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ESTIMATION OF A CENTRALITY PARAMETER AND RANDOM SAMPLING TIME

Part I. Necessary Conditions for Optimality

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A weakly stationary discrete time process is sampled according a renewal process. A centrality parameter is estimated. We establish necessary conditions for the samplig law to be optimal w.r.t. a criterion based on the estimator's asymptotic variance.

0. INTRODUCTION

Random sampling scheme in the determination of the values of parameters indexing observations has been studied in several publications ([4]).

The random sampling time scheme is defined as a renewal process $T = (t_n)_{n\in\mathcal{N}}$, *independent* in probability of the process modelling the observations.

The results presented here deal with the estimation of centrality parameters θ of real discrete time weakly stationary random processes. As an estimator we choose the empirical mean or the Gauss-Markov estimator, based on n observations of the sampled process.

Our purpose is the determining of a sampling distribution L_0, optimal with respect to an asymptotic quadratic criterion in the set $\mathcal{P}_m$, the sampling rate of which is large enough.

The main difficulty in this problem consists in the *non convexity* of the function we have to optimize. One is then within the framework of a differential programming concerned with convex constraints; it makes possible to prove a *necessary* condition of Kuhn-Tucker type ([10], [12]).

There is a small number of publications about parameter estimation by random sampling time methods; we may mention as examples works about estimation of diffusion processes parameters ([15], [14], [7]).

1. DEFINITIONS AND NOTATIONS

Afterwards $X = (X_n)_{n\in\mathscr{Z}}$ will be a real weakly stationary discrete time random process with mean θ, covariance function C_X and correlation function ϱ_X.

Let us denote by $T = (t_n)_{n\in\mathscr{N}}$ with $t_0 = 0$ a renewal process on $\mathscr{N}$, independent in probability of X, which we choose to call the Sample Process. The probability distribution L of $t_{n+1} - t_n$ is defined by:

$$L_j = \mathsf{P}(\{t_{n+1} - t_n = j\}), \quad j \in \mathscr{N}^* = \mathscr{N} - \{0\} \tag{1}$$
$$L_0 = 0$$

(the process always moves forward!)

The process which has undergone sampling, which we choose to call the Sampled Process $\tilde{X} = (\tilde{X}_n)_{n\geq 0}$, is defined by $\tilde{X}_n = X_{t_n}$, $n \in \mathscr{N}$; it is also a real weakly stationary random process with mean θ and with covariance function $C_{\tilde{X}}$ defined by:

$$C_{\tilde{X}}(0) = C_X(0)$$
$$C_{\tilde{X}}(k) = \sum_{j=k}^{\infty} L^{*k}(j)\, C_X(j), \quad k \in \mathscr{N} \tag{2}$$

- L^{*k} is the distribution of $t_{n+k} - t_n$, $n \geqq 0$, $k \geqq 1$; we set $L^{*k}(i) = 0$ for $k \geqq i$; $*$ represents the convolution.
- $Q_L(m) = \sum_{k=1}^{m} L^{*k}(m)$, $m \in \mathscr{N}^*$, is the potential distribution associated with T; $Q_L(m) \leqq 1$ for all $m \in \mathscr{N}^*$.
- l^1 is the set of real sequences $x = (x_n)$ such that $\sum |x_n| < \infty$.
- l^∞ is the set of bounded real sequences; it is the dual space of l^1.
- $\langle , \rangle$ is the scalar product of the duality (l^1, l^∞).
- $\mathscr{P}$ is the set of probability distribution on $\mathscr{N}$.
- $\mathscr{P}_m = \{L \in \mathscr{P} \mid \sum_{j\in\mathscr{N}} jL_j \leqq m\}$, $\quad m \geqq 1$.
- δ_i^j is the Kronecker symbol.
- $e_i = (\delta_i^j)_{j\in\mathscr{N}}$, $\quad i \in \mathscr{N}$.
- δ_j is the Dirac distribution on $j \in \mathscr{N}$.
- $\operatorname{supp} L = \{j \in \mathscr{N} \mid L(j) \neq 0\}$ is the support of L.
- Card A is the cardinality of a set A.
- L^1 is the set of real sequences $X = (X_j)$ such that $\sum_{j=1}^{\infty} j|X_j| = \|X\|_{L^1} < +\infty$.
- L^∞ is the dual space of L^1.
- $L^\infty_+ = \{Z \in L^\infty \mid Z_j \geqq 0, j \in \mathscr{N}\}$.
- $\langle\!\langle X, Z \rangle\!\rangle = \sum_{j=1}^{\infty} jX_jZ_j$ is the scalar product of the duality (L^1, L^∞).
- $\Phi_L(z) = \sum_{j=1}^{\infty} L_j z^j$ is the z-transform of the distribution L.
- $\hat{C}_{\tilde{X}}(z)$ is the z-transform of $C_{\tilde{X}}$.

