(\mathscr{P}\,2)$ in a different way than it was done in Sec. 5. Namely, if we ensure that $\tilde{L}(\cdot, r, v^*)$ attains its minimum for a couple (r, v^*) and if we perform this minimization, then we solve actually a perturbed problem of the type (9.24). Our task is now to iterate on couples (r, v^*) in such a way, that the corresponding shifts will tend to the zero vector possibly without $r \to \infty$.

However, even in the differentiable case we disclose the following difficulties: The single perturbed problems possess Kuhn-Tucker vectors, but there need not exist any Kuhn-Tucker vector of the unperturbed original problem. In such a case $(\mathscr{P}\,2)$ is not ψ_1-stable by Cor. 9.7.1 and we may look merely for a certain approximation of the zero shift (even theoretically). From this point of view it is important to know various "regularity" conditions guaranteeing the existence of a Kuhn-Tucker vector. Probably the most general condition of this type for f and q being Fréchet differentiable has been proved in [63]:

Let $\hat{x}$ be a solution of $(\mathscr{P}\,2)$ which is *regular*, i.e.

$$\vartheta \in \operatorname{int}\,(q(\hat{x}) + \nabla q(\hat{x})\,(A - \hat{x}) + D)\,. \tag{9.25}$$

Then there exists at least one Kuhn-Tucker vector of $(\mathscr{P}\,2)$ at $\hat{x}$.

Of course, conditions guaranteeing the ψ_1-stability of $(\mathscr{P}\,2)$ are still stronger and more complicated to verify than the regularity condition (9.25). In the finite-dimensional case and for inequality constraints they have been investigated by Rockafellar:

Proposition 9.8. Suppose in the finite-dimensional case that $(\mathscr{P}\,2)$ has a unique solution $\hat{x}$ satisfying the weak form of the second-order sufficient optimality conditions and that the quadratic growth condition holds. Then $(\mathscr{P}\,2)$ is ψ_1-stable for all r sufficiently large.

For the proof of the assertion and the formulation of the weak form of the second-order sufficient optimality condition see [72], [73].

It is supposed that the sufficient optimality conditions of [64] can be used in an infinite-dimensional version of Prop. 9.8, but we cannot provide the appropriate reference.

We turn now our attention to numerical methods for solving ($\mathscr{P}$ 2) by way of ($\mathscr{D}$ 2) which have proved themselves to be very useful in many nonconvex problems.

Proposition 9.9. Let ($\mathscr{P}$ 2) be inf-stable and $\{r_n, v_n^*\} \subset \mathbb{R}_+ \times \boldsymbol{H}$ be such a sequence that $r_n \geqq \delta > 0$ for all n and

(9.26) $$\lim_{n\to\infty} G(r_n - \delta, v_n^*) = \alpha(\equiv \inf(\mathscr{P} 2)) .$$

Suppose furthermore that

(9.27) $$\tilde{L}(x_n, r_n, v_n^*) \leqq G(r_n, v_n^*) + \gamma_n$$

for all n with $\gamma_n \to 0$. Then, $\{x_n\}$ is an asymptotically feasible sequence and $\operatorname{dist}(v_n^*/2r_n, -D^*) \to 0$. If $\{v_n^*\}$ happens to be bounded, $\{x_n\}$ is ASMS for ($\mathscr{P}$ 2).

Proof. Let

(9.28) $$p_n = (q(x_n) - v_n^*/2r_n)^{D^*} + v_n^*/2r_n .$$

Therefore,

$$\tilde{L}(x_n, r_n, v_n^*) = f(x_n) + \delta_A(x_n) - \langle v_n^*, p_n\rangle + r_n\|p_n\|^2 =$$
$$= f(x_n) + \delta_A(x_n) - \langle v_n^*, p_n\rangle + (r_n - \delta)\, \|p_n\|^2 + \delta\|p_n\|^2 \geqq G(r_n - \delta, v_n^*) + \delta\|p_n\|^2.$$

By (9.27)

$$\alpha - G(r_n - \delta, v_n^*) + \gamma_n \geqq \delta\|p_n\|^2$$

so that $p_n \to \vartheta$ by (9.26). From (9.28) $p_n - q(x_n) = (v_n^*/2r_n - q(x_n))^D$, $\operatorname{dist}(-q(x_n), D) \leqq \|p_n\| \to 0$ which implies the asymptotic feasibility of $\{x_n\}$. Analogously, $\operatorname{dist}(v_n^*/2r_n, -D^*) \leqq \|p_n\| \to 0$.

It remains to prove that $\{x_n\}$ is ASMS for ($\mathscr{P}$ 2). As $\{v_n^*\}$ is bounded, $\langle v_n^*, p_n\rangle$ converges to zero. Thus,

$$\varliminf_{n\to\infty} r_n\|p_n\|^2 \leqq \alpha - \varlimsup_{n\to+\infty} f(x_n) .$$

But, according to the Everett theorem and our assumptions

$$\varliminf_{n\to\infty} f(x_n) \geqq \varliminf_{n\to\infty} h(p_n) = \alpha$$

so that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \tilde{L}(x_n, r_n, v_n^*) = \alpha . \qquad \square$$

The preceding assertion is the fundamental convergence result for a number of numerical methods. We introduce here only one representative belonging to the effective class of so called "shifted-increased" algorithms:

Given $r_0 > 0$, $k > 1$, $\delta \in (0, 1)$, $S_0 > 0$ and $\xi > 0$, determine a sequence $\{x_n, r_n, v_n^*\}$, $n = 0, 1, 2, \ldots$, according to

(9.29) $$x_n \in \arg\inf_{x\in X} \tilde{L}(x, r_n, v_n^*), \quad p_n = (q(x_n) - v_n^*/2r_n)^{D^*} + v_n^*/2r_n .$$

If $\|p_n\| \leqq S_n$ set

(9.30) $$v^*_{n+1} = -2r_n(q(x_n) - v^*_n/2r_n)^{D*}, \quad r_{n+1} = r_n, \quad S_{n+1} = \delta S_n.$$

If $\|p_n\| > S_n$ set

(9.31) $$v^*_{n+1} = (-2r_n/k)(q(x_n) - v^*_n/2r_n)^{D*}, \quad r_{n+1} = kr_n, \quad S_{n+1} = S_n.$$

If $\|p_n\| \leqq \xi$ stop.

Remark. Observe that (9.30) is nothing else than

$$v^*_{n+1} = v^*_n + 2r_n \nabla_{v*}\tilde{L}(x_n, r_n, v^*_n),$$

i.e. we have the same multiplier-update as (5.11). Therefore, if r_n is kept constant, we apply then the steepest ascent method to the dual cost G (with respect to v^*) with the fixed step length $2r_n$.

The above algorithm is actually a combination of algorithm (5.10), (5.11) and the well-known quadratic exterior penalty method. If ($\mathscr{P}2$) is only inf-stable, the algorithm is convergent by Prop. 9.9. If ($\mathscr{P}2$) is ψ_1-stable, the penalty increase in (9.31) is applied only as many times as it is necessary to ensure the convergence. This algorithm has been many times successfully applied in various static and dynamic optimization problems, cf. e.g. [61], [79]. Of course, we have to secure that solutions of intermediate minimization (9.29) exist. Concerning purely shifted algorithms of the type (5.10), (5.11), the conditions guaranteeing their convergence in the non-convex case are of a rather complicated nature, cf. [38].

Almost all results of the preceding part of this section are taken (may be in a slightly modified form) from [72] and [38]. There is a number of other works dealing with the same or similar problems, cf. e.g. [46], [47], [71]. For numerical studies see e.g. [75].

The above theory will now be illustrated on a simple example dealing with the time optimal control of a heating of a bar with state-space and terminal state constraints. This problem cannot be directly written in the form ($\mathscr{P}3$) – we have first to remove the free terminal time by the following standard transformation: We set $\boldsymbol{U} = \mathbb{R} \times \boldsymbol{H}^1 [0, 1, \mathbb{R}^2]$ with a typical element $u = (\tau, T)$, $T = (T^0, T^1)$, $\boldsymbol{Y} = \boldsymbol{H}^{1,2}(Q) \times \boldsymbol{H}^1(\Omega)$, where $Q = (0, 1) \times (0, 1)$, $\Omega = (0, 1)$ and by $\boldsymbol{H}$ the obvious Sobolev spaces of the Hilbert type are denoted.

(9.32) $$J(u, y) = \tau.$$

Λ assigns to a certain control couple (τ, T) the solution y and the terminal state $y(1, \cdot)$ of the linear parabolic equation

(9.33) $$\frac{\partial y}{\partial t} = \tau\sigma \frac{\partial^2 y}{\partial x^2} \quad \text{a.e. in } Q,$$

with the initial condition

(9.34) $$y(0, \cdot) = \vartheta \quad \text{a.e. in } \Omega,$$

and the boundary condition

(9.35) $\dfrac{\partial y}{\partial x}(\cdot, x) = (-1)^x \alpha_0(y(\cdot, x) - T^x)$ a.e. in $(0, 1)$ for $x = 0, 1$,

where σ, α_0 are given positive constants. The constraining sets M, N are given by

(9.36)
$$M = [0, \tau_0] \times$$
$$\times \{T \in \boldsymbol{H}^1[0, 1, \mathbb{R}^2] \mid 0 \leqq T^x(0) \leqq C_0, C_1 \leqq \dot{T}^x/\tau \leqq C_2, x = 0, 1\}$$
for τ_0, C_0, C_1, C_2 given constants,

(9.37) $N = \{y \in \boldsymbol{H}^{1,2}(Q) \times \boldsymbol{H}^1(\Omega) \mid y \leqq \not{f}_0$ on $Q, a \leqq y(1, \cdot) \leqq b$ on $\Omega\}$

for a given function $\not{f}_0$ on Q and constants $0 < a < b$ which are prescribed.

As in Sec. 5 we transform the above problem to the mathematical programming form $(\mathscr{P}2)$, where $\boldsymbol{X} = \boldsymbol{U}$, $\boldsymbol{Z} = \boldsymbol{L}_2(Q) \times \boldsymbol{L}_2(\Omega) \times \boldsymbol{L}_2(\Omega)$,

$$D = \{z = (z_1, z_2, z_3), z_1 \in \boldsymbol{L}_2(Q), z_2, z_3 \in \boldsymbol{L}_2(\Omega) \mid z_1(t, x) \geqq 0$$
$$\text{for a.e. } (t, x) \in Q, z_2(x), z_3(x) \geqq 0 \text{ for a.e. } x \in \Omega\},$$
$$A = M, \quad f = J(u, \Lambda u) = \tau,$$

and

$$q = \begin{bmatrix} \Lambda_1 u - \not{f}_0 \\ \Lambda_2 u - b \\ a - \Lambda_2 u \end{bmatrix},$$

where $\Lambda_1 : u \mapsto y$ and $\Lambda_2\, u \mapsto y(1, \cdot)$.

If there exists a control $\bar{u} \in M$ such that $-q(\bar{u}) \in D$ (i.e. the process $(\bar{u}, \bar{y})$ is feasible), all requirements of Prop. 9.5. are satisfied (the weak continuity of q is a consequence of the appropriate variational formulation of the state equation) so that our problem is inf-stable and hence also ψ_1-normal. The dual problem attains for $v^* = (\lambda, \mu, \nu) \in \boldsymbol{Z}$ the form

(9.38)
$$\inf_{u \in M} \tilde{L}(u, r, \lambda, \mu, \nu) \to \sup$$
subj. to
$$(r, \lambda, \mu, \nu) \in \mathbb{R}_+ \times \boldsymbol{L}_2(Q) \times \boldsymbol{L}_2(\Omega) \times \boldsymbol{L}_2(\Omega),$$

where

$$\tilde{L}(u, r, \lambda, \mu, \nu) = \tau - \int_Q \lambda \max\{\Lambda_1 u - \not{f}_0, \lambda/2r\}\, dx\, dt +$$
$$+ r \int_Q \max^2\{\Lambda_1 u - \not{f}_0, \lambda/2r\}\, dx\, dt - \int_\Omega \mu \max\{\Lambda_2 u - b, \mu/2r\}\, dx +$$
$$+ r \int_\Omega \max^2\{\Lambda_2 u - b, \mu/2r\}\, dx - \int_\Omega \nu \max\{a - \Lambda_2, u, \nu/2r\}\, dx +$$
$$+ r \int_\Omega \max^2\{a - \Lambda_2 u, \nu/2r\}\, dx + \delta_M(u).$$

The arguments t, x have been dropped for the sake of simplicity. This problem may be solved e.g. by the algorithm (9.29)–(9.31). However, we have to be able to solve the intermediate minimization in the step (9.29), namely the following optimal control problems without state-space and terminal state constraints

$$\Theta(\tau, T) \equiv \tilde{L}(\tau, T, r, \lambda, \mu, \nu) - \delta_M(\tau, T) \to \inf \quad \text{subj. to} \quad (\tau, T) \in M \tag{9.39}$$

for fixed dual variables r, λ, μ, ν. These problems may be solved by a gradient technique – the appropriate partial derivatives are computed in Props. 9.10, 9.11 below. For a function y defined a.e. in Q for which $y(1, \cdot)$ has some sense (e.g. as a trace) we denote in the rest of Sec. 9 $y(1, \cdot) = y_1$.

Proposition 9.10. Let $\bar{u} = (\bar{\tau}, \bar{T}) \in M$ and $\bar{y}$ be the corresponding state variable. Let ω be the solution of the adjoint equation

$$\frac{\partial \omega}{\partial t} + \sigma\bar{\tau}\frac{\partial^2 \omega}{\partial x^2} = -2r\left(\bar{y} - \not{f}_0 - \lambda/2r\right)^+ \quad \text{a.e. in } Q \tag{9.40}$$

with the terminal condition

$$\omega_1 = 2r[(\bar{y}_1 - b - \mu/2r)^+ - (a - \bar{y}_1 - \nu/2r)^+] \tag{9.41}$$

and the boundary condition

$$\frac{\partial \omega}{\partial x}(\cdot, x) = (-1)^x \alpha_0\, \omega(\cdot, x) \quad \text{a.e. in } (0, 1) \quad \text{for} \quad x = 0,1\,. \tag{9.42}$$

Then

$$\nabla_T \Theta(\bar{\tau}, \bar{T}) = \bar{\tau}\sigma\alpha_0(\omega(\cdot, 0), \omega(\cdot, 1))\,.$$

Proof. We introduce the function

$$\bar{L}(y, y_1) = \Theta(u) \tag{9.43}$$

for $y = \Lambda_1 u$, $y_1 = \Lambda_2 u$ and observe that due to Lemma 9.2

$$\nabla_y \bar{L}(y, y_1) = 2r(y - \not{f}_0 - \lambda/2r)^+\,, \tag{9.44}$$
$$\nabla_{y_1} \bar{L}(y, y_1) = 2r[(y_1 - b - \mu/2r)^+ - (a - y_1 - \nu/2r)^+]\,.$$

Denoting $\Xi = \Lambda(\bar{\tau}, \cdot)$ for fixed $\bar{\tau}$, the adjoint operator Ξ^* is given as follows: For $(y^*, z^*) \in \boldsymbol{L}_2(Q) \times \boldsymbol{L}_2(\Omega)$ we solve the equation

$$\frac{\partial \omega}{\partial t} + \sigma\bar{\tau}\frac{\partial^2 \omega}{\partial x^2} = -y^*$$

with the terminal condition $\omega_1 = z^*$ and the boundary condition (9.42). Then $\Xi^*(y^*. z^*) = \bar{\tau}\sigma\alpha_0(\omega(\cdot, 0),\ \omega(\cdot, 1))$ and it suffices to apply the rule concerning Fréchet derivatives of composite functions. □

Proposition 9.11. Let $\bar{u} = (\bar{\tau}, \bar{T}) \in M$ and $\bar{y}$ be like in Prop. 9.10. Let ω be the solution of the adjoint equation

$$\frac{\partial \omega}{\partial t} = \sigma \bar{\tau} \frac{\partial^2 \omega}{\partial x^2} + \sigma \frac{\partial^2 \bar{y}}{\partial x^2} \quad \text{a.e. o in } Q \tag{9.45}$$

with the initial condition $\omega(0, \cdot) = \vartheta$ in Ω and the boundary condition (9.42). Then

$$\nabla_\tau \Theta(\bar{\tau}, \bar{T}) = 1 + 2r \int_Q (\bar{y} - \not{y}_0 - \lambda/2r)^+ \omega \, dx \, dt +$$
$$+ 2r \int_\Omega [(\bar{y}_1 - b - \mu/2r)^+ - (a - \bar{y}_1 - \nu/2r)^+] \, \omega_1 \, dx .$$

Proof. Denoting $\Sigma = \Lambda(\cdot, \bar{T})$ for fixed $\bar{T}$, $(d\Sigma/d\tau)(\bar{\tau}) = (\omega, \omega_1)$ where ω is given by the above adjoint equation. It remains to use the rule concerning derivatives of composite functions and the gradients of $\bar{L}$ given by (9.44). □

In the above problem we have implicitly used many important details concerning properties of solutions of parabolic equations which are mostly mentioned in [11] and [81]. Even if the controlled system is linear, the theory of Sec. 5 could not be applied due to the disconvexifying role of the minimum time objective.