2. THE PROBLEM

The question is to estimate the mean θ of $X = (X_n)_{n \in \mathscr{Z}}$ a real weakly stationary random process, by random sampling time of observations, with a renewal process T defined in (1).

For the estimation of θ we use N observations the instants of observation of which are random sampled and measured during a research.

The chosen estimator is the empirical mean $\hat{\theta}_L(N)$ denoted:

$$\hat{\theta}(N) = \frac{1}{N} \sum_{n=1}^{N} \tilde{X}_n .$$

The quadratic criterion is:

$$a(L) = \lim_{N \to \infty} (N \operatorname{var} \hat{\theta}(N)) . \tag{3}$$

The problem is the determination of a distribution L of the sampler Process T step that minimizes $a(L)$, where L belongs to a subset of $\mathscr{P}$ further specified. Now we introduce several lemmas.

Lemma 2.1. If $C_X \in l^1$, then:

i) $C_{\tilde{X}} \in l^1$.

ii) $$\lim_{N \to \infty} (N \operatorname{var} \hat{\theta}(N)) = \hat{C}_{\tilde{X}}(1) = \sum_{n=-\infty}^{+\infty} [C_{\tilde{X}}(|n|)] = C_X(0) [1 + 2\langle \varrho_X, Q_L \rangle] . \tag{4}$$

Proof. i) From (2) we prove that:

$$\forall p \geqq 1: \sum_{n=1}^{p} |C_{\tilde{X}}(n)| \leqq \sum_{n=1}^{p} [\sum_{k=n}^{\infty} L^{*n}(k) |C_X(k)|] \leqq \sum_{k=1}^{\infty} [|C_X(k)| \sum_{i=1}^{p} L^{*i}(k)]$$

Now: $\sum_{i=1}^{p} L^{*i}(k) \leqq Q_L(k) \leqq 1$.

Consequently: $\forall p \geqq 1: \sum_{n=1}^{p} |C_{\tilde{X}}(n)| \leqq \sum_{n=1}^{\infty} |C_X(n)|$ and i) is proved and more: $\|C_{\tilde{X}}\|_{l_1} \leqq \leqq \|C_X\|_{l_1}$.

ii) We know (cf. [1], p. 578) that:

$$\lim_{N \to \infty} (N \operatorname{var} \hat{\theta}_N = \sum_{n=-\infty}^{+\infty} [C_{\tilde{X}}(n)]$$

and the last equality is immediately proved. □

Let $m \in \mathscr{R}_+$ and $\mathscr{P}_m = \{L \in \mathscr{P} \mid \sum_{j=1}^{\infty} j L_j \leqq m\}$ be the set of sampling distributions with a sampling rate higher than $1/m$ with $m \geqq 1$. Now we prove that if the sampling rate is higher than $1/m$, there is an optimal sampling distribution:

Lemma 2.2. If $C_X \in l^1$, then the restriction to $\mathscr{P}_m$ of the function $F: \mathscr{P} \mapsto \mathscr{R}$ defined by:

$$F : L \mapsto \langle \varrho_X, Q_L \rangle \tag{5}$$

is at its minimum.

Proof. We know that $\mathscr{P}_m$ is a compact subset of l^1 (cf. [5]); hence is is enough to show the continuity of F with regard to the norm of l^1.

$$|\langle \varrho_X, Q_L \rangle - \langle \varrho_X, Q_M \rangle| \leqq \sum_{i=1}^{\infty} |\varrho_X(i)| \, |Q_L(i) - Q_M(i)|.$$

Now: $|\varrho_X(i)| \leqq 1$ and $Q_L(i) \leqq 1$ for all $i \in \mathscr{N}$ and $L \in \mathscr{P}_m$, hence

$$|\langle \varrho_X, Q_L \rangle - \langle \varrho_X, Q_M \rangle| \leqq \sum_{i=1}^{N} |Q_L(i) - Q_M(i)| + 2 \sum_{i=N+1}^{\infty} |\varrho_X(i)| . \tag{6}$$

Now $\varrho_X \in l^1$, hence we can choose N such that the second term on the right side of (6) should be arbitrarily small. Moreover $Q_L(i)$ is a polynomial expression of $(L(j))_{1 \leqq j \leqq i}$, consequently:

$$\lim_{\|L-M\|_{l^1} \to 0} \left[\sum_{i=1}^{N} |Q_L(i) - Q_M(i)| \right] = 0 . \qquad \square$$

Lemma 2.2 justifies the choice of the optimality criterion. Indeed we have to solve the problem of minimizing $a(L)$ if $L \in \mathscr{P}_m$.

Definition 2.1. Let $L_0 \in \mathscr{P}_m$; the sampling distribution L_0 is *optimal* if:

$$L_0 = \arg \operatorname{Inf} \{ C_X(0) (1 + 2\langle \varrho_X, Q_L \rangle) \mid L \in \mathscr{P}_m \} \tag{7}$$

The real question in the search of L_0 lies in the *non convexity* of the function F defined in (5). Consequently the application of Kuhn-Tucker's results (cf. [12]), only leads, in the general case, to a *necessary condition* for the optimality of L_0.

3. MAIN POINTS

3.1 Necessary condition of Optimality for L_0

When the correlation function ϱ_X converges towards zero, with an exponential rate, we give a simple expression for the first and second order derivatives of the function F for every element of a l^1-sphere C_a including $\mathscr{P}$ and defined by

$$C_a = \{ x \in l^1 \mid \|x\|_{l^1} < 1/a \} , \quad a \in]0, 1[. \tag{8}$$

The next proposition sets up the rules of computation for the first and second order derivatives; their existence inside the l^1-sphere C_a, defined in (8) is connected with a rate of convergence towards zero of the correlation function of X. We have to introduce the assumption:

$\mathbf{H}_1$. There exists $\alpha \in \mathbb{R}_+^*$, $\alpha < a$ such that $\varrho_X(k) = O(\alpha^{|k|})$.

Proposition 3.1. Under $\mathbf{H}_1$, the function F is twice Frechet-differentiable on C_a.

i) $F'_x \in \mathscr{L}(l^1, \mathscr{R})$ is the continuous linear operator such that:

$$F'_x(y) = \langle \varrho, Q'_x * y \rangle \quad \text{with} \quad y \in l^1 \quad \text{and} \quad Q'_x = \sum_{k=1}^{\infty} y x^{*(k-1)} \tag{9}$$

ii) $F''_x \in \mathscr{L}(l^1, \mathscr{L}(l_1, \mathscr{R}))$ is the continuous bilinear operator such that:

$$F''_x(y, z) = \langle \varrho, Q''_x * z * y \rangle \quad \text{with} \quad (y, z) \in l^1 \times l^1 \quad \text{and}$$

$$Q''_x = \sum_{k=1}^{\infty} k(k-1)\, x^{*(k-2)} \tag{10}$$

Proof. (See the appendix.)

Remark 3.1. All the results of Proposition 3.1 are preserved if L^1 takes the place of l^1.

The Frechet-differentiability properties, consequently the strict differentiability (cf. [2]), lead us to present a necessary condition of existence of an optimal distribution L_0, under a Kuhn-Tucker form.

Let $b \in]1, 1/a[$, $x = (x_j)_{j \in \mathscr{N}^*}$, an element of C defined by

$$C = \{x \in L^1 / \sum_{j=1}^{\infty} x_j = 1, \|x\|_{l^1} \leqq b\} .$$

It is easy to prove that C is a closed convex subset of L^1 included in C_a. Let $N_C(x)$ be the normal cone at $x \in C$ and for the restriction to C of F (5), we define $\nabla F_x = [(\nabla F_x)(j)]_{j \in \mathscr{N}^*} \in L^\infty$ by

$$F'_x(V) = \langle\langle V, \nabla F_x \rangle\rangle , \quad V \in L^1 . \tag{11}$$

Finally, for each distribution of probability x contained in $\mathscr{P}_m$, we define $h(x)$ by:

$$h(x) = (-x, \sum_{j=1}^{\infty} j x_j - m) . \tag{12}$$

This function formalizes the constraints.