10. Some other augmented Lagrangians

In spite of their efficiency and successful applications to many problems, shifted penalties possess also two „not so nice" properties, namely

(i) the restriction that $\boldsymbol{Z}$ must be Hilbert;

(ii) $L(x, r, v^*)$ is not twice differentiable in x and p^* even if the functions f and q are; it may cause some difficulties in the intermediate minimizations with respect to x.

The former disadvantage may be removed by generalizing of $\psi_1(p, w) = \langle v^*, p \rangle + rb(p)$, where $\boldsymbol{V}$ is supposed to be a reflexive Banach space and b is a suitable function, cf. [14]. Augmented Lagrangians with better differentiability properties are derived in this section.

Let us suppose that $\boldsymbol{V}$ is a l.c.s. and $b[\boldsymbol{V} \to \mathbb{R}]$ is a nonnegative convex function with $b(\vartheta) = 0$ satisfying the coercivity condition: For any $\mathcal{O} \in \mathfrak{U}_V(\vartheta)$

$$\inf \{ b(p) \mid p \in \boldsymbol{V} \setminus k\mathcal{O} \} \to +\infty \quad \text{for} \quad k \to +\infty . \tag{10.1}$$

Let $\boldsymbol{W} = \mathbb{R}_+ \times \boldsymbol{V}$ with a typical point (r, v) and consider the generalized pairing

$$\psi(p, w) = b(rp + v) . \tag{10.2}$$

Lemma 10.1. For all $\delta \in \mathbb{R}$ and $a \in \boldsymbol{V}$ there is $\gamma > 0$ such that

$$b(\gamma p) \geqq b(p + a) + \delta \quad \text{for all} \quad p \in \boldsymbol{V} \setminus Q ,$$

where $Q \subset \boldsymbol{V}$ is a set satisfying condition (10.1) with $\mathcal{O}$ replaced by Q.

Proof. Suppose by contradiction that for all integers $k > 1$ there is $p_k \in V \setminus Q$ such that

$$b(kp_k) < b(p_k + a) + \delta .$$

Let $\varphi_k(\lambda) = b(\bar{p}_k(\lambda))$, where $\bar{p}_k(\lambda) \equiv \lambda(p_k + a) + (1 - \lambda) kp_k$. φ_k is finite and convex on $\mathbb{R}$, $\varphi_k(1) = b(p_k + a)$ and $\varphi_k(0) = b(kp_k)$. Hence

$$\varphi_k(1) - \varphi_k(0) > -\delta$$

which implies

$$\varphi_k(\lambda) \geqq \varphi_k(0) - \lambda\delta = b(kp_k) - \lambda\delta \quad \text{for} \quad \lambda \geqq 1 .$$

We have $\bar{p}_k(\lambda_k) = \lambda_k a$ for $\lambda_k = k/(k - 1)$. Therefore,

$$b\left(\frac{k}{k-1} a\right) = \varphi_k\left(\frac{k}{k-1}\right) \geqq b\,(kp_k) - \frac{k}{k-1}\,\delta .$$

As $kp_k \notin kQ$, $\underline{\lim}_{k\to\infty} b(kp_k) = +\infty$. Thus,

$$b(a) = \lim_{k\to\infty} b\left(\frac{k}{k-1} a\right) \geqq \underline{\lim}_{k\to\infty} \left[b(kp_k) - \frac{k}{k-1}\,\delta\right] = +\infty$$

which contradicts the finiteness of b. □

Proposition 10.1. The generalized pairing (10.2) is sharp.

Proof. Let $\bar{p}$, $(\bar{r}, \bar{v})$ correspond to $\bar{x}$, $\bar{y}$ in Def. 8.2, respectively, and $\mathcal{O}, \hat{\mathcal{O}}$ and μ be like in Def. 8.2. For an arbitrary $\hat{r}$ let $\hat{v} = -\hat{r}\bar{p}$. Then

$$\psi(p, \hat{w}) = b(\hat{r}(p - \bar{p})) \geqq 0$$

with the equality at $p = \bar{p}$, so that Ineq. (8.5) is satisfied. Concerning (8.4), we want to find some $\hat{r}$ such that

$$b(\hat{r}(p - \bar{p})) \geqq b(\bar{r}p + \bar{v}) + \mu \quad \forall p \in V \setminus \hat{\mathcal{O}},\, \hat{\mathcal{O}} \subset \mathcal{O} , \tag{10.3}$$

or equivalently

$$b(\hat{r}q) \geqq b(\bar{r}q + \bar{r}\bar{p} + \bar{v}) + \mu \quad \forall q \in V \setminus (\hat{\mathcal{O}} - \bar{p}) .$$

Without any loss of generality we may assume that $\bar{r} \neq 0$ so that we want to find a scalar $\hat{r}$ satisfying the inequality

$$b\left(\frac{\hat{r}}{\bar{r}} q\right) \geqq b\,(q + \bar{r}\bar{p} + \bar{v}) + \mu \quad \forall q \in V \setminus \frac{1}{\bar{r}}\,(\hat{\mathcal{O}} - \bar{p}) .$$

But the existence of such $\hat{r}$ is guaranteed by the preceding lemma the assumptions of which are ensured by the growth property of b. □

Remark. Ineq. (10.3) is satisfied for any $\mu \in \mathbb{R}$, hence also for any $\mu' = \mu - b(\bar{v})$.

Therefore, the generalized pairing

$$\psi_2(p, w) = b(rp + v) - b(v)$$

is also sharp.

Henceforth it will be assumed that V is a reflexive Banach space and we shall apply the class of minorants generated by ψ_2 to the problem $(\mathscr{P}2)$ with the same perturbations as in the preceding section and $Z = V$. We assume that X is a Banach space and the assumptions (ii), (iii) and (iv) of Sec. 9 are fulfilled. The augmented Lagrangian

$$(10.4) \quad L(x, r, v) = f(x) + \delta_A(x) + \inf_{p \in V} [\delta_D(p - q(x)) + b(rp + v)] - b(v) .$$

Of course, we have to choose the function b with respect to V, D and q in such a way that the minimization on the right-hand side of (10.4) could be explicitly solved and L would exhibit some nice properties like the second order differentiability.

Example 10.1. $b = (1/\alpha) \|\cdot\|^\alpha$, $\alpha \geqq 2$. Then

$$(10.5) \quad L(x, r, v) = f(x) + \delta_A(x) + \frac{1}{\alpha} \|(-r\, q(x) - v)^D + r\, q(x) + v\|^\alpha - \frac{1}{\alpha} \|v\|^\alpha .$$

If V is a Hilbert space,

$$L(x, r, v) = f(x) + \delta_A(x) + \frac{1}{\alpha} \|(r\, q(x) + v)^{D^*}\|^\alpha - \frac{1}{\alpha} \|v\|^\alpha .$$

Note that the projection $(-r\, q(x) - v)^D$ in (10.5) need not be attained at a unique point (provided V is not strictly convex).

Example 10.2. $V = \mathbb{R}^m$, $D = \mathbb{R}^m_+$ and $b : z \mapsto \sum_{i=1}^{m} \tilde{b}(z^i)$, where $\tilde{b}[\mathbb{R} \to \mathbb{R}_+]$ is a convex function such that $\tilde{b}(\tilde{z}) = 0$ iff $\tilde{z} = 0$.

$$(10.6) \quad L(x, r, v) = f(x) + \delta_A(x) + \sum_{i=1}^{m} [\tilde{b}((r\, q^i(x) + v^i)^+) - \tilde{b}(v^i)] .$$

We cannot give any general results concerning the differentiability yet. Merely for V being a Hilbert space, Lemma 9.2 may be utilized, giving in particular for b from Example 10.1

$$(10.7) \quad \nabla_v L(x, r, v) = \|(r\, q(x) + v)^{D^*}\|^{\alpha - 2} (r\, q(x) + v)^{D^*} - \|v\|^{\alpha - 2} v .$$

However, we may obtain even better differentiability properties that those of Lagrangian (9.13) as it will be shown later.

The dual problem attains the form

$$(\mathscr{D}2) \qquad \begin{aligned} &G(r, v) = \inf_{x \in X} L(x, r, v) \to \sup \\ &\text{subj. to} \\ &\qquad (r, v) \in \mathbb{R}_+ \times V . \end{aligned}$$

The form of ψ_2 causes a significant disadvantage with respect to the dual problem of Sec. 9: Neither L nor G are generally concave with respect to the dual variable v.

Concerning the behaviour of both L and G with respect to $r \in \mathbb{R}_+$, we also cannot prove any monotonicity like in the case of shifted penalties. Some relations of the type (9.15) may be obtained for a concrete function b; however, (9.17) does not hold even for $b = \frac{1}{2}\|\cdot\|^2$ (merely for $v = \vartheta$ which is the method of exterior quadratic penalties).

We bring first some results concerning the global solutions of $(\mathscr{P}2)$ and $(\mathscr{D}2)$ which can be directly derived using the generalized perturbational theory of Sec. 8.

Definition 10.1. We say that $(\mathscr{P}2)$ satisfies the *b-growth condition* if there is an $r \geqq 0$ such that $L(x, r, \vartheta)$ is bounded below as a function of x $(\equiv G(r, 0) > -\infty)$.

Clearly,

$$\inf_{x \in X} L(x, r, \vartheta) = \inf_{p \in V} [h(p) + b(rp)]$$

so that the b-growth condition holds iff there exist real numbers $\bar{\xi} \geqq 0$ and $\bar{k}$ such that

$$h(p) \geqq \bar{k} - b(\bar{\xi} p) \quad \forall p \in V .$$

If the b-growth condition does not hold, the ψ_2-minorant set of h is empty and hence $G \equiv -\infty$.

Proposition 10.2. Suppose that $(\mathscr{P}2)$ is inf-stable and satisfies the b-growth condition. Then $(\mathscr{P}2)$ is ψ_2-normal.

Proof. Apply Prop. 8.7 to $(\mathscr{P}2)$ with Φ given by (9.12) and $\psi = \psi_2$. □

Concerning the inf-stability of $(\mathscr{P}2)$, the appropriate assertions can be found in the preceding section.

Proposition 10.3. Let there exist a couple $\hat{w} = (\hat{r}, \hat{v}) \in \mathbb{R}_+ \times V$ such that

$$h(p) \geqq h(\vartheta) - b(\hat{r}p + \hat{v}) + b(\hat{v}) \quad \forall p \in V . \tag{10.8}$$

Then,

$$\alpha = \beta = G(\hat{r}, \hat{v}) ,$$

i.e. $\hat{w}$ is a solution of $(\mathscr{D}2)$.

Proof. It is a consequence of Prop. 8.9. □

The ψ_2-stability condition (10.8) is hardly verifiable as well as the ψ_1-stability condition of Prop. 9.6. Mangasarian [65] has proved a result similar to Prop. 9.8: If $(\mathscr{P}2)$ is finite-dimensional, the functions f, q are twice continuously differentiable and a couple $(\hat{x}, \hat{v}^*)$ satisfies the second order sufficient optimality conditions with the strict complementarity, then there exist a couple $(\hat{r}, \hat{v})$ and a neighbourhood

$\mathcal{O} \subset \mathfrak{U}_X(\hat{x})$ such that

$$L(\hat{x}, \hat{r}, v) \leqq L(\hat{x}, \hat{r}, \hat{v}) \leqq L(x, \hat{r}, \hat{v}) \quad \forall v \in V, \quad x \in \mathcal{O}, \tag{10.9}$$

provided b satisfies some additional requirements (to those listed previously). The relation between $\hat{v}^*$ and $(\hat{r}, \hat{v})$ is given by (10.12) below.

This result could probably be extended to infinite dimensions but as well as (10.8) it would hardly give us any information before we start the solution of $(\mathscr{D}2)$.

Besides the desirable couples $(\hat{x}, \hat{w})$ which are saddle points of L with the saddle value equal α there may be further saddle points $(\tilde{x}, \tilde{w})$ with generally different saddle values corresponding to solutions of $(\mathscr{P}2)$ with additional constraints

$$x \in \mathcal{O} \in \mathfrak{U}_X(\tilde{x}). \tag{10.10}$$

The majority of saddle point seeking algorithms terminate at the stationary points of L; therefore it is important to ask about the significance of these stationary points with respect to the original problem. The appropriate assertion has been proved for finite-dimensional problems in [65]. We prove here (with respect to an application in optimal control) a generalized version with $Z = V = L_\alpha[0, 1, \mathbb{R}^m]$, $1 < \alpha < \infty$, and $D = \{p \in L_\alpha[0, 1, \mathbb{R}^m] \mid p^i(t) \geqq 0$ a.e. on $[0, 1]$, $i = 1, 2, \ldots, m\}$. A "Sobolev" variant of this assertion can be proved in the same manner using the projection derived in [21].

Suppose that $\tilde{b}[\mathbb{R} \to \mathbb{R}]$ is such that

(i) $\tilde{b}$ is finite, nonnegative, strictly convex with $\tilde{b}(0) = 0$;

(ii) $\tilde{b}$ is continuously differentiable on $\mathbb{R}$;

(iii) there exist constants k_1, k_2 such that $|\tilde{b}(p)| \leqq k_1 + k_2|p|^\alpha$ for each $p \in \mathbb{R}$;

(iv) there exist constants k_3, k_4 such that $|\nabla\tilde{b}(p)| \leqq k_3 + k_4|p|^{\alpha-1}$ for each $p \in \mathbb{R}$;

(v) there exist $v_0 \in L_\alpha[0, 1]$ such that $\langle \nabla\tilde{b}(v), v - v_0 \rangle / \|v\| \to +\infty$ if $\|v\| \to +\infty$.

Assumption (iii) ensures that the integral functional $\int_0^1 \tilde{b}(v(t))\, dt$ is defined over $L_\alpha[0, 1]$. Its convexity together with requirements (iv), (v) imply that the map $v \mapsto \nabla\tilde{b}(v)$ is surjective due to the well-known Browder-Minty theorem, cf. [8]. We shall consider the function

$$b(p) = \int_0^1 \sum_{i=1}^m \tilde{b}(p^i(t))\, dt$$

defined over $L_p[0, 1, \mathbb{R}^m]$ and conclude that b satisfies the assumptions imposed at the beginning of this section. Moreover, ∇b is bijective due to the strict coercivity of $\tilde{b}$ and $\nabla b(\vartheta) = \vartheta$.

The corresponding Lagrangian attains the form

$$L(x, r, v) = f(x) + \delta_A(x) + \int_0^1 \sum_{i=1}^m [\tilde{b}((r\, q^i(x)(t) + v^i(t))^+) - \tilde{b}(v^i(t))]\, dt. \tag{10.11}$$

Henceforth we shall suppose that f and q are continuously Fréchet differentiable over X and denote $\bar{L}: (x, r, v) \mapsto L(x, r, v) - \delta_A(x)$.

Proposition 10.4. Let $\hat{v}^* \in \boldsymbol{L}_{\alpha'}[0, 1, \mathbb{R}^m]$ $((1/\alpha) + (1/\alpha') = 1)$, be a Kuhn-Tucker vector of $(\mathscr{P}2)$ at a point $\hat{x}$ which is a solution of $(\mathscr{P}2)$. Define $\hat{v} \in \boldsymbol{L}_\alpha [0, 1, \mathbb{R}^m]$ by

$$(10.12) \qquad -\nabla\tilde{b}(\hat{v}^i(t)) = \frac{\hat{v}^{*i}(t)}{\hat{r}} \quad \text{for a.e.} \quad t \in [0, 1], \quad i = 1, 2, \ldots, m,$$

where $\hat{r}$ is any positive number. Then,

$$(10.13) \qquad \langle \nabla_x \bar{L}(\hat{x}, \hat{r}, \hat{v}), x - \hat{x} \rangle \geqq 0 \quad \forall x \in A, \quad \nabla_v L(\hat{x}, \hat{r}, \hat{v}) = \vartheta .$$

Conversely, if (10.13) holds, then $\hat{v}^*$ given by (10.12) is a Kuhn-Tucker vector of $(\mathscr{P}2)$ at the solution $\hat{x}$.

Remark. In the above assertion we utilize the obvious generalization of Def. 9.3 to reflexive spaces.

Proof of Prop. 10.4. Eq. (10.12) provides us with a unique $\hat{v}$ due to the assumptions being imposed. For the sake of simplicity we shall set $m = 1$ and drop the variable t in the notation whenever it cannot lead to any confusion. Evidently,

$$(10.14) \qquad \nabla_v L(\hat{x}, \hat{r}, \hat{v}) \colon t \mapsto \nabla\tilde{b}((\hat{r}\, q(\hat{x})(t) + \hat{v}(t))^+) - \nabla\tilde{b}(\hat{v}(t)) .$$

Denoting $I_1 = \{t \in [0, 1] \mid q(\hat{x})(t) = 0\}$ and $I_2 = [0, 1] \setminus I_1$, we see that

$$\nabla_v L(\hat{x}, \hat{r}, \hat{v})(t) = \nabla\tilde{b}(\hat{v}(t)) - \nabla\tilde{b}(\hat{v}(t)) = \vartheta \quad \text{if} \quad t \in I_1, \quad \text{cf. (9.21)}$$

and

$$\nabla_v L(\hat{x}, \hat{r}, \hat{v})(t) = \nabla\tilde{b}((q(\hat{x})(t))^+) - \nabla\tilde{b}(0) = \vartheta \quad \text{if} \quad t \in I_2, \quad \text{cf. (9.22).}$$

This is exactly the second assertion of (10.13). Further,

$$(10.15) \qquad \begin{aligned} \langle \nabla_x \bar{L}(\hat{x}, \hat{r}, \hat{v}), x - \hat{x} \rangle &= \left\langle \nabla f(\hat{x}) + \int_0^1 \hat{r}(\nabla q(\hat{x}))^* \nabla\tilde{b}((\hat{r}\, q(\hat{x}) + \hat{v})^+)\, \mathrm{d}t, x - \hat{x} \right\rangle = \\ &= \left\langle \nabla f(\hat{x}) - \int_0^1 (\nabla q(\hat{x}))^* \hat{v}^*\, \mathrm{d}t, x - \hat{x} \right\rangle \geqq 0 \end{aligned}$$

because of the same idea as above and (9.23).

Conversely, let $\nabla_v L(\hat{x}, \hat{r}, \hat{v}) = \vartheta$. Then for a.e. $t \in [0, 1]$ either $\hat{r}\, q(\hat{x})(t) + \hat{v}(t) = \hat{v}(t) \geqq 0$, or $\hat{r}\, q(\hat{x})(t) + \hat{v}(t) < 0$, $\hat{v}(t) = 0$. This implies further that $-q(\hat{x}) \in D$, $\hat{v}^* \in -D^*$ and condition (9.22) is satisfied because of the properties of $\tilde{b}$. Ineq. (10.15) completes the proof. □

Prop. 10.4 relates the couples $(\hat{x}, \hat{v}^*)$ satisfying the Kuhn-Tucker conditions to couples $(\hat{x}, \hat{w})$, constituting the stationary points of L. Indeed, (for $m = 1$) we have

$$\nabla_r L(\hat{x}, \hat{r}, \hat{v}) = \int_0^1 \nabla\tilde{b}((\hat{r}\, q(\hat{x})(t) + \hat{v}(t))^+)\, q(\hat{x})(t)\, \mathrm{d}t = 0$$

as well. Therefore, when seeking stationary points of L, we could set r equal to any positive constant and keep it unchanged during the iteration process. However, in most so called multiplier methods we look for saddle points of L by maximizing of $G(r, v)$ and therefore, r must be chosen sufficiently large to increase the chance, that a ψ_2-subgradient of h at ϑ with this value of r really exists. This situation is depicted in Fig. 2 for $b = |\cdot|^2$.

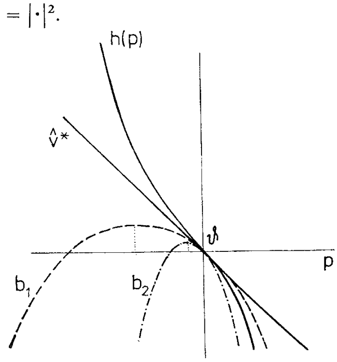

Fig. 2. $b_1 \notin \partial_{\psi_2} h(\vartheta)$, $b_2 \in \partial_{\psi_2} h(\vartheta)$.

The relation of the ψ_2-stability to the existence of Kuhn-Tucker vectors is (according to Prop. 10.4) the same as for the ψ_1-stability: If $(\mathscr{P}2)$ does not possess any Kuhn-Tucker vector, $(\mathscr{P}2)$ is not ψ_2-stable.

In the presence of the stability, the following assertion may be utilized in any saddle-point seeking algorithm:

Proposition 10.5. Let $V = L_\alpha\,[0, 1, \mathbb{R}^m]$, $2 \leqq \alpha < \infty$ and $b = (1/\alpha)\,\|\cdot\|^\alpha$. Let $\hat{w} = (\hat{r}, \hat{v})$ be a solution of $(\mathscr{D}2)$. Then for any $\delta > 0$ the couple

$$\tilde{w} = (\tilde{r}, \tilde{v}) = \left(\hat{r} + \delta, \left(\frac{\hat{r}}{\hat{r} + \delta}\right)^{\frac{1}{\alpha - 1}} \hat{v}\right)$$

is also a solution of $(\mathscr{D}2)$. Moreover, any

$$\hat{x} \in \arg\inf_{x \in X} L(x, \tilde{r}, \tilde{v}) \tag{10.16}$$

is a solution of $(\mathscr{P}2)$.

Proof. If $(\hat{r}, \hat{v})$ is a solution of $(\mathscr{D}2)$, then the ψ_2-minorant

$$\hat{\lambda}(p) = h(\vartheta) - \frac{1}{\alpha}\,\|\hat{r}p + \hat{v}\|^\alpha + \frac{1}{\alpha}\,\|\hat{v}\|^\alpha$$

supports h at ϑ. The function

$$\tilde{\lambda}(p) = h(\vartheta) - \frac{1}{\alpha}\,\|\tilde{r}p + \tilde{v}\|^\alpha + \frac{1}{\alpha}\,\|\tilde{v}\|^\alpha$$

has the same value and the gradient at $p = \vartheta$. Let us examine the difference

$$\Delta_{\hat{v}}(p) = \hat{\lambda}(p) - \tilde{\lambda}(p) = \frac{1}{\alpha}\left(\|\tilde{r}p + \tilde{v}\|^\alpha - \|\hat{r}p + \hat{v}\|^\alpha + \|\hat{v}\|^\alpha - \|\tilde{v}\|^\alpha\right).$$

$\Delta_{\hat{v}}$ is continuously differentiable and coercive, and possesses only one stationary point, namely $p = \vartheta$. Let V' be $\mathbb{R}^k$ equipped with the norm $|||\cdot||| : y \mapsto (\sum_{i=1}^{k} (\beta_i y_i)^\alpha)^{1/\alpha}$ with weights $\beta_i > 0$, $i = 1, \ldots, k$. For p, $\tilde{v}$, $\hat{v}$ we denote by $\tilde{\Delta}_{\hat{v}}$ the functions given in the same way as $\Delta_{\hat{v}}$ with the norm $|||\cdot|||$. Clearly $\tilde{\Delta}_{\hat{v}}$ attains at $p = \vartheta$ strongly its minimum over V'. The same holds also for $L_\alpha[0, 1, \mathbb{R}^m]$, $2 \leqq \alpha < \infty$. Indeed, let for some $\bar{p} \neq \vartheta$ $\Delta_{\hat{v}}(\bar{p}) = -\varepsilon < 0$. Without any lack of generality we may assume that $\bar{p}$ is a simple function (i.e. the range of $\bar{p}$ is a finite set). Certainly, there exists a simple function $\bar{v}$ sufficiently closely approximating $\hat{v}$ such that

$$(10.17)\quad \Delta_{\bar{v}}(\bar{p}) = \frac{1}{\alpha}\left[\|\tilde{r}\bar{p} + \left(\frac{\hat{r}}{\tilde{r}}\right)^{1/(\alpha-1)} \bar{v}\|^\alpha - \|\hat{r}\bar{p} + \bar{v}\|^\alpha + \|\bar{v}\|^\alpha\left(1 - \left(\frac{\hat{r}}{\tilde{r}}\right)^{\alpha/(\alpha-1)}\right)\right] \leqq -\frac{\varepsilon}{2}.$$

We denote by $\{E_i\}_{i=1}^m$ the canonical partitioning of $[0, 1]$ corresponding to both $\bar{p}$ and $\bar{v}$. On a finite-dimensional space $V_{\bar{p},\bar{v}}$ generated by $\text{sp}\,(\chi_{E_1}, \chi_{E_2}, \ldots, \chi_{E_m})$, $\Delta_{\bar{v}} > 0$ whenever $p \neq \vartheta$ because of the above reasoning. However, this is a contradiction with (10.17) so that $\Delta_{\hat{v}} \geqq 0$ on $L_\alpha[0, 1, \mathbb{R}^m]$. If there would be any nonzero global minimizer $\tilde{p} \in L_\alpha[0, 1, \mathbb{R}^m]$ we would have $\nabla\Delta_{\hat{v}}(\tilde{p}) = 0$. But ϑ is the only stationary point so that $\tilde{\lambda}(p) < \hat{\lambda}(p) \leqq h(p)$ for all $p \neq \vartheta$ which was to be proved. □

Remark. It is hoped that the assertion of Prop. 10.5 could probably be extended to a larger class of functions b and spaces V.

We turn now our attention briefly to the numerical solution of $(\mathscr{D}2)$. Mangasarian proposes in his original work two methods, the second of which is a typical multiplier method of the type investigated in Secs. 4, 5 and 9. However, to prove the local convergence of this method, he uses another augmented Lagrangian, having the same stationary points as L.

A typical "multiplier" algorithm for the solution of $(\mathscr{D}2)$ attains the following form:

Given $r > 0$, $v_0 \in V$.

k-th step: 1. Given $v_k \in V$, determine $x_k \in A$ such that

$$(10.17)\qquad L(x_k, r, v_k) = \inf_{x \in X} L(x, r, v_k).$$

2. Compute a "suitable" stepsize ϱ_k and set

$$(10.18)\qquad v_{k+1} = v_k + \varrho_k \nabla_v L(x_k, r, v_k).$$

Stepsizes $\{\varrho_k\}$ may be chosen in many ways: If the intermediate minimization (10.17) is very time-consuming, we may take (under some additional requirements) any positive sequence satisfying the conditions

$$(10.19)\qquad \varrho_k \to 0, \quad \sum_{k=0}^{\infty} \varrho_k = +\infty, \quad \frac{\varrho_{k+1}}{\varrho_k} \to 1,$$

cf. [69]. Various other stepsize choices are also proposed in this work. The above algorithm is merely "conceptual" because the minimization (10.17) mostly cannot be carried out exactly in practice. Therefore, a class of ε-quasigradient methods has been developed, where in (10.17) one performs only an approximate minimization

$$L(x_k, r, v_k) \leqq \inf_{x \in X} L(x, r, v_k) + \varepsilon_k \tag{10.20}$$

for a given positive sequence $\{\varepsilon_k\}$ satisfying the condition

$$\sum_{k=0}^{\infty} \varepsilon_k < \infty , \tag{10.21}$$

cf. also [69]. Generally, for the maximization of $G(r, v_k)$ any numerical method for the unconstrained maximization of a nonconcave locally Lipschitz functional suffices. The problem is that most such methods work in $\mathbb{R}^n$ or in Hilbert spaces and their extension to reflexive Banach spaces may be problematic, cf. e.g. [12], [13], [52]. The minimizations (10.17) or (10.20) supply for the maximization the quasigradients, the ε-quasigradients (see [69] for the definitions) or the directional derivatives needed by the main routine. We return once more to this subject in Secs. 11, 12 devoted to nonsmoth optimization.

Once a solution $\hat{w}$ of ($\mathscr{D}2$) with $\hat{v} \in D$ has been computed (theoretically), the points from $\arg\inf_{x \in X} L(x, \hat{r}, \hat{v})$ are suspicious to be local solutions of ($\mathscr{P}2$). Supposing $\bar{x} \in \arg\inf_{x \in X} L(x, \hat{r}, \hat{v})$ is feasible with respect to the constraint $-q(x) \in D$ and $\hat{v}^i(t)\, q^i(\bar{x})(t) = 0$ for a.e. $t \in [0, 1]$ (cf. Prop. 10.4), then $\bar{x}$ is indeed a local minimizer of ($\mathscr{P}2$). Alternatively, the assertion of Prop. 10.5 may also be exploited.

We show now the form of ($\mathscr{D}2$) on a simple optimal control example convertible (as in Sec. 5) to the mathematical programming form ($\mathscr{P}2$). We have to minimize

$$J(u, x) = \varphi(x(T)) + \int_0^T \Psi(u(t), x(t))\, \mathrm{d}t \to \inf$$

subj. to

$$\begin{aligned} \dot{x}(t) &= F(x(t), u(t)) \quad \text{a.e. on } [0, T] , \\ x(0) &= x_0 \\ x^1(t) &\leqq \mathscr{x}(t) \in \boldsymbol{L}_3\,[0, T] \quad \text{a.e. on} \quad [0, T] , \\ u &\in \boldsymbol{L}_\infty\,[0, T, \mathbb{R}^l] , \end{aligned} \tag{10.22}$$

where T is the fixed terminal time, $x \in \boldsymbol{C}_0[0, 1, \mathbb{R}^n]$ denotes the state vector, u is the control vector, $\varphi[\mathbb{R}^n \to \mathbb{R}]$, $\Psi[\mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}]$ and $F[\mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n]$ are continuously differentiable.

We denote by Λ the nonlinear operator which assigns to each control vector u the corresponding solution of the differential equation in (10.22) with the given initial condition. On setting $A = \boldsymbol{X} = \boldsymbol{U}$, $\boldsymbol{Z} = \boldsymbol{L}_3[0, 1]$, $D = \{z \in \boldsymbol{L}_3[0, 1] \mid z(t) \geqq 0$

a.e. on $[0, 1]\}$,

$$q(u) = (\Lambda u)^1 - x\,,$$

and

$$f(u) = J(u, \Lambda u)\,,$$

we have rewritten (10.22) into the form ($\mathscr{P}2$). We use now the dualisation of Example 10.1 with $\alpha = 3$ providing us with the Lagrangian

$$(10.23)\quad L(u, r, v) = f(u) + \tfrac{1}{3}\int_0^T ((r((\Lambda u)^1 - x) + v)^+)^3 \,\mathrm{d}t - \tfrac{1}{3}\int_0^T |v|^3 \,\mathrm{d}t\,.$$

In each minimization step (10.17) we solve the unconstrained optimal control problem

$$\varphi(x(T)) + \int_0^T [\Psi(u(t), x(t)) + \tfrac{1}{3}((r(x^1(t) - x(t)) + v(t))^+)^3 - \tfrac{1}{3}|v(t)|^3] \,\mathrm{d}t \to \inf$$

subj. to

$$(10.24)\qquad \dot{x}(t) = F(x(t), u(t)) \quad \text{a.e. on} \quad [0, T]$$

$$x(0) = \vartheta\,,$$

$$u \in L_\infty[0, T, R^l]\,.$$

As in the example of Sec. 5 we may use some gradient method for its solution.

Proposition 10.6. Let $\Theta_{r,v}[L_\infty[0, 1, R^l] \to R]$ be the functional given by

$$\Theta_{r,v}(u) = L(u, r, v)\,.$$

Let a function $\varkappa$ be the solution of the adjoint equation

$$(10.25)\quad \dot{\varkappa}(t) = -\left[\frac{\partial F(x(t), u(t))}{\partial x}\right]^* \varkappa(t) + rd\,((r(x^1(t) - x(t)) + v(t))^+)^2$$

a.e. backwards on $[0, T]$ with $d = (1, 0, 0, \ldots, 0) \in R^n$ and the terminal condition

$$(10.26)\qquad \varkappa(T) = -\nabla\varphi(x(T))\,.$$

Then, the functional $\Theta_{r,v}$ is Fréchet differentiable over $L_\infty[0, 1, R^l]$ and

$$(10.27)\quad \nabla\Theta_{r,v}(u)(t) = -\left[\frac{\partial F(x(t), u(t))}{\partial u}\right]^* \varkappa(t) + \nabla_u \Psi(u(t), x(t)), \quad t \in [0, 1]\,.$$

Proof. See [16] and [70].

Remark. If $\dot{x}^1$, $\dot{v}$, $\dot{x}$ are continuous then the right-hand side of Eq. (10.25) is continuously differentiable in t. Thus, Eq. (10.25) may behave numerically substantially better than the adjoint equation for the Lagrangian of Sec. 9. One could use even some 2-nd order methods in the intermediate minimization (10.24) in some cases; however, in optimal control problems the computation of Hessians of $\Theta_{r,v}$ is rather cumbersome. Nevertheless, in usual sufficiently smooth mathematical programs the usage of 2-nd order methods in (10.24) may substantionally speed up the computing time.