Proposition 3.2. A necessary condition for $\bar{x} = \arg \operatorname{Inf} \{F(x) \mid x \in C, h(x) < 0\}$ is: there exists $(r, q) \in L_+^\infty \times \mathscr{R}_+$ such that:

$$\begin{aligned} &\text{i)} \quad r_j \bar{x}_j = 0 \quad \text{for all} \quad j \in \mathscr{N}^* \\ &\text{ii)} \quad q(\sum_{j=1}^{\infty} j \bar{x}_j - m) = 0 \\ &\text{iii)} \quad -\sum_{j=1}^{\infty} [\nabla F_{\bar{x}}(j) - r_j - q]\, e_j \in N_C(\bar{x}) \end{aligned} \tag{13}$$

Proof. We apply results of differentiable programming under a convex set (cf. [10]). See the appendix. □

Theorem 3.1. (Characterization of L_0.) Let $L \in \mathscr{P}_m$. A necessary condition for L to be optimal is: there exist $k \in \mathbb{R}$, $p \in L_+^\infty$ with $\sup_{j \in \mathscr{N}^*} p_j / j < \infty$, $q \in \mathscr{R}_+$ such that:

i) $p_j L_j = 0$ for all $j \in \mathcal{N}^*$

ii) $q(\sum_{j=1}^{\infty} jL_j - m) = 0$

iii) $g_L(j) - p_j + jq + k = 0$ for all $j \in \mathcal{N}^*$ (14)
where g_L is defined by:

$$g_L(j) = j(\nabla F_L)(j) = \frac{\partial \langle \varrho_X, Q_L \rangle}{\partial L(j)} \quad \text{for all } j \in \mathcal{N}^* . \tag{15}$$

Proof. It is a simple consequence of the previous results, using the properties of the distribution L. Indeed, if $L \in \mathscr{P}_m$, then $L \in C$; the normal cone at $L \in C$

$$N_C(L) = \{Z \in L^\infty \mid \forall x \in C, \langle\!\langle x - L, Z \rangle\!\rangle \leqq 0\}$$

is included in the one dimensional vector subspace of L^∞ generated by: $e_\infty = \sum_{j=1}^{\infty} j^{-1} e_j$; hence every element of $N_C(L)$ is colinear to e_∞.

Using that $(\nabla F_L)(j) = j^{-1} g_L(j)$, and noting $jr_j = p_j$ we conclude. □

Before drawing the outcomes of Theorem 3.1, we will give some properties of $(g_L(n))_{n \in \mathcal{N}}$, i.e. the gradient of F.

Corollary 3.1. Under H_1:

i)
$$g_L(k) = \sum_{i=0}^{\infty} \varrho_X(k+i)\, a(i) \quad \text{for all } k \geqq 1 \tag{16}$$

and

$$a(0) = 1\,, \quad a(n) = \sum_{j=1}^{n} (j+1)\, L^{*j}(n) \quad \text{for all } n \geqq 1\,. \tag{17}$$

In addition:

$$0 \leqq a(n) \leqq n + 1 \quad \text{for all } n \geqq 0\,. \tag{18}$$

ii) $\lim_{n \to \infty} g_L(n) = 0$.

Proof. i) relations (16) and (17) are consequences of the equality:

$$\sum_{n=1}^{\infty} \varrho_X(n) \left[\sum_{k=1}^{n} k(L^{*(k-1)} * M)(n)\right] =$$
$$= \sum_{n=1}^{\infty} \varrho_X(n) \left[M(n) + \sum_{k=2}^{n} \left(\sum_{p=1}^{n-k+1} L^{*(k-1)}(n-p)\, M(p)\right)\right].$$

We notice that:

$$\forall\, n \geqq 1: a(n) = Q_L(n) + \sum_{k=0}^{n-1} L(n-k)\, a(k)\,, \quad Q_L(n) \leqq 1$$

$$\sum_{k=0}^{n-1} (k+1)\, L(n-k) \leqq n \sum_{k=0}^{n-1} L(n-k) \leqq n$$

and (18) is proved by recurrence.

ii) is the outcome of i) and of the convergence, with exponential rate, of $\varrho_X(n)$

towards zero, because

$$|g_L(k)| \leqq \sum_{i=0}^{\infty} |\varrho_X(k+i)| (i+1) \leqq \sum_{i=0}^{\infty} |\varrho_X(k+i)| \, |k+i| = \sum_{j=k}^{\infty} j |\varrho_X(j)|$$

consequently $|g_L(k)|$ is smaller than the remainder of a convergent series. □

Corollary 3.2. Let L_0 be an optimal sampling distribution.

i) The points $\{n, g_{L_0}(n)\}_{n \geqq 1} \subset \mathscr{R}^2$ are lined up a line of slope smaller than zero for the elements of supp L_0, and above the line for all other points.

ii) If $\sum_{j=1}^{\infty} j \, L_0(j) < m$, then

a) $\forall s \in \operatorname{supp} L_0 \colon g_{L_0}(s) = -k$, a negative constant, and

b) $\forall n \in \mathscr{N}^* - \operatorname{supp} L_0 \colon g_{L_0}(n) \geqq g_{L_0}(s)$

iii) If supp L_0 is not finite

a) $\forall s \in \operatorname{supp} L_0 \colon g_{L_0}(s) = 0$

b) $\forall n \geqq 1 \colon g_{L_0}(n) \geqq 0$

Proof. i) is obvious by (13, i), because $p_n = 0$ if $n \in \operatorname{supp} L_0$ and $p_n \geqq 0$ if not.

ii) a) $\sum_{j=1}^{\infty} jL_j - m \neq 0$, consequently $q = 0$ in (14). Let $s \in \operatorname{supp} L_0$: $p_s = 0$ in (14) consequently $g_{L_0}(s) + k = 0$ for all $s \in \operatorname{supp} L_0$.

b) for all n, $g_{L_0}(n) - p_n + k = g_{L_0}(n) - p_n - g_{L_0}(s) = 0$, with $p_n \geqq 0$.

iii) by (ii), Corollary 3.1 and (i), Corollary 3.2. □

Conclusion. In a nice paper [6] the authors discuss some criterions that can be used for the comparison of non random sampling schemes. But there are only a few papers that try to look for optimal sampling at least as far as a random sampling scheme is concerned (cf. e. g. [15], [7]).

The criterion we choose is connected with the asymptotic variance of the estimator, the sampling rate is supposed to be large enough. Other criterions could give results, different from the one we proved. It would be worth to study them.

APPENDIX

1. Proof of Proposition 3.1.

a) F is well defined for every $x \in C_a$.

Let $x \in C_a$; and write $\varrho = \varrho_X$. Indeed:

$$F(x) = \sum_{\tau=1}^{\infty} \varrho(\tau) \, Q_x(\tau) \, .$$

$$|Q_x(\tau)| \leqq \sum_{k=1}^{\tau} |x^{*k}(\tau)| \leqq \sum_{k=1}^{\tau} \|x^{*k}\| \leqq \sum_{k=1}^{\tau} \|x\|^k, \text{ hence:}$$

$$\cdot Q_x(\tau)| \leqq \begin{cases} \dfrac{\|x\|^{\tau+1} - 1}{\|x\| - 1} & \text{if} \quad \|x\| < 1 \\ \tau\|x\|^{\tau} & \text{if} \quad 1 \leqq \|x\| < 1/a \end{cases}$$

If $\|x\| < 1$, then obviously $F(x)$ is well defined; if not

$$|F(x)| \leqq \sum_{\tau=1}^{\infty} |\varrho(\tau)| \, \tau\|x\|^{\tau} \leqq \sum_{\tau=1}^{\infty} \tau \frac{1}{a^{\tau}} \, O(\alpha^{\tau}) < \infty \,, \quad \text{by} \quad \mathrm{H}_1 .$$

b) F' defined by (8) is a linear and continuous function.

The linearity is obvious, we prove that F' is bounded, i.e.

$$\sup_{\|y\| \leqq 1} |\langle \varrho, Q'_x * y \rangle| < \infty$$

i) if $\|x\| < 1$, $\|Q'_x\| < \infty$ and:

$$|\langle \varrho, Q'_x * y \rangle| \leqq \|\varrho\| \, \|Q'_x * y\| \leqq \|\varrho\| \, \|Q'_x\| \, \|y\| < \infty \, .$$

ii) if $\|x\| \in]1, 1/a[$:

$$|(\sum_{k=1}^{\tau} kx^{*(k-1)} * y)(\tau)| \leqq \sum_{k=1}^{\tau} k\|k\|^{k-1} \, \|y\| \leqq O(\tau^2) \, \|x\|^{\tau-1} \, \|y\| \,,$$

and hence:

$$\sup_{\|y\| \leqq 1} \sum_{\tau=1}^{\infty} |\varrho(\tau)| \, [|(Q'_x * y)(\tau)|] \leqq \sup_{\|y\| \leqq 1} \sum_{\tau=1}^{\infty} |\varrho(\tau)| \, O(\tau^2) \, \|x\|^{\tau-1} \, \|y\| < \infty \, .$$

c) F'' defined by (9) is bilinear and continuous.