The up-date step (10.18) attains the form

$$(10.28) \quad v_{k+1}(t) = v_k(t) + \varrho_k[((r(x^1(t) - x(t)) + v(t))^+)^2 - (v(t))^2 \operatorname{sgn}(v(t))], \quad t \in [0, 1].$$

The classes ψ_1 and ψ_2 of minorants by far do not exhaust considerable variety of various "nonconvex" Lagrangians which have appeared in the literature in recent twenty years. In some cases we even do not need the sharpness property. We return to some other classes of minorants in the third part of the paper devoted to nonsmooth problems. However, the complete treatment of this subject would enormously exceed the limited extent of the sequel. Therefore, we refer to papers [40], [41], [59], [66], where another augmented Lagrangians are studied. Many more references may be found in [71].

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PART III: NONSMOOTH PROBLEMS

Introduction

In the last part of the supplement our attention will be devoted to dualisation of two kinds of nonconvex extremal problems, namely

(i) problems in which some function (functional) appearing in the objective or the constraints is nonsmooth or nondifferentiable;

(ii) problems, the extremal value functional of which is not ψ-subdifferentiable (Def. 8.3) for any $\psi(\cdot, w)$ smooth at ϑ.

In both cases we face generally nonsmooth dual problems solvable exclusively thanks to the modern rapidly developing theory of nonsmooth optimization. In this theory, the first step was in defining a new concept coinciding with the notion of gradient in the differentiable case and coinciding with the notion of subdifferential in the convex case. A number of such concepts have been introduced with an increasing generality, cf. [92] and [106]. These "generalized gradients" and the appropriate calculi are fundamental in the theory of nonsmooth extremal problems (optimality conditions, existence questions) as well as for numerical methods, cf. [12], [69], [88], [89], [90], [92], [96], [98], [106], [111].

We confine ourselves here to such extremal problems and dualisations in which the generalized gradient of Clarke ([86]) and its extension to l.s.c. functions over Banach spaces suffices for describing local properties of all important functions whenever it is required. Taking this into account, we have collected in Sec. 11 only those results of nonsmooth analysis needed in the rest of the paper (Secs. 12, 13). This material simultaneously enables to apply a "bundle" method to the solution of dual problems, cf. [12], [96], where, in the typical case, one needs to be able to supply at least one element of the generalized gradient of Clarke at each point. If one wishes to use another numerical methods, he has to consult the appropriate works, cf. e.g. [52], [69], [98], [107], [115].

In Sec. 12 we will examine the dualisation of $(\mathscr{P}2)$ derived by means of the Lindberg's theory with

$$\psi(p, w) = \psi_3'(p, w) = w\|p\|$$

and perturbations (4.1). This approach leads in fact to a class of nondifferentiable exact penalties. It may generally be applied to so called *calm* problems. (Calmness is slightly weaker than ψ_3'-stability, cf. Prop. 11.15). However, although the calmness (or ψ_3'-stability) requirements are rather weak, there are still some (even simple) problems, where they are not satisfied. In such cases, a dualisation of Sec. 12 may be examined. It was originally proposed in [110] and in this way we may obtain optimality conditions even for some noncalm problems. On the other hand, the numerical solution of the dual problem may be very complicated, cf. [109], [110].

In addition to the notation introduced previously we shall denote $\nabla^s f(x)$ the strict derivative of a functional (mapping) f at x, $\overline{\text{co}}^* A$ is the $w(X^*, X)$-closed convex hull of a set $A \subset X^*$, $\text{cl}^* A$ is the $w(X^*, X)$-closure of a set $A \subset X^*$, $\mathscr{R}(\mathscr{A})$ is the range of an operator $\mathscr{A}$.

11. Nonsmooth analysis

This section is a nonconvex counterpart of Sec. 1 and, correspondingly, the aim is to provide a reader with those main definitions and assertions of nonsmooth analysis which will be used in the last two sections. Main part is devoted to the analysis of locally Lipschitz functions. We mention also briefly the case of completely general functions.

Throughout the whole rest of the paper we confine ourselves to Banach spaces and their topological duals (denoted by means of an asterisk).

Definition 11.1. Let $f[\boldsymbol{X} \to \mathbb{R}]$ be Lipschitz around $x_0 \in \boldsymbol{X}$. The *directional derivative of Clarke* of f at x_0 in the direction h, is given by

$$f^0(x; h) \equiv \overline{\lim_{\substack{x \to x_0 \\ \lambda \to 0_+}}} \frac{f(x + \lambda h) - f(x)}{\lambda} .$$

It possesses a number of very convenient properties, namely:

(i) The function $h \mapsto f^0(x_0; h)$ is finite, positively homogeneous and subadditive on $\boldsymbol{X}$, and satisfies

$$|f^0(x_0; h_1) - f^0(x_0; h_2)| \leqq K\|h_1 - h_2\| ,$$

for each $h_1, h_2 \in \boldsymbol{X}$, where K is a Lipschitz constant of f near x_0;

(ii) $f^0(x; h)$ is upper semicontinuous (u.s.c.) as a function of (x, h);

(iii) $f^0(x; -h) = (-f)^0 (x; h)$.

Definition 11.2. Let $f[\boldsymbol{X} \to \mathbb{R}]$ be Lipschitz around $x_0 \in \boldsymbol{X}$. The *generalized gradient* of f at x_0, denoted $\partial f(x_0)$ is the subset of $\boldsymbol{X}^*$ given by

$$\partial f(x_0) = \{\xi \in \boldsymbol{X}^* \,|\, f^0(x_0; h) \geqq \langle \xi, h \rangle \;\; \forall h \in \boldsymbol{X}\} .$$

This set has also remarkable properties implied by the properties of $f^0(x_0; h)$:

(i) $\partial f(x_0)$ is a nonempty, convex, $w(\boldsymbol{X}^*, \boldsymbol{X})$-compact subset of $\boldsymbol{X}^*$ and

$$\|\xi\|_{\boldsymbol{X}^*} \leqq K \quad \forall \xi \in \partial f(x_0) ;$$

(ii)
$$f^0(x_0; h) = \delta^*_{\partial f(x_0)}(h) \quad \forall h \in \boldsymbol{X} .$$

Let us remark, furthermore, that $\partial f(x_0)$ coincides with the classical subdifferential (Def. 1.7) if f happens to be convex and Lipschitz near x_0. It justifies our notation. $\partial f(x_0)$ reduces to a singleton (strict derivative) if f is strictly differentiable at x_0. If f is Lipschitz near x_0 and admits a Gâteaux (or Hadamard, or strict, or Fréchet derivative) $\nabla^G f(x_0)$, then $\nabla^G f(x_0) \in \partial f(x_0)$.

Remark. We say that a function f, Lipschitz about x_0, admits at x_0 the *strict* *derivative* $\nabla^S f(x_0) \in X^*$ provided that for each direction $h \in X$

$$\lim_{\substack{x \to x_0 \\ \lambda \to 0_+}} \frac{f(x + \lambda h) - f(x)}{\lambda} = \langle \nabla^S f(x_0), h \rangle \, ,$$

cf. [89]. In the same way we define the strict derivative of a mapping.

We turn now our attention to the calculus and bring some basic rules. However, we have to introduce the concept of regularity previously.

Definition 11.3. $f[X \to \mathbb{R}]$ is said to be *regular* at x_0 provided for all h $Df(x_0; h)$ exists and equals $f^0(x_0; h)$.

Regularity is implied e.g. by the strict differentiability or convexity. A finite linear combination (by nonnegative scalars) of functions regular at x_0 is regular at x_0. If f admits a Gâteaux derivative $\nabla^G f(x_0)$ and is regular at x_0, then

$$\partial f(x_0) = \nabla^G f(x_0) \, .$$

Proposition 11.1. Let functions $f, f_1, \ldots, f_n[X \to \mathbb{R}]$ be Lipschitz near $x_0 \in X$ and s be any scalar. Then,

$$\partial(sf)(x_0) = s \, \partial f(x_0) \, ; \tag{11.1}$$

$$\partial\Big(\sum_{i=1}^{n} f_i\Big)(x_0) \subset \sum_{i=1}^{n} \partial f_i(x_0) \, . \tag{11.2}$$

Equality holds in (11.2) if either all but at most one of the functions f_i are strictly differentiable at x_0, or each f_i is regular at x_0.

The so called chain rules hinge mostly on the following mean-value theorem of Lebourg.

Proposition 11.2. Let x and y be points in X, and suppose that f is Lipschitz on an open set containing the line segment $[x, y]$. Then there exists a point u in (x, y) such that

$$f(y) - f(x) \in \langle \partial f(u), y - x \rangle \, .$$

We now state the chain rule I that pertains to the following situation: $f = g \circ F$, where $F[X \to \mathbb{R}^n]$ and $g[\mathbb{R}^n \to \mathbb{R}]$. We assume that each component F^i is Lipschitz near x_0 and that g is Lipschitz near $F(x_0)$.

Proposition 11.3. (Chain rule I.) One has

$$\partial f(x_0) \subset \overline{\text{co}}^* \Big\{ \sum_{i=1}^{n} \alpha^i \xi_i \;\Big|\; \xi_i \in \partial F^i(x_0), \alpha \in \partial g(F(x_0)) \Big\} \, . \tag{11.3}$$

Equality holds in (11.3) under anyone of the following additional hypotheses:

(i) g is regular at $F(x_0)$, each F^i is regular at x_0, and every element α of $\partial g(F(x_0))$ has nonnegative components;

(ii) g is strictly differentiable at $F(x_0)$ and $n = 1$. (In this case the $\overline{\text{co}}^*$ is superfluous.)
(iii) g is regular at $F(x_0)$ and F is strictly differentiable at x_0. (In this case, again, the $\overline{\text{co}}^*$ is superfluous.)

Proposition 11.4. (Chain rule II.) Let F be a map from $\boldsymbol{X}$ to another Banach space $\boldsymbol{Y}$. Suppose that F is strictly differentiable at x_0 and that $g[\boldsymbol{Y} \to \mathbb{R}]$ is Lipschitz near $F(x_0)$. Then, for $f = g \circ F$ one has

$$(11.4) \qquad \partial f(x_0) \subset (\nabla^S F(x_0))^* \, \partial g(F(x_0)) \, .$$

Equality holds in (11.4) if either g (or $-g$) is regular at $F(x_0)$ or $\nabla^S F(x_0)$ is surjective.

Props. 11.3, 11.4 enable us to compute generalized gradients of a great variety of composite functions like e.g. products, quotiens, poinwise maxima etc.

Proposition 11.5. Let $f_i[\boldsymbol{X} \to \mathbb{R}]$, $i = 1, 2, \ldots, n$ be a finite collection of functions, Lipschitz near x_0. Let

$$f = \max_{1 \leqq i \leqq n} f_i$$

and

$$I(x) = \{i \in \{1, 2, \ldots, n\} \mid f(x) = f_i(x)\} \, .$$

Then

$$(11.5) \qquad \partial f(x_0) \subset \text{co} \, \{\partial f_i(x_0) \mid i \in I(x_0)\} \, .$$

Equality holds in (11.5) if f_i is regular at x_0 for each $i \in I(x_0)$.

Let $f[\boldsymbol{X}_1 \times \boldsymbol{X}_2 \to \mathbb{R}]$ be Lipschitz near $(x_1, x_2) \in \boldsymbol{X}_1 \times \boldsymbol{X}_2$. In the same way as it was done in Sec. 1 in the case of the subdifferential we may introduce *partial generalized gradients* $\partial_{x_1} f(x_1, x_2)$, $\partial_{x_2} f(x_1, x_2)$. If f is regular at $x = (x_1, x_2)$, then

$$(11.6) \qquad \partial f(x_1, x_2) \subset \partial_{x_1} f(x_1, x_2) \times \partial_{x_2} f(x_1, x_2) \, .$$

Applications in calculus of variations and optimal control require to study the generalized gradients of integral functionals. Here we have a result of this sort:

Proposition 11.6. Let $f[[0, 1] \times \mathbb{R}^m \to \mathbb{R}]$ and

$$(11.7) \qquad J(x) = \int_0^1 f(t, x(t)) \, \mathrm{d}t$$

be an integral functional defined over $L_\infty[0, 1, \mathbb{R}^m]$. We suppose that

(i) J is (finitely) defined at a point x_0;
(ii) there exists $\varepsilon > 0$ and a function $k \in L_1[0, T]$ such that, for a.e. $t \in [0, 1]$ for all $v_1, v_2 \in x_0(t) + B_m^\varepsilon(\vartheta)$

$$(11.8) \qquad |f(t, v_1) - f(t, v_2)| \leqq k(t) \, |v_1 - v_2|_m \, ;$$

(iii) the mapping $t \to f(t, v)$ is measurable for each $v \in \mathbb{R}^m$.

Then J is Lipschitz in a neighbourhood of x_0 and one has

$$(11.9) \qquad \partial J(x_0) \subset \int_0^1 \partial f_t(x_0(t))\,dt\,,$$

where $\partial f_t(x_0(t))$ is the generalized gradient of the function $f(t, \cdot)$ at $x_0(t)$. Further, if $f(t, \cdot)$ is regular at $x_0(t)$ for a.e. t, then J is regular at x_0 and equality holds in (11.9).

Remark. The interpretation of (11.9) is as follows: $\partial J(x_0)$ consists of those linear functionals over $L_\alpha[0, 1, \mathbb{R}^m]$ that

$$\langle \xi, \cdot \rangle = \int_0^1 \langle \xi_t, \cdot \rangle_m\,dt\,,$$

where $\xi_t \in \partial f_t(x_0(t))$ a.e. in $[0, 1]$ and for any $v \in L_\alpha[0, 1, \mathbb{R}^m]$ the function $t \mapsto \langle \xi_t, v(t) \rangle_m$ belongs to $L_1[0, 1]$.

Remark. If J is defined over $L_\alpha[0, 1, \mathbb{R}^m]$, $\alpha \in [1, \infty)$ and the function k in (11.8) belongs to $L_{\alpha^*}[0, 1]$ $(1/\alpha + 1/\alpha^* = 1)$, then the assertion of Prop. 11.6 remains true. Moreover, any element of $\partial J(x_0)$ belongs to $L_{\alpha^*}[0, 1, \mathbb{R}^m]$.

In Prop. 11.5 we have discussed pointwise maxima of a finite collection of locally Lipschitz functions. Many applications like computer-aided design or some exact penalties require an infinite-dimensional version of this assertion.

Proposition 11.7. Suppose that P is a compact subset of a Banach space Y, $f[X \times P \to \mathbb{R}]$ is continuous and, for every $p \in P$, the function $\varphi_p : x \mapsto f(x, p)$ is Lipschitz with a Lipschitz constant L. If

$$(11.10) \qquad \forall p \in P \quad \forall x \in X \quad \varphi_p^0(x; h) = \varlimsup_{\substack{y \to x \\ \lambda \to 0_+ \\ q \to p}} \frac{f(y + \lambda h, q) - f(y, q)}{\lambda}\,,$$

then for $g(x) = \max_{p \in P} f(x, p)$

$$(11.11) \qquad g^0(x; h) \leqq \sup_{p \in P(x)} \varphi_p^0(x; h)\,,$$

where $P(x) = \{p \in P \mid f(x, p) = g(x)\}$. If the functions φ_p are regular at x, the equality holds in (11.11).

There are important geometric concepts associated with the generalized gradient. One possible bridge is the distance function $d_A(\cdot) = \operatorname{dist}(\cdot, A)$, which is clearly globally Lipschitz.

Definition 11.4. Let $A \subset X$ and $x \in A$. A vector $v \in X$ is tangent to A at x provided $d_A^0(x; v) = 0$. The set of all tangents to A at x is termed the *tangent cone* of A at x and denoted $T_A(x)$.

It is an immediate consequence of the properties of $d_A^0(x; v)$ that $T_A(x)$ is a closed convex cone always containing the zero vector. We define the *normal cone* of A

at x by

(11.12) $$N_A(x) = -(T_A(x))^* = \{\xi \in X^* \mid \langle \xi, v \rangle \leqq 0 \quad \forall v \in T_A(x)\}\,.$$

It may be shown that

(11.13) $$N_A(x) = \mathrm{cl}^* \{ \bigcup_{\lambda \geqq 0} \lambda\, \partial d_A(x)\}\,.$$

If A is convex, then $N_A(x)$ coincides with the cone of normals in the sense of convex analysis.

The comparison of T_A with the classical contigent cone, cf. [106] enables to define the regularity of sets consistently with Def. 11.3. We recall that a vector $v \in X$ belongs to the contingent cone $K_A(x)$ of $A \subset X$ at x iff, for all $\varepsilon > 0$, there exist t in $(0, \varepsilon)$ and a point $w \in v + B_X^\varepsilon(\vartheta)$ such that $x + tw \in A$. Clearly, $T_A(x) \subset K_A(x)$.

Definition 11.5. A set A is *regular* at x provided $T_A(x) = K_A(x)$.

Proposition 11.8. Let f be Lipschitz near x. Then
(i) the epigraph of $f^0(x, \cdot)$ is $T_{\mathrm{epi}\, f}(x, f(x))$;
(ii) f is regular at x iff epi f is regular at $(x, f(x))$.

The above assertion indicates the way how to extend the notion of generalized gradient and the notion of regularity to non-lipschitzian functions.

Definition 11.6. Let $f[X \to \mathbb{R}_{cl}]$ be finite at a point x. We define $\partial f(x)$ to be the set of all $\xi \in X^*$ (if any) for which

(11.14) $$(\xi, -1) \in N_{\mathrm{epi}\, f}(x, f(x))\,.$$

We say that f is *regular* at x provided epi f is regular at $(x, f(x))$.

It follows that $\partial f(x)$ is a $w(X^*, X)$ closed subset of X^*, not necessarily compact and nonempty as in the Lipschitz case. However, $\partial f(x) \neq \emptyset$ if f attains a local minimum at x.

Proposition 11.9. Let $f[X \to \mathbb{R}_{cl}]$ attain its local minimum at x. Then, $\vartheta \in \partial f(x)$.

The generalized gradient for non-lipschitzian functions may be defined along the same lines as in the Lipschitz case (Defs. 11.1, 11.2), cf. [89], [106]. A further analysis of this concept has lead to the class of directionally Lipschitz functions. In what follows the expression

$$(y, \alpha) \downarrow_f x$$

shall mean that $(y, \alpha) \in \mathrm{epi}\, f$, $y \to x$, $\alpha \to f(x)$.

Definition 11.7. Let $f[X \to \mathbb{R}_{cl}]$ be finite at x_0. We say that f is *directionally Lipschitz* at x_0 with respect to $h \in X$ if

$$f^+(x_0; v) = \overline{\lim_{\substack{(x,\alpha)\downarrow_f x_0 \\ v \to h \\ \lambda \to 0_+}}} \frac{f(x + \lambda v) - \alpha}{\lambda} < +\infty\,.$$

f is *directionally Lipschitz* at x_0 if f is directionally Lipschitz at x_0 with respect to at least one $h \in \boldsymbol{X}$.

Remark. $\delta^*_{\partial f(x)}(h) < \infty$ whenever f is directionally Lipschitz at x with respect to h.

A convenient measure of the degree to which a given function f fails to be Lipschitz near a point x (where $f(x)$ is finite) is provided by the *asymptotic generalized gradient* of f at x, denoted $\partial^\infty f(x)$, defined by

$$\partial^\infty f(x) = \{\xi \in \boldsymbol{X}^* \mid (\xi, 0) \in N_{\mathrm{epi}}(x, f(x))\} . \tag{11.15}$$

Of course, the calculus of directionally Lipschitz functions is somewhat sparser than in the Lipschitz case. We present here the analogues of Props. 11.1, 11.4.

Proposition 11.10. Suppose that f_1 is finite at x and f_2 is Lipschitz near x. Then one has

$$\partial(f_1 + f_2)(x) \subset \partial f_1(x) + \partial f_2(x) \tag{11.16}$$

with equality for f_1, f_2 being regular at x.

Proposition 11.11. Let F be a map from $\boldsymbol{X}$ to another Banach space $\boldsymbol{Y}$. Suppose that F is strictly differentiable at x_0 and that $g[\boldsymbol{Y} \to \mathbb{R}]$ is finite and directionally Lipschitz at $F(x_0)$ with

$$\mathscr{R}(\nabla^S F(x_0)) \cap \operatorname{int}\{v \mid \delta^*_{\partial g(F(x_0))}(v) < \infty\} \neq \emptyset . \tag{11.17}$$

Then for $f = g \circ F$ Incl. (11.4) remains true with equality for g regular at $F(x_0)$.

The proofs of Props. 11.1–11.6 and 11.8–11.11 may be found e.g. in [89], the proof of Prop. 11.7 in [83]. In these works and e.g. also in [86], [87], [88], [92] an interested reader can find many more important results from the locally Lipschitz analysis.

For further studies it will be convenient to introduce within the framework of problem $(\mathscr{P}2)$ a locally Lipschitz program

$$(\mathscr{P}2_L) \qquad \begin{gathered} f(x) \to \inf \\ \text{subj. to} \\ q^i(x) \leqq 0 , \quad i = 1, 2, \ldots, n , \\ x \in A , \end{gathered}$$

where functions $f, q^i[\boldsymbol{X} \to \mathbb{R}]$, $i = 1, 2, \ldots, n$, are supposed to be locally Lipschitz.

In [88] the first-order necessary optimality condition for $(\mathscr{P}2_L)$ have been derived with the help of the following important assertion:

Proposition 11.12. Let $f[\boldsymbol{X} \to \mathbb{R}]$ be Lipschitz near $\hat{x} \in \boldsymbol{X}$ with a Lipschitz constant K. Let $\hat{x} \in C \subset \boldsymbol{X}$ and suppose that f attains a minimum over C at $\hat{x}$. Then for any $\hat{K} \geqq K$ there is a neighbourhood $\mathcal{O} \in \mathscr{U}_{\boldsymbol{X}}(\hat{x})$ over which $\hat{x}$ minimizes the function

$$g(y) = f(y) + \hat{K}\, d_C(y) . \tag{11.18}$$

Using this result and the variational principle of Ekeland, cf. [8], Clarke stated the "Fritz John type" of optimality conditions for $(\mathscr{P}2_L)$ as follows:

Proposition 11.13. If $\hat{x}$ is a local solution of $(\mathscr{P}2_L)$, then for every k sufficiently large there exist numbers r_0, r_i, $i = 1, 2, \ldots, n$, not all zero, such that

$$\begin{aligned} &r_i \geqq 0\,, \quad i = 0, 1, \ldots, n\,;\\ (11.19) \qquad &r_i\, q^i(\hat{x}) = 0\,, \quad i = 1, 2, \ldots, n\,;\\ &\vartheta \in r_0\, \partial f(\hat{x}) + \sum_{i=1}^{n} r_i\, \partial q^i(\hat{x}) + k\, \partial d_A(\hat{x})\,. \end{aligned}$$

As in the differentiable case, we need a constraint qualification to ensure the *normality* of $(\mathscr{P}2_L)$, i.e. the existence of a multiplier sets $(r_0, r_1, \ldots, r_n)$ satisfying (11.19) with $r_0 > 0$ for the appropriate solution $\hat{x}$ of $(\mathscr{P}2_L)$. This normality must not be interchanged with the normality of convex programs, cf. [8] which is termed inf-stability in this paper. The well-known Mangasarian-Fromowitz constraint qualification, cf. [97], may be reformulated as follows:

Proposition 11.14. Let $\hat{x}$ be a local solution of $(\mathscr{P}2_L)$ and, moreover, let there exist a direction $h \in T_A(\hat{x})$ such that

$$(q^j)^0(\hat{x}; h) < 0 \quad \text{for all} \quad j \in I(\hat{x}) := \{i \in \{1, 2, \ldots, n\} \mid q^i(\hat{x}) = 0\}\,.$$

Then in any multiplier vector $(r_0, r_1, \ldots, r_n)$ satisfying (11.19) $r_0 > 0$.

Also some other well-known constraint qualifications may be modified in the above way. Alternatively, we may ensure the validity of (11.19) with $r_0 \neq 0$ by means of a "calmness" requirement.

Definition 11.7. Let $\hat{x}$ be a solution of $(\mathscr{P}2_L)$. We say that $(\mathscr{P}2_L)$ is *calm* at $\hat{x}$ provided there exist $\varepsilon > 0$ and $M > 0$ such that

$$(11.20) \qquad f(x) - f(\hat{x}) + M|p|_n \geqq 0 \quad \forall p \in B_n^\varepsilon(\vartheta)\,, \quad x \in C(p) \cap B_X^\varepsilon(\hat{x})\,,$$

where $C(p) = \{x \in A \mid q^i(x) \leqq p^i, i = 1, 2, \ldots, n\}$.

Proposition 11.15. Let $h(p) = \inf_{x \in C(p)} f(x)$, i.e. h is the extremal-value function of $(\mathscr{P}2_L)$ with respect to (finite-dimensional) perturbations (4.1). Let $h(\vartheta)$ be finite, and

$$\varliminf_{p \to \vartheta} \frac{h(p) - h(\vartheta)}{|p|_n} > -\infty\,.$$

Then, for any solution $\hat{x}$ of $(\mathscr{P}2_L)$, $(\mathscr{P}2_L)$ is calm at $\hat{x}$.

In agreement with [89] we shall term the problem $(\mathscr{P}2_L)$ *calm*, if the hypotheses of the preceding assertion are satisfied. The proofs of the implication „calmness $\Rightarrow$ $\Rightarrow$ normality" can be found in [88], [89]. It can be shown that requiring a problem of the type $(\mathscr{P}2_L)$ to be calm is less stringent than most of the customary constraint qualifications. Moreover, if h is finite on a neighbourhood $\mathcal{O} \in \mathcal{U}_m(\vartheta)$, then for almost all $p \in \mathcal{O}$ the problems

$$\begin{array}{ll} & f(x) \to \inf \\ (11.21) \qquad \text{subj. to} & \\ & x \in C(p) \end{array}$$

are calm. Indeed, h is on $\mathcal{O}$ nonincreasing as a function of each component p^i separately. But such functions are known to be differentiable a.e. and the existence of $\nabla h(p)$ implies evidently the calmness of (11.21). On the other hand, calmness is a rather hardly verifiable property and we are mostly forced to examine whether h is Lipschitz near zero. Strong results of this kind have been obtained for $\boldsymbol{X}$ being finite-dimensional, cf. [84], [89], [91]. A simplified part of them relevant for our investigations is summarized in Prop. 11.16 below.

We denote Ξ the set of solutions of $(\mathscr{P}2_L)$ and $M_k^1(\hat{x})$, $M_k^0(\hat{x})$ the sets of numbers $(r_0, r_1, \ldots, r_n)$ satisfying the conditions (11.19) for some $\hat{x} \in \Xi$, $k > 0$ with $r_0 > 0$, $r_0 = 0$, respectively. Moreover, we denote

$$M_k^1(\Xi) = \bigcup_{x \in \Xi} M_k^1(x), \quad M_k^0(\Xi) = \bigcup_{x \in \Xi} M_k^0(x).$$

Proposition 11.16. Let in $(\mathscr{P}2_L)$ $\boldsymbol{X}$ be finite-dimensional, A be compact, $h(\vartheta)$ be finite (h given by perturbations (4.1)) and $M_k^0(\Xi) = \{\vartheta\}$ for some $k > K$ being a Lipschitz constant for $[f, q^1, \ldots, q^n]$ on a neighbourhood of $\hat{x}$. Then h is Lipschitz near ϑ and

$$\partial h(\vartheta) \subset \overline{\text{co}}\, M_k^1(\Xi).$$

Thus, if the Mangasarian-Fromowitz constraint qualification is satisfied at all points $\hat{x} \in \Xi$, h is Lipschitz near ϑ. However, to ensure the local calmness of $(\mathscr{P}2_L)$ the following more modest requirement suffices:

Corollary 11.16.1. Let $\hat{x}$ be a local solution of $(\mathscr{P}2_L)$ and $M_k^0(\hat{x}) = \{\vartheta\}$ for some $k > K$. Then $(\mathscr{P}2_L)$ is calm at $\hat{x}$.

For the proof see [89], Cor. 5 of Th. 6.5.2.

In Banach spaces, the known results are substantially weaker due to the difficulties with a realistic compactness assumption. The assertion below has been proved by Aubin and Clarke, see [83].

Proposition 11.17. Let in $(\mathscr{P}2_L)$ $\boldsymbol{X}$ and $\boldsymbol{Z}$ be Banach, $q \in \mathscr{L}[\boldsymbol{X}, \boldsymbol{Z}]$, A be convex and bounded and f be Lipschitz on A. If

$$\vartheta \in \text{int}\,(q(A) - D), \tag{11.22}$$

then $h(p) = \inf\{f(x) \mid x \in A, p - q(x) \in D\}$ is Lipschitz near ϑ.

For another results of this kind, see [105], [112].

The concept of calmness is closely connected to a certain group of exact penalty methods discussed in the next section. With respect to the extent and orientation of the supplement we could state only a few basic assertions, mostly in a simplified form. These results belong to the fundamentals of the modern theory of nonsmooth optimization and we have to refer the interested reader to the above cited works for a complete general treatment of these matters. Moreover, this theory is still under a rapid development and we may expect further important results on this field in the next years.

12. A class of nonsmooth exact penalties

Let (according to the notation introduced in Sec. 8) V be a l.c.s., $W = R_+$ and $\varkappa[V \to \mathbb{R}]$ be such a functional that $\varkappa(\vartheta) = 0$ and

$$\inf_{p \notin \mathcal{O}} \varkappa(p) = \delta_{\mathcal{O}} > 0 \quad \forall \mathcal{O} \in \mathscr{U}_V(x_0)\,. \tag{12.1}$$

It is easily seen that

$$\psi_3(p, w) = w\,\varkappa(p) \tag{12.2}$$

is sharp at ϑ. Let us first consider the problem ($\mathscr{P}2$) and the perturbations (4.1) so that

$$\Phi(x, p) = f(x) + \delta_A(x) + \delta_D(p - q(x))\,. \tag{12.3}$$

The Lagrangian attains the form

$$L(x, w) = f(x) + \delta_A(x) + w \inf_{v \in D} \varkappa(v + q(x))$$

and we apparently need to impose some further assumptions concerning the „penalty" functional $\varkappa$. We will distinguish two cases:

(i) $V = Z = \mathbb{R}^m$, $D = \mathbb{R}^m_+$, $\varkappa(p) = \sum_{i=1}^{m} \varkappa^i(p^i)$ and $\varkappa^i[\mathbb{R} \to \mathbb{R}_+]$ are non-decreasing on $\mathbb{R}_+$, $i = 1, \ldots, m$;

(ii) $V = Z$ is a normed space and $\varkappa = \|\cdot\|$.

In the former case

$$L(x, w) = f(x) + \delta_A(x) + w \sum_{i=1}^{m} \varkappa^i((q^i(x))^+)\,, \tag{12.4}$$

and in the latter

$$L(x, w) = f(x) + \delta_A(x) + w \operatorname{dist}(-q(x), D)\,. \tag{12.5}$$

If Z is a Hilbert space, then

$$L(x, w) = f(x) + \delta_A(x) + w\|(q(x))^{D^*}\|\,. \tag{12.6}$$

As most used nonsmooth exact penalties are of the form (ii), the rest of this section is devoted to this type of dualisation. We denote $\psi'_3(p, w) = w\|p\|$ and we shall suppose that f and q are locally Lipschitz.

Proposition 12.1. For the extremal value function $h(p) = \inf_{x \in X} \Phi(x, p)$ let

$$\inf_{p \in V} \frac{h(p) - h(\vartheta)}{\|p\|} = -\bar{w} > -\infty . \tag{12.7}$$

Then all $w > \bar{w}$ are solutions of the dual problem

$$(\mathscr{D}2) \qquad \text{subj. to} \quad \begin{aligned} &\inf_{x \in X} L(x, w) \to \sup \\ &w \in \mathbb{R}^+ \end{aligned}$$

and for these w's the minimizers of $L(\cdot, w)$ are the actual (global) solutions of $(\mathscr{P}2)$.

Proof. $w \in \partial^{\cap}_{\psi_3'} h(\vartheta)$ whenever $w > \bar{w}$. Hence, $(\mathscr{P}2)$ is ψ_3'-stable, and w is a solution of $(\mathscr{D}2)$ according to Prop. 8.9. Clearly, for $w = \bar{w}$

$$\inf_{x \in X} L(x, \bar{w}) = \inf_{p \in V} [h(p) + \bar{w}\|p\|]$$

is attained also at $p = \vartheta$. If $w > \bar{w}$, this infimum is attained solely at $p = \vartheta$. □

The property (12.7) is very close to the (global) calmness defined in the preceding section. In fact, if e.g. $(\mathscr{P}2_L)$ is calm and $f + \delta_A$ is bounded from below, then $(\mathscr{P}2_L)$ is ψ_3'-stable. However, as we wish to solve nonconvex problems $(\mathscr{P}2)$ by way of $(\mathscr{D}2)$, i.e. by the single unconstrained minimization of $L(x, w)$ over x for a sufficiently large penalty parameter w, it is more important to relate local minima of f and $L(\cdot, w)$. This question has been studied in many works since 1967, cf. e.g. [94], [104], [117].

We generalize first the notion of local calmness to the case of general mathematical program $(\mathscr{P}2)$.

Definition 12.1. Let $\hat{x}$ be a solution of $(\mathscr{P}2)$. We say that $(\mathscr{P}2)$ is *calm* at $\hat{x}$ provided there exist $\varepsilon > 0$ and $M > 0$ such that

$$f(x) - f(\hat{x}) + M\|p\|_Z \geqq 0 \quad \forall p \in B^{\varepsilon}_Z(\vartheta), \quad x \in \{y \in A \mid p - q(y) \in D\} . \tag{12.8}$$

Proposition 12.2. Let Z be Hilbert and $\hat{x}$ solve $(\mathscr{P}2)$ which is calm at $\hat{x}$. Then, for some $w > 0$, $\hat{x}$ provides a local minimum for the Lagrangian $L(\cdot, w)$ given by (12.5).

Proof. We take a fixed real number $w > M$. If the assertion is false, then for each integer k there is a point $x_k \in A \cap B^{1/k}_X(\hat{x})$ such that

$$f(x_k) + w\|(q(x_k))^{D^*}\| < f(\hat{x}) .$$

Necessarily, $\|(q(x_k))^{D^*}\| > 0$. It follows

$$\|(q(x_k))^{D^*}\| = \|-q(x_k) - (-q(x_k))^D\| \leqq \|-q(x_k) - (-q(\hat{x}))\| \to 0 .$$

Let us define $p_k = q(x_k) + (-q(x_k))^D$. Then x_k is feasible with respect to the constraint

$$p_k - q(x_k) \in D$$

and simultaneously $\|p_k\| = \|(q(x_k))^{D^*}\|$. Hence,

$$f(x_k) - f(\hat{x}) + w\|p_k\| < 0$$

which is the contradiction with Def. 12.1. □

Unfortunately, we have no means to verify the local calmness in the general case of $(\mathscr{P}2)$ with the only exception if the localized extremal-value functional

$$h_{\hat{x}}(p) = \inf\{f(x) \mid x \in A \cap \mathscr{O}, \quad p - q(x) \in D\}$$

is Lipschitz near ϑ for some suitably chosen $\mathscr{O} \in \mathscr{U}_X(\hat{x})$ guaranteeing that $f(\hat{x}) = h_{\hat{x}}(\vartheta)$. In such considerations, Prop. 11.17 may be applied. However, generally, we have to confine ourselves to locally Lipschitz programs $(\mathscr{P}2_L)$ defined over finite-dimensional spaces X. Then, Cor. 11.16.1 implies the following statement:

Proposition 12.3. Let $\hat{x}$ be a (local) solution of $(\mathscr{P}2_L)$ and let at $\hat{x}$ the Mangasarian-Fromowitz constraint qualification be satisfied. Then, for some w, $\hat{x}$ provides a (local) minimum for the Lagrangian

$$L(x, w) = f(x) + \delta_A(x) + w|(q(x))^+|_n . \tag{12.9}$$

Proof. The constraint qualification (Prop. 11.14) implies that $M_k^0(\hat{x}) = \{\vartheta\}$ for all $k > 0$, hence also for $k > K$ (the Lipschitz constant for $[f, q^1, \ldots, q^n]$ near $\hat{x}$). Cor. 11.16.1 implies then the calmness of $(\mathscr{P}2_L)$ at $\hat{x}$ and it remains to apply Prop. 12.2. □

Remark. In fact, the assertion of Prop. 12.3 was already extended for X being a Banach space. This extension was achieved using the apparatus of Levitin-Miljutin-Osmolovskii approximations in [94].

In numerical solutions of $(\mathscr{D}2)$ the choice of a proper value for w is extremely important. If w is too small, the minizers of $L(\cdot, w)$ need not satisfy the inequality constraints, if w is too large, the minimization of $L(\cdot, w)$ behaves numerically very badly. We have also sometimes a certain freedom in the choice of Z and hence in the penalty term dist $(-q(x), D)$. The case of Z being Hilbert is favourable not only for the elegant expression of the above distance (see (12.6)), but also with respect to the numerical solution of $(\mathscr{D}2)$.

Proposition 12.4. Let Z be Hilbert and $q[X \to Z]$ be strictly differentiable at $\hat{x}$. Then the function $\varphi(x) = \|(q(x))^{D^*}\|$ is strictly differentiable at $\hat{x}$ provided $-q(\hat{x}) \notin D$ with

$$\nabla^S\varphi(\hat{x}) = (\nabla^S q(\hat{x}))^* \frac{(q(\hat{x}))^{D^*}}{\|(q(\hat{x}))^{D^*}\|} .$$

If $-q(\hat{x}) \in D$,

$$\partial\varphi(\hat{x}) = (\nabla^S q(\hat{x}))^* (B_Z^1(\vartheta) \cap D^* \cap \{z\}^\perp) .$$

Proof. The proof consists in an application of Chain rule II (Prop. 11.4) and the following simple

Lemma 12.5. Let Z be Hilbert. Then the function $\psi(z) = \|(z)^{D^*}\|$ is Fréchet differentiable if $(z)^{D^*} \neq \vartheta$ with

$$\nabla\psi(z) = \frac{(z)^{D^*}}{\|(z)^{D^*}\|}. \tag{12.10}$$

If $(z)^{D^*} = \vartheta$

$$\partial\psi(z) = B_Z^1(\vartheta) \cap D^* \cap \{z\}^{\perp}. \tag{12.11}$$

Proof. ψ may be rewritten into the form $\psi(z) = \sqrt{(2\tfrac{1}{2}\|(z)^{D^*}\|^2)}$. If $(z)^{D^*} \neq \vartheta$, it suffices to apply Lemma 9.2 which proves Eq. (12.10). Concerning Eq. (12.11)

$$\xi \in \partial\psi(\vartheta) \Leftrightarrow \langle \xi, h\rangle \leq \|(h)^{D^*}\| \quad \forall h \in Z.$$

An analysis of the above inequality and the fact that for an aibitrary positively homogeneous convex function $\upsilon[Z \to \mathbb{R}]$ and $z \in Z$ $\partial\upsilon(z) = \{\xi \in \partial\upsilon(\vartheta) | \langle \xi, z\rangle = \upsilon(z)\}$ proves the result. □

By a concrete evaluation of generalized gradients of the penalty term in non-Hilbert spaces, Props. 11.5, 11.6 and 11.7 are frequently utilized provided the projection $(-q(x))^D$ exists.

We demonstrate now the usage of this exact penalty approach on two examples of optimal control problems: in the first Z is Hilbert, in the second $Z = C_0$. The first example is the same as in Sec. 10, i.e. problem (10.22), where, however, $\mathscr{x} \in H^1[0, T]$ and also the system equation is such that $x \in H^1[0, T, \mathbb{R}^n]$. Furthermore, let

$$J(u, x) = \int_0^T \psi(u)\,dt.$$

where $\psi[\mathbb{R}^l \to \mathbb{R}]$ is a regular locally Lipschitz function.

We denote now $s = (x^1 - \mathscr{x})^{D^*}$ (for its computation see [21]) and observe that

$$L(u, w) = \int_0^T \psi(u)\,dt + w\sqrt{\left(|s(0)|^2 + \int_0^T \dot{s}^2\,dt\right)}$$

provided x corresponds to u with respect to the system equation. For the minimization of L with respect to u for some fixed suitably chosen w a bundle method of the type [96] can be applied. Then, one needs a following statement:

Proposition 12.6. Let $\bar{u} \in L_\infty[0, T, \mathbb{R}^l]$, $\bar{x} \in H^1[0, T, \mathbb{R}^n]$ be the corresponding trajectory, and

$$\alpha = \vartheta \quad \text{if} \quad s = \vartheta$$

$$\alpha = \frac{s}{\|s\|} \quad \text{otherwise.}$$

Let $\varkappa$ be the solution of the adjoint (integral) equation

$$\varkappa(t) + \int_0^t \left[\frac{\partial F(\bar{x}(\tau), \bar{u}(\tau))}{\partial x}\right]^* \varkappa(\tau)\,d\tau = w\,d(t), \quad d = (\alpha, 0, \ldots, 0)^*,$$

with the terminal condition

$$\int_0^1 \left[\frac{\partial F(\bar{x}(t), \bar{u}(t))}{\partial x}\right]^* \varkappa(t)\, \mathrm{d}t = \vartheta .$$

Then,

$$\partial\psi(\bar{u}) + \left[\frac{\partial F(\bar{x}, \bar{u})}{\partial u}\right]^* \varkappa \in \partial_u L(\bar{u}, w) .$$

The proof is merely a combination of the Chain rule II with the general scheme of [16], how to construct adjoint equations. A discrete-time version of Prop. 12.6 has been proved in [102].

The second example is the time-optimal control problem of a nonlinear parabolic system with constraints on stresses. It describes e.g. the seaking of such time-optimal heating of a body which does not damage it. The problem can be posed as follows (due to its complexity we do not use the notation of $(\mathscr{P}3)$):

(12.12a) $$\tau \to \min$$

subj. to

(12.12b) $$\begin{cases} \dfrac{1}{\tau}\dfrac{\partial\gamma(q)}{\partial t} = \Delta q & \text{on } Q = (0,1) \times \Omega , \\ \dfrac{\partial q}{\partial n} = g(q, T) & \text{on } S = (0,1) \times \partial\Omega , \\ q(0, \cdot) = q_0 & \text{on } \Omega , \end{cases}$$

(12.12c) $$\begin{cases} \displaystyle\sum_{j=1}^{3} \frac{\partial\sigma_{ij}(v)}{\partial x_j} = \frac{\partial\beta(q)}{\partial x_i} & \text{on } \Omega \text{ for each } t \in (0,1), \quad i \in \{1,2,3\} , \\ \displaystyle\sum_{j=1}^{3} \sigma_{ij}(v)\, n_j = \beta(q)\, n_i & \text{on } \partial\Omega \text{ for each } t \in (0,1), \quad i \in \{1,2,3\} , \end{cases}$$

(12.12d) $$\varphi(q) \geqq |E(v)|_r \quad \text{for each } (t, x) \in Q ,$$

(12.12e) $$\frac{\partial T}{\partial t} \in [0, c\tau] \quad \text{on } (0,1), \quad T(0) = T_0 , \quad T(1) = T_1 ,$$

where Ω is a sufficiently smooth domain in $\mathbb{R}^3$, $\sigma_{ij}(v) = 2\mu\, e_{ij}(v) + \lambda\delta_{ij} \sum_{k=1}^{3} e_{kk}(v)$, $i, j = 1, 2, 3$, δ_{ij} is the Kronecker symbol and $e_{ij}(v) = \frac{1}{2}((\partial v_i/\partial x_j) + (\partial v_j/\partial x_i))$, $i, j = 1, 2, 3$. $\lambda \geqq 0$, $\mu > 0$, $c > 0$, $k_0 > 0$, $k_1 > 0$, $T_0 \geqq 0$ and $T_1 \geqq 0$ are given constants, $\beta[\mathbb{R} \to \mathbb{R}_+]$, $\gamma[\mathbb{R} \to \mathbb{R}_+]$, $g[\mathbb{R}^2 \to \mathbb{R}]$, $\varphi[\mathbb{R} \to \mathbb{R}_+]$ are given sufficiently smooth functions, $\varphi > k_1$, $\nabla\gamma > k_1$, on $\mathbb{R}$. $g(q, T) = k_0 T^4 + \eta(q)$, where $\eta < 0$, $\nabla\eta < 0$ on $\mathbb{R}_+$, $\lim_{q \to +\infty} \eta(q) = -\infty$ (g corresponds to the Stefan-Boltzmann boundary value condition). E is a linear differential operator from $C_1(\Omega, \mathbb{R}^3)$ into $C_0(\Omega, \mathbb{R}^s)$;

in the general 3-dimensional case $s = 6$ and

$$E : v \mapsto \left(\sqrt{2}\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right), \sqrt{2}\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_3}{\partial x_3}\right), \sqrt{2}\left(\frac{\partial v_2}{\partial x_2} - \frac{\partial v_3}{\partial x_3}\right), \right.$$
$$\left. \sqrt{3}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right), \sqrt{3}\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right), \sqrt{3}\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right)\right). \tag{12.13}$$

(If Ω possesses some special symmetries, smaller s and simpler E can be chosen.)

The appropriate solution q of the system equation (12.12b) is regular in the classical sense (cf. [95]). Combining appropriate results from [99], [100], [113], one can prove the same assertion for (12.12c).

Denoting by Υ the operator assigning q to given (T, τ) according to (12.12b) and Γ the operator assigning v to given q according to (12.12c), we can reformulate the problem (12.12) into the form corresponding to $(\mathscr{P}2)$ as follows:

$$\tau \to \min$$

subj. to

$$\frac{\partial T}{\partial t} \in \langle 0, c\tau \rangle \quad \text{on} \quad \langle 0, 1 \rangle\,, \quad T(0) = T_0\,, \quad T(1) = T_1\,, \tag{12.14}$$
$$\psi(\omega(T, \tau)) \leqq 0 \quad \text{on} \quad Q\,,$$

where ω: $(T, \tau) \mapsto (\varphi \circ \Upsilon(T, \tau), E \circ \Gamma \circ \Upsilon(T, \tau))$, $(T, \tau) \in \boldsymbol{C}_{0,1}(0, 1) \times \mathbb{R}$ is differentiable and $\psi : (y_1, y_2) \mapsto |y_2|_s - y_1$, $(y_1, y_2) \in \mathbb{R} \times \mathbb{R}^s$ is non-diferentiable. Thus in the notation of $(\mathscr{P}2)$ (cf. page 5 of the supplement) $A := \{(T, \tau) \in \boldsymbol{C}_{0,1}(0, 1) \times \mathbb{R} \mid (\partial T/\partial t) \in \in [0, c\tau]$ a. e. in $[0, 1]$, $T(0) = T_0$, $T(1) = T_1$, $\tau \in [0, \tau_0]\}$ for a fixed $\tau_0 >$ inf of problem (12.14) (in such a manner we preserve some compactness of A). D will be the cone of nonnegative functions on Q e.g. considered in $\boldsymbol{C}_0(Q)$ or $\boldsymbol{L}_2(Q)$. In this context, the arguments of the Arzelà-Ascoli type can be used to prove the existence of a solution of the problem (12.14).

Due to the non-smooth character of the problem, the exact penalty method described in this section is adequate to find its solution. The Lagrangian has the form

$$L(T, \tau, w) = \tau + \delta_A(T, \tau) + w \operatorname{dist}(\psi \circ \omega(T, \tau), -D)\,. \tag{12.15}$$

If we take $\boldsymbol{Z} = \boldsymbol{C}_0(Q)$ or $\boldsymbol{L}_2(Q)$, $\operatorname{dist}(\psi \circ \omega(T, \tau), -D) = \|(\psi \circ \omega(T, \tau))^+\|_{\boldsymbol{Z}}$. Thus the second factor of the third term of L is a composition of ω and the function $\Theta : y \mapsto \|(\psi(y))^+\|_{\boldsymbol{Z}}$, $y \in \boldsymbol{Z}^{s+1}$.

For the use of the described bundle methods, it is necessary to find an element of the generalized gradient $\partial_{T,\tau} L$. To obtain it we need to find $\mathscr{X} \equiv \mathscr{X}_{T,\tau} \equiv (\nabla\omega(T, \tau))^*$ and an element of $\partial\Theta$. The operator $\mathscr{X}$ has the form

$$\mathscr{X} : (h, k) \mapsto (\nabla\Upsilon(T, \tau))^* \circ (\nabla\varphi(\Upsilon(T, \tau))^* h + (\nabla\Gamma(\Upsilon(T, \tau)))^* \circ E^* k)\,, \tag{12.16}$$
$$h \in \boldsymbol{Z}^*\,, \quad k \in (\boldsymbol{Z}^s)^*\,.$$

The operator $(\nabla \Upsilon(T, \tau))^*$ assigns the vector $((\partial g/\partial T)(q, T)\tau\tilde{w}/s, \int_Q \Delta q \tilde{w}\, \mathrm{d}x\, \mathrm{d}t)$ to a given real function f on Q, where $q = \Upsilon(T, \tau)$ and $\tilde{w}$ is the solution of the equation

$$(12.17) \quad \nabla\gamma(q)\frac{\partial \tilde{w}}{\partial t} + \tau\,\Delta\tilde{w} = -f \quad \text{on } Q, \quad \tilde{w}(1, \cdot) = 0 \text{ on } \Omega\,, \quad \frac{\partial \tilde{w}}{\partial n} = \frac{\partial g}{\partial q}(q, T)\,.$$

The operator $(\nabla\Gamma(q))^*$ assigns the function $\nabla\beta(q)$ div z to a given function $e[Q \to \mathbb{R}^3]$, where z solves the problem

$$(12.18) \quad \sum_{j=1}^{3} \frac{\partial \sigma_{ij}(z)}{\partial x_j} = -e \quad \text{on} \quad \Omega\,, \quad \sum_{j=1}^{3} \sigma_{ij}(z)\, n_j = 0 \quad \text{on} \quad \partial\Omega\,, \quad i = 1, 2, 3$$
$$\text{for} \quad t \in \langle 0, 1 \rangle\,.$$

Of course, if $\boldsymbol{Z} = C_0(Q)$, e and f need not be functions generally, but (12.17) and (12.18), respectively, remain formally true.

In case $\boldsymbol{Z} = C_0(Q)$, $\partial\Theta$ has the following form: denote $Mf := \{\zeta_0 \in Q \mid f(\zeta_0) = \sup_{\zeta \in Q} f(\zeta)\}$ for a function f defined on Q. Let us remark that $Mf \neq \emptyset$ for $f \in \psi \circ \omega(A)$. It holds (supposing $s = 6$)

$$(12.19) \quad \begin{aligned} \partial\psi^+(y_1, y_2) &= \left(\frac{y_2}{|y_2|_6}, -1\right) \operatorname{sign}\,(|y_2|_6 - y_1)^+ \quad \text{if} \quad 0 \neq |y_2|_6 \neq y_1 \\ &= \left\{\alpha\left(\frac{y_2}{|y_2|_6}, -1\right) \,\middle|\, \alpha \in \langle 0, 1 \rangle\right\} \quad \text{if} \quad 0 \neq |y_2|_6 = y_1 \\ &= \{(c, -1) \operatorname{sign}\,(-y_1)^+ \mid c \in B_6^1(\vartheta)\} \quad \text{if} \quad y_2 = \vartheta\,, \quad y_1 \neq 0 \\ &= \{\alpha(c, -1) \mid \alpha \in \langle 0, 1 \rangle\,, \quad c \in B_6^1(\vartheta)\} \quad \text{if} \quad |y_2|_6 = y_1 = 0 \end{aligned}$$

for ψ from (12.14). To compute it, we have used Prop. 11.3. For $\zeta \in Q$ let us denote by μ_ζ the element of $\boldsymbol{Z}^*$ having the form $\langle \mu_\zeta, f \rangle = f(\zeta)$, $f \in \boldsymbol{Z}$. We denote furthermore by $P(Mf)$ the set of all probability measures on Mf. Using Prop. 11.3 and 11.7, we obtain

$$(12.20) \quad \partial\Theta(h) = \left\{\int_{M\psi^+(h)} \varphi_\zeta(h)\, \mu_\zeta\, \mathrm{d}\nu \mid \varphi_\zeta(h) \in \partial\psi^+(h(\zeta)),\, \nu \in P(M\,\psi^+(h))\right\}, \quad h \in \boldsymbol{Z}^7\,,$$

cf. also [89].

Remark. Optimal control problems are typically defined over the Cartesian product of two spaces. It would be very convenient to work with partial generalized gradients; however, it is possible only in some very special cases, cf. [102].

13. A dual treatment of noncalm problems

There exist very simple nonconvex programs e.g. of the type ($\mathscr{P}$ 2), where f, q are continuously differentiable, but they are noncalm and hence any dualisation presented previously is not applicable. Additionally, necessary optimality conditions of the type of Prop. 11.13 cannot be applied, because we cannot ensure that $r_0 > 0$ and it is well possible that they do not hold at any solution point of the program in question. Smith and VandeLinde have proposed in [110] a dual treatment of a large class of such problems. This approach enables to obtain sufficient optimality conditions of a saddle-point type which may be applied under very weak assumptions. Concerning the numerical solution of the dual problems, as to our knowledge no technique based on generalized gradients (like bundle methods in the Lipschitzian case) is available. The reason lies in the unboundedness of generalized gradients which destroys the convergence proofs. Therefore, one has to confine to methods of the type [114] not using any gradient information and they are, naturally, not very successful.

In this section we will present the theory of [110] in the duality framework of Sec. 8. It will be again applied to the mathematical programming problem ($\mathscr{P}$2), where D is a convex cone with the vertex at the origin. We show that the sharpness of $\psi(p, w)$ at $p = \vartheta$ is not always necessary if one wants to derive a usable duality scheme.

Assume that the constraint space $\boldsymbol{Z}$ in ($\mathscr{P}$2) is normed, $\boldsymbol{V} = \boldsymbol{Z}$ and Φ is given by (12.3). Then clearly the extremal-value function h is nonincreasing with respect to the partial ordering given by D. Let $\boldsymbol{W} = \{(v_0, v_1) \mid v_0 \geqq 0,\ v_1 > 0\}$ and

$$\psi_4(p, w) = v_0 \|(p)^D\|^{v_1} . \tag{13.1}$$

Definition 13.1. ψ is said to be *D-sharp* at $\vartheta \in \boldsymbol{V}$ if for all $\bar{w} \in \boldsymbol{W}$, all neighbourhoods $\mathcal{O}$ of $-D$, all $\mu \in \mathbb{R}$ and all $\varepsilon > 0$ there is a $\hat{w} \in \boldsymbol{W}$ and a neighbourhood $\hat{\mathcal{O}}$ of $-D$, $\hat{\mathcal{O}} \subset \mathcal{O}$ such that

$$\psi(p, \hat{w}) \geqq \psi(p, \bar{w}) + \mu \quad \forall p \in \boldsymbol{V} \setminus \hat{\mathcal{O}} \tag{13.2}$$

$$\psi(p, \hat{w}) \geqq -\varepsilon \qquad \forall p \in \hat{\mathcal{O}} \tag{13.3}$$

and

$$\psi(p, w) = 0 \quad \text{on} \quad -D \times \boldsymbol{W} . \tag{13.4}$$

We remark that by the neighbourhood of a set $A \subset \boldsymbol{V}$ we mean an arbitrary set $\mathcal{O}$ containing $B_V^\varepsilon(A)$ for some $\varepsilon > 0$ where

$$B_V^\varepsilon(A) = \{v \in V \mid \operatorname{dist}(v, A) \leqq \varepsilon\} .$$

Clearly, for $D = \vartheta$ the D-sharpness of ψ at ϑ becomes the sharpness of ψ at ϑ in the sense of Def. 8.2.

This generalization may be exploited as follows:

Proposition 13.1. Let $h(\vartheta)$ be finite, h possess some ψ-minorant and satisfy the condition

(13.5) $\quad \forall \varepsilon > 0 \quad \exists \eta > 0$ such that $x \in B_V^\eta(-D) \Rightarrow h(x) \geqq h(\vartheta) - \varepsilon$.

Moreover, let ψ be D-sharp at ϑ. Then h is lower ψ-regular at ϑ.

Proof. We may proceed exactly as in the proof of Prop. 8.4 with ψ_ε given by

$$\psi_\varepsilon(p) = \begin{cases} h(\vartheta) - 2\varepsilon - \psi(p, \hat{w}) & \text{for} \quad p \in X \setminus (-D) \\ h(\vartheta) - 2\varepsilon & \text{for} \quad p \in -D\,. \end{cases}$$

□

Remark. Condition (13.5) is satisfied e.g. if h is uniformly l.s.c. at $\partial(-D)$, i.e.

$$\forall \varepsilon > 0 \quad \exists \eta > 0 \quad \text{such that} \quad \forall x_0 \in \partial(-D) \quad \forall x \in B_V^\eta(x_0)$$
$$h(x) - h(x_0) \geqq -\varepsilon\,.$$

Let us now investigate the D-sharpness of ψ_4 given by (13.1).

Proposition 13.2. Let for any $p \in V$ the following implications hold:

(13.6) $\quad \forall\ \xi > 0 \quad \exists\ \delta > 0 \quad$ such that $\operatorname{dist}(p, -D) \geqq \xi \Rightarrow \|(p)^D\| > \delta$,

(13.7) $\quad p \in -D \Rightarrow (p)^D = \vartheta$.

Then the function ψ_4 defined by (13.1) is D-sharp at ϑ.

Proof. Clearly, it remains to prove only that for a suitable couple $((\hat{v}_0, \hat{v}_1)$ and $\hat{\mathcal{O}} \subset \mathcal{O}$

$$\hat{v}_0\|(p)^D\|^{\hat{v}_1} \geqq \bar{v}_0\|(p)^D\|^{\bar{v}_1} + \mu \quad \forall p \in V \setminus \hat{\mathcal{O}}\,,$$

where $\bar{w} = (\bar{v}_0, \bar{v}_1)$, $\hat{w} = (\hat{v}_0, \hat{v}_1)$ and all other symbols remain as in Def. 13.1. But indeed, if for all $p \in V \setminus \hat{\mathcal{O}}$ $\operatorname{dist}(p, -D) \geqq \xi > 0$, then it suffices to set

$$\hat{v}_0 = \bar{v}_0 + \frac{\mu}{\delta^{\bar{v}_1}}, \quad \hat{v}_1 = \bar{v}_1$$

due to the condition (13.6). □

Remark. If V is a Hilbert space, conditions (13.6), (13.7) are implied by $D = D^*$. Indeed, due to Lemma 5.1 we have then for any $p \in H$

$$p = (p)^D + (p)^{-D} \quad \text{and dist} \quad (p, -D) = \|(p)^D\|\,.\,.$$

Also for $V = C_0(K)$ or $L_q(\Omega)$, $q \in [1, +\infty]$, where $K, \Omega \subset \mathbb{R}^n$ are compact and measurable, respectively, conditions (13.6), (13.7) hold provided D is the cone of nonnegative functions and we set $(x)^D = x^+$, where $x^+ : t \mapsto (x(t))^+$.

By definition

(13.8) $\quad L(x, w) = f(x) + \delta_A(x) + v_0 \inf_{v \in D} \|(v + q(x))^D\|^{v_1}$

and again, as in Sec. 12, some additional requirements are needed to remove the infimum of the right-hand side.

Proposition 13.3. Let the condition (13.7) be satisfied and, moreover, the function $\|(\cdot)^D\|$ be subaditive. Then

$$\inf_{v \in D} \|(v + q(x))^D\| = \|(q(x))^D\| . \tag{13.9}$$

Proof. For all $v \in D$

$$\|(v + q(x) - v)^D\| \leqq \|(v + q(x))^D\| + \|(-v)^D\| = \|(v + q(x))^D\| . \qquad \square$$

Corollary 13.3.1. The equality (13.9) holds provided either
(i) V is Hilbert and $D = D^*$,
or
(ii) $V = C_0(K, \mathbb{R}^n)$ or $L_q(\Omega, \mathbb{R}^n)$, $q \in [1, +\infty]$, D is the cone of nonnegative (vector-valued) functions and $(x)^D = x^+$ (componentwise).

In the rest of the section it is supposed that one of the assumptions of the preceding corollary is satisfied. The dual problem attains then the form

$$(\mathscr{D}2) \qquad \begin{gathered} \inf_{x \in A} [f(x) + v_0 \|(q(x))^D\|^{v_1}] \rightarrow \sup \\ \text{subj. to} \\ (v_0, v_1) \in W . \end{gathered}$$

An eventual duality gap vanishes provided h satisfies the conditions of Prop. 13.1 – however, they may be rather hardly verifiable. Fortunately, the investigated dualisation is the only nonconvex one in our supplement which possesses a rather simple stability criterion expressed in terms of f, q and D and hence in many cases directly verifiable. Therefore, we turn our attention to the question of ψ_4-stability of $(\mathscr{P}2)$ (with respect to Φ given by (12.3).

Definition 13.2. Problem $(\mathscr{P}2)$ is *exponentially stable* if $h(\vartheta)$ is finite and

$$\inf_{p \neq \vartheta} \frac{h(p) - h(\vartheta)}{\|p\|^\alpha} > -\infty .$$

Proposition 13.4. There exists a scalar $\hat{v}_0 \geqq 0$ such that $(\hat{v}_0, \hat{v}_1) \in \partial^{\cap}_{\psi_4} h(\vartheta)$ iff problem $(\mathscr{P}2)$ is exponentially stable with $\alpha = \hat{v}_1$.

Proof. We will prove only the "sufficiency" part of the statement. For the rest we refer to [110]. Let us consider first the Hilbert case.

For $p \in -D$ we have $h(p) \geqq h(\vartheta)$. Also for such p $\psi_4(p, w) = 0$ $(D = D^*)$ and we immediately have the subgradient inequality

$$h(p) \geqq h(\vartheta) - \psi_4(p, w) \quad \text{for} \quad p \in -D \text{ and } w \in W .$$

Let $p \in (Z \setminus (-D))$. Then, clearly $p - (p)^{-D} \in D$ and $h(p - (p)^{-D}) \leqq h(p)$ by the monotonicity of h. Consequently,

$$\frac{h(p) - h(\vartheta)}{\|(p)^D\|^{\vartheta_1}} \geqq \frac{h(p - (p)^{-D}) - h(\vartheta)}{\|(p)^D\|^{\vartheta_1}} = \frac{h((p)^D) - h(\vartheta)}{\|(p)^D\|^{\vartheta_1}} > -\infty$$

by the exponential stability of ($\mathscr{P}2$).

Concerning the requirement (ii) of Cor. 13.3.1, realize that for $x^- : t \mapsto \min(0, x(t))$

$$p - (p)^{-D} = p - (p)^- = p^+ = (p)^D$$

so that one may proceed along the same lines as in the Hilbert case. □

In what follows we exploit similarly as in Sec. 11 the notation $d_B(x) = \operatorname{dist}(x, B)$ for a set $B \subset V$. Moreover, we denote

$$S_p = \{x \in A \mid p - q(x) \in D\}.$$

In the following two lemmas it suffices to suppose that V is a normed linear space only.

Lemma 13.1. For every convex set $K \subset V$ and each $v_0 \in V$

$$d_K(v_0) = \max_{v^* \in B^1_{V^*}(\vartheta)} [\langle v^*, v_0\rangle - \delta^*_K(v^*)]. \tag{13.10}$$

Proof. For each $v^* \in B^1_{V^*}(\vartheta)$ $\langle v^*, v_0\rangle - \delta^*_K(v^*) \leqq d_K(v_0)$. The equality for a suitable $v^*_0 \in B^1_{V^*}(\vartheta)$ we prove for $v_0 = \vartheta$ (it corresponds to a suitable shift of the coordinates). Due to the appropriate separation theorem, there is $\tilde{v}^* \in V^*$ such that $\inf_{v \in Q_0} \langle \tilde{v}^*, v\rangle_V = \sup_{v \in K} \langle \tilde{v}^*, v\rangle_V = \varkappa < 0$ for $Q_0 := \{v \in V \mid \|v\|_V \leqq d_K(v_0)\}$.

Taking $v^*_0 = \tilde{v}^*[d_K(v_0)/-\varkappa]$ we obtain $v^*_0 \in B^1_{V^*}(\vartheta)$ and $\delta^*_K(v^*_0) = -d_K(v_0)$. □

Lemma 13.2. Let K be a convex cone in V. Then d_{-K} is nondecreasing with respect to the ordering generated on V by K.

Proof. Let $v_2 - v_1 \in K$. By Lemma 13.1 there is $v^*_1 \in B^1_{V^*}(\vartheta)$ such that $d_{-K}(v_1) = \langle v^*_1, v_1\rangle_V - \delta^*_{-K}(v^*_1)$. As $\delta^*_{-K} = \delta_{K^*}$,

$$d_{-K}(v_2) \geqq \langle v^*_1, v_2\rangle_V \geqq \langle v^*_1, v_1\rangle_V = d_{-K}(v_1).$$ □

Proposition 13.5. Let V be a reflexive strictly convex Banach space (instead of supposition (i) of Cor. 13.3.1) or let supposition (ii) of Cor. 13.3.1 hold. Then ($\mathscr{P}2$) is exponentially stable iff there is $\alpha \in (0, 1]$ such that

$$\inf_{\substack{x \in A \\ -q(x) \notin D}} \frac{f(x)}{(d_{-D}(q(x)))^\alpha} > -\infty. \tag{13.11}$$

Proof. We shall suppose $h(\vartheta) = 0$ and define $A_0 := \{x \in A \mid f(x) < 0\}$. Suppose $A_0 \neq \emptyset$ (the case $A_0 = \emptyset$ is trivial). Clearly $A_0 \cap S_\vartheta = \emptyset$ and

$$\inf_{p \in V} \frac{h(p)}{\|p\|^\alpha_V} = \inf_{p \notin -D} \inf_{x \in A_0 \cap S_p} \frac{f(x)}{\|p\|^\alpha_V} \equiv \Xi.$$

As by Lemma 13.2. $\|p\|_V \geqq d_{-D}(p) \geqq d_{-D}(q(x))$ for each $x \in S_p$,

$$\Xi = \inf_{p\in V} \frac{h(p)}{\|p\|_V^\alpha} \geqq \inf_{p\notin -D} \inf_{x\in A_0} \frac{f(x)}{(d_{-D}(q(x)))^\alpha} \geqq \inf_{x\in A\setminus S_\vartheta} \frac{f(x)}{(d_{-D}(q(x)))^\alpha}.$$

Conversely,

$$\inf_{x\in A\setminus S_\vartheta} \frac{f(x)}{(d_{-D}(q(x)))^\alpha} \geqq \inf_{x\in A\setminus S_\vartheta} \frac{h(q(x))}{\|q(x) - (q(x))^{-D}\|_V^\alpha} \geqq \inf_{x\in A\setminus S_\vartheta} \frac{h(q(x) - (q(x))^{-D})}{\|q(x) - (q(x))^{-D}\|_V^\alpha} \geqq \inf_{p\in V} \frac{h(p)}{\|p\|_V^\alpha},$$

because $q(x) - (q(x))^{-D} \geqq q(x)$. □

Remark. Proposition 13.5 holds even for the case V is a normed linear space ([110]). Of course, then the projection does not exist in general.

Prop. 8.10 provides us now with the following valuable necessary and sufficient optimality conditions:

Proposition 13.6. Let condition (13.11) hold with $\alpha = \hat{v}_1$. Then for a suitable finite scalar $\hat{v}_0 \geqq 0$ $\hat{x}$ solves $(\mathscr{P}2)$ iff

$$L(\hat{x}, v_0, v_1) \leqq L(\hat{x}, \hat{v}_0, \hat{v}_1) \leqq L(x, \hat{v}_0, \hat{v}_1) \qquad (13.12)$$
$$\text{for all} \quad x \in \boldsymbol{X}, \quad v_0 \geqq 0, \quad v_1 > 0.$$

Proof. The exponential stability of $(\mathscr{P}2)$ with $\alpha = \hat{v}_1$ ensured by (13.11) implies the existence of $\hat{v}_0 \geqq 0$ such that $(\hat{v}_0, \hat{v}_1) \in \partial^\frown_{\psi_4} h(\vartheta)$ due to Prop. 13.4. Hence $(\hat{v}_0, \hat{v}_1)$ solves $(\mathscr{D}2)$ and an eventual duality gap vanishes. Therefore, by definition, $\Phi(\hat{x}, \cdot)$ is lower ψ_4-regular at ϑ $(h(\vartheta) = \Phi(\hat{x}, \vartheta)$, $h^\frown(\hat{v}_0, \hat{v}_1) = \Phi^\frown(\vartheta, \hat{v}_0, \hat{v}_1))$ and the assertion of (8.10) completes the proof. □

It is evident that (similarly as in the dualisations of Secs. 9 and 12) if $(\hat{v}_0, \hat{v}_1)$ solves $(\mathscr{D}2)$ then $(v_0, \hat{v}_1)$ solves $(\mathscr{D}2)$ as well provided $v_0 > \hat{v}_0$.

Proposition 13.7. Let $(\hat{v}_0, \hat{v}_1)$ solve $(\mathscr{D}2)$, $v_0 > \hat{v}_0$ and $\hat{x} \in \arg\inf_{x\in X} L(x, v_0, \hat{v}_1)$. Then $\hat{x}$ solves $(\mathscr{P}2)$.

Proof. For $v_0 = \hat{v}_0$ the infimum of $h(p) + v_0\|(p)^D\|^{\hat{v}_1}$ is attained at $p = \vartheta$ by the assumption. For $v_0 > \hat{v}_0$ this infimum cannot be attained outside $-D$. Thus $-q(\hat{x}) \in D$. □

We demonstrate now the application of Prop. 13.6 on a simple mathematical program from [110] which does not possess any Kuhn-Tucker vector in the sense of Def. 9.3. Solve

$$f(x) = \tfrac{1}{3} - x \to \inf$$
$$\text{subj. to} \qquad (13.13)$$
$$q(x) = 9x(3x^2 - 3x + 1) - 1 \leqq 0, \quad x \in \mathbb{R}.$$

This problem is exponentially stable for $\alpha = \frac{1}{3}$. Due to Prop. 13.4 $(v_0, \frac{1}{3})$ solves the corresponding dual problem, provided v_0 is found by solving

$$\max_{v_0 \geqq 0} \inf_x \{\tfrac{1}{3} - x + v_0[(9x(3x^2 - 3x + 1) - 1)^+]^{1/3} \mid x \in \mathbb{R}\} .$$

Noting that $q(x) = 27(x - \frac{1}{3})^3$, the above expression becomes the form

$$\max_{v_0 \geqq 0} \inf_x \{\tfrac{1}{3} - x + 3v_0 (x - \tfrac{1}{3})^+ \mid x \in \mathbb{R}\} = \max_{v_0 \geqq 0} \begin{Bmatrix} -\infty & \text{if} & v_0 < \frac{1}{3} \\ 0 & \text{if} & v_0 \geqq \frac{1}{3} \end{Bmatrix},$$

It can be easily seen that e.g. $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a saddle point of the Lagrangian; hence $\hat{x} = \frac{1}{3}$ is a solution of (13.13) due to Prop. 13.6. Alternatively, we may take any $v_0 \geqq \frac{1}{3}$ and minimize

$$L(x, v_0, \tfrac{1}{3}) = \tfrac{1}{3} - x + 3v_0(x - \tfrac{1}{3})^+$$

which provides us again with $\hat{x} = \frac{1}{3}$ due to Prop. 13.7.

We close this last section of our supplement with a simple optimal control problem on which we illustrate not only the application of the presented dualisation, but simultaneously a simple use of the extended calculus of Clarke. So, let in $(\mathscr{P}3)$ $\boldsymbol{U} = \boldsymbol{L}_\infty[0, T, \mathbb{R}^m], \boldsymbol{Y} = \mathbb{R}^n$,

$$J(u) = \int_0^T f_0(u(t))\,\mathrm{d}t$$

with $f_0[\mathbb{R}^m \to \mathbb{R}]$, Λ assign to each $u \in \boldsymbol{U}$ the terminal state $y(T)$ of the differential equation

$$\dot{y}(t) = f(y(t), u(t))$$

with the initial condition $y(0) = a$ and $f[\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n]$, $M = \boldsymbol{U}$ and $N = \{z \in \mathbb{R}^n \mid -q(z) \in \mathbb{R}_+\}$, $q[\mathbb{R}^n \to \mathbb{R}]$. We suppose that functions f_0, f and q are continuously differentiable and transcribe the problem (as usual) into the mathematical programming form

$$\begin{aligned} &J(u) \to \inf \\ \text{subj. to}\quad & \\ &q(y(1)(u)) \leqq 0 \\ &u \in L_\infty[0, 1, \mathbb{R}^m] . \end{aligned} \tag{13.14}$$

Provided (13.14) is exponentially stable for some $\alpha \in (0, 1]$ we may solve it by way of its dual

$$\max_{v_0 \geqq 0} \inf_{u \in \boldsymbol{U}} \{J(u) + v_0[(q(y(1)(u)))^+]^\alpha\} , \tag{13.15}$$

and the existence of maximizing $\hat{v}_0 \geqq 0$ is ensured by Prop. 13.4.

Proposition 13.8. Let $(\hat{v}_0, \alpha)$, $\alpha \in (0,1)$ solve (13.15) and $\hat{y}$ be the trajectory corresponding to a control $\hat{u}$ with respect to the system equation. Let the system be locally controllable at $(\hat{u}, \hat{y})$, i.e. we assume that for any $h(T) \in \mathbb{R}^n$ there is $k \in \boldsymbol{L}_\infty[O, T, \mathbb{R}^n]$

such that $h(T)$ is the terminal state of the linearized equation

$$\dot{h}(t) = \frac{\partial f(\hat{y}(t), \hat{u}(t))}{\partial y} h(t) + \frac{\partial f(\hat{y}(t), \hat{u}(t))}{\partial u} k(t) \quad \text{a.e. on } [0, 1], \; h(0) = \vartheta .$$

Finally, let $\nabla q(\hat{y}(T)) \neq \vartheta$. Then the following condition is necessary for $\hat{u}$ to be a solution of (13.14):

$$\nabla f_0(\hat{u}(t)) - \frac{\partial f(\hat{y}(t), \hat{u}(t))^T}{\partial u} p(t) = \vartheta \quad \text{on} \quad [0, T], \tag{13.16}$$

where p is the solution of the adjoint equation

$$\dot{p}(t) = -\frac{\partial f(\hat{y}(t), \hat{u}(t))^T}{\partial y} p(t) \quad \text{a.e.} \quad \text{on } [0, T] \tag{13.17}$$

backwards on $[0, T]$ from the terminal condition

$$p(T) = \begin{cases} \vartheta & \text{if} \quad q(\hat{y}(T)) < 0 \\ -\beta \, \nabla q(\hat{y}(T)) & \text{if} \quad q(\hat{y}(T)) = 0 \end{cases} \tag{13.18}$$

for some $\beta \in [0, \infty]$.

Proof. Let us denote for $x \in \mathbb{R}$ $\varphi(x) = v_0(x^+)^\alpha$. Evidently, $\varphi^0(0; 1) = D\,\varphi(0; 1) = +\infty$, $\varphi^0(0; -1) = D\,\varphi(0; -1) = 0$.

The rest consists in combining the chain rule of Prop. 11.11 with the standard way of constructing the adjoint equations in the same way as in [102] for the optimal control problem with the objective

$$J(u) + \varphi(q(y(T)(u))) . \qquad \square$$

Remark. As $D\,\varphi(0; 1) = +\infty$, we could solve by way of (13.15) even problems where $\nabla q(\hat{y}(T)) = \vartheta$ for $\hat{y}$ being an optimal trajectory. However, in such a case, we cannot use even the extended calculus to state the optimality conditions in the form (13.16)–(13.18) because the requirements of Prop. 11.11 cannot be met.

CONCLUSION

As we have already mentioned at various places of the supplement, there is a lot of significant duality concepts and important duality applications in mathematical programming which could not be described here due to the extent limitations. Let us still mention e.g. the duality in semi-infinite programming (cf. [85]), in discrete programming (cf. [116]) or the important applications of Lagrange dualisation in linear and quadratic programming. Also our reference list is by far not exhaustive; it is a small fraction of the literature devoted to duality theory and its applications. On the other hand both the convex and nonconvex perturbational theories of duality have been discussed in a sufficient extent so that any reader could, with the help of them, construct and investigate his own general or specialized duality schemes.

During the whole paper we have emphasised the usage of single dualisations for the numerical solution of extremal problems by way of their (unconstrained or easily constrained) duals. Let us conclude, therefore, by a few remarks concerning the actual numerical solution. In Institute of Information Theory and Automation (ÚTIA ČSAV) a package of codes has been collected based on three "core" routines for the actual minimization:

(i) COGOMP – for the minimization of continuously differentiable functions with respect to linear constraints;

(ii) CONWOL – for the unconstrained minimization of convex continuous functions;

(iii) BOREPS – for the minimization of weakly semismooth (cf. [98]) functions in presence of constraints in the form of upper and lower bounds.

The first code has been constructed by Z. Schindler on the basis of Beale's conjugate gradient method. The second and the third code are typical bundle methods and have been written by Cl. Lemaréchal. Besides these "core" routines the package contains a number of auxiliary codes for treating the state-space constraints in optimal control problems with systems described by ordinary differential equations. They use the Fenchel dualisation, shifted penalty approach or exact penalties with norms in $\boldsymbol{H}^1$, $\boldsymbol{C}_0, \boldsymbol{L}_1$. Some numerical experience collected by solving practical problems can be found in [61], [91], [92], [93]. [108]. The package is currently under development as we wish to be able to solve distributed-parameter constrained control problems as well. We also wish to add still one "core" routine for minimization of quasidifferentiable functions based on the ideas of [107], [115] especially with respect to nonsmooth optimal control problems.

APPENDIX

As the notion of the rotundity is not sufficiently widely known, we will give the proof of Prop. 1.11. Without a loss of generality we suppose $u = \vartheta, u^* = \vartheta, f(\vartheta) = 0$ throughout the proof and, moreover $\nabla^G f^*(\vartheta) = \vartheta$ under the differentiability assumption.

For f weakly rotund at ϑ relative to $\vartheta \in V^*$ we have $f^*(\vartheta) = 0$, $f^* \geqq 0$ on V^*, hence $\vartheta \in \partial f^*(\vartheta)$. Let for some $v^* \in V^*$, $\eta > 0$ and a positive sequence $\lambda_n \downarrow 0$ $\eta \leqq \leqq f^*(\lambda_n v^*)/\lambda_n$. The rotundity yields the existence of $\mu > 0$ such that

$$\overline{\lim_{\substack{v \in V \\ |\langle v^*, v\rangle| \to +\infty}}} \; [|\langle \mu v^*, v\rangle| - f(v)] < 0 \,.$$

Hence for every $\lambda \in \langle 0, \mu\rangle$ $f^*(\lambda v^*) \in \mathbb{R}$. We take a sequence $\{v_n\} \subset V$ such that $f^*(\lambda_n v^*) \leqq \lambda_n(\langle v^*, v_n\rangle + 1/n)) - f(v_n)$, $n \geqq n'_0$. As

$$(*) \qquad \mu f(v_n)/\lambda_n \leqq \mu\langle v^*, v_n\rangle - \mu\eta + \mu/n \,, \quad n \in \mathbb{N} \,,$$

then there is $K \in \mathbb{R}$ and $n_0 \geqq n_0'$ such that $K \geqq \mu\langle v^*, v_n \rangle \geqq \frac{1}{2}\mu\eta$, $n \in \mathbb{N}$, $n \geqq n_0$. Thus there is δ such that $f(v_n) \geqq \delta$, hence $\lim_{n \to +\infty} \mu f(v_n)/\lambda_n = +\infty$ which contradicts $(*)$. In the converse case clearly $f^*(\vartheta) = 0$. Let for a subset $\{u_1^*, \ldots, u_r^*\} \subset V^*$, $\delta > 0$ and for each $n \in \mathbb{N}$ an element $v_n \in V$ exist such that $f(v_n) \leqq 1/n$ and

$$(**) \qquad \langle u_j^*, v_n \rangle \geqq \delta \quad \text{for a suitable} \quad j = j_n \,.$$

Then there is $j_0 \in \{1, \ldots, r\}$ such that $(**)$ holds for an infinite subset $M \subset \mathbb{N}$. Then $f^*(\lambda u_{j_0}^*) \geqq \sup_{n \in M} [\lambda\langle u_{j_0}^*, v_n \rangle - f(v_n)] \geqq \lambda\delta$, hence $D f^*(\vartheta; u_{j_0}^*) \geqq \delta$.

Let V be a Banach space and $\nabla f^*(\vartheta) = \vartheta$. Let there are $\eta > 0$ and $\{v_n\} \subset V$ such that $\|v_n\| \geqq \eta$ and $f(v_n) \leqq 1/n^2$. Taking $v_n^* \in V^*$ with $\|v_n^*\| = 1/n$ and $\langle v_n^*, v_n \rangle = \eta/n$, we obtain

$$\frac{f^*(v_n^*)}{\|v_n^*\|} \geqq \langle v_n^*/\|v_n^*\|, v_n \rangle - f(v_n)/\|v_n^*\| \geqq \eta - 1/n \,.$$

Conversely, if f is norm rotund at ϑ relative to ϑ, then $f^*(\vartheta) = 0, f^* \geqq 0$ on V^*. Put $\varphi(\mu) = \inf \{f(v) \mid v \in V, \|v\| \geqq \mu\}$. We have

$$0 \leqq \varliminf_{v^* \to 0} \frac{f^*(v^*)}{\|v^*\|} \leqq \varlimsup_{v^* \to 0} \frac{f^*(v^*)}{\|v^*\|} = \inf_{\varepsilon > 0} \sup_{\lambda \in (0, \varepsilon)} \sup_{v^* \in B^1_{V^*}(\vartheta)} \sup_{v \in V} [\langle v^*, v \rangle - f(v)/\lambda] \leqq$$

$$\leqq \inf_{\varepsilon > 0} \sup_{\mu \geqq 0} \sup_{\lambda \in (0, \varepsilon)} \left(\mu - \frac{\varphi(\mu)}{\lambda}\right) = \inf_{\varepsilon > 0} \sup_{\mu \geqq 0} \left(\mu - \frac{\varphi(\mu)}{\varepsilon}\right).$$

From the rotundity assumption there is δ_n such that $\varphi(\mu) \geqq \delta_n$ for $\mu \geqq 1/n$. As φ is positive, increasing and convex on $\mathbb{R}_+$, $\varphi(\alpha/n) \geqq \alpha\delta_n$ for $\alpha \geqq 1$. Thus for $\varepsilon < n\delta_n$ and $\mu \geqq 1/n$ $\varphi(\mu) \geqq n\mu\delta_n \geqq \varepsilon\mu \geqq \varepsilon(\mu - 1/n)$. If $\mu \leqq 1/n$, then $\varepsilon(\mu - 1/n) \leqq 0 \leqq \varphi(\mu)$. Hence, $\inf_{\varepsilon > 0} \sup_{\mu \geqq 0} (\mu - \varphi(\mu)/\varepsilon) = 0$. □

In [1] more general assertions than Props. 1.10, 1.11 have been proved with the help of a specially developed apparatus.

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