We have to prove that: $\sup_{\|y\| \leqq 1} \{ \sup_{\|z\| \leqq 1} |\langle \varrho, Q''_x * y * z \rangle| \} < \infty$ i.e. that F'', obviously bilinear, is bounded.

$$|\langle \varrho, Q''_x * y * z \rangle| \leqq \sum_{\tau=1}^{\infty} |\varrho(\tau)| \, |(Q''_x * y * z)|; \quad \text{moreover}$$

$$(Q''_x * y * z)(\tau) = \sum_{k=2}^{\tau} k(k-1)(x^{*(k-2)} * y * z)(\tau)$$

$$|(Q''_x * y * z)(\tau)| \leqq \sum_{k=2}^{\tau} k(k-1) \, \|x\|^{k-1} \quad \text{because} \quad \|y\| \leqq 1 \,, \quad \|z\| \leqq 1 \, .$$

Now in the same way as in *b*), we investigate the cases where $\|x\| < 1$ and $x \in$ $\in]1, 1/a[$.

d) Calculus of derivatives.

1) *First order derivative.*

Let: $a_1(y) = [1/\|y\| \, \langle \varrho, Q_{x+y} \rangle - \langle \varrho, Q_x \rangle - \langle \varrho, \sum_{k=1}^{\infty} k \, x^{*(k-1)} * y \rangle|]$ and prove that $\lim_{\|y\| \to 0} a_1(y) = 0$.

We note that for $k \geqq j \geqq 2$: $\binom{k}{j} \leqq k^2 \binom{k-2}{j-2}$, hence if $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$:

$$(a + b)^k - a^k - ka^{k-1}b \leqq b^2k^2(a + b)^{k-2} \tag{A1}$$

and

$$\frac{1}{\|y\|} \|(x + y)^{*k} - x^{*k} - kx^{*(k-1)} * y\| = \frac{1}{\|y\|} \left\| \sum_{j=2}^{k} \binom{k}{j} y^{*j} * x^{*(k-j)} \right\| \leqq$$
$$\leqq \frac{(\|x\| + \|y\|)^k - \|x\|^k - k\|x\|^{k-1} \|y\|}{\|y\|}$$

and by (A1) we can write:

$$\frac{1}{\|y\|} \|(x + y)^{*k} - x^{*k} - kx^{*(k-1)} * y\| \leqq \|y\| \, k^2 \|x + y\|^{(k-2)} . \tag{A2}$$

We now majorate $a_1(y)$:

$$a_1(y) \leqq \frac{1}{\|y\|} \sum_{\tau=2}^{\infty} \{|\varrho(\tau)| \, |(Q_{x+y} - Q_x - \sum_{k=2}^{\infty} kx^{*(k-1)} * y)(\tau)|\} \leqq$$
$$\leqq \sum_{\tau=2}^{\infty} \left\{ |\varrho(\tau)| \sum_{k=2}^{\tau} \frac{\|(x + y)^{*k} - x^{*k} - kx^{*(k-1)} * y\|}{\|y\|} \right\} \leqq$$
$$\leqq \|y\| \cdot (\sum_{\tau=2}^{\infty} |\varrho(\tau)|) \cdot (\sum_{k=2}^{\tau} k^2(\|x\| + \|y\|)^{k-2}) \quad \text{by (A1)}$$
$$\leqq \|y\| \cdot (\sum_{\tau=2}^{\infty} |\varrho(\tau)|) \, . \, O(\tau^3) \, . \, (\|x\| + \|y\|)^{\tau-2}$$

and as $\|y\| \to 0$, we can find a neighbourhood of x such that $\|x\| + \|y\| < 1/a$, then by H_1

$$\sum_{\tau=2}^{\infty} |\varrho(\tau)| \, O(\tau^3) \, [\|x\| + \|y\|]^{\tau-2} < +\infty$$

consequently: $\lim_{\|y\| \to 0} a_1(y) = 0$.

2) *Second order derivative.*

Let

$$a_2(y) = \frac{1}{\|y\|} \{ \sup_{\|y\| \leqq 1} |F'_{x+y}(z) - F'_x(z) - \langle \varrho, Q''_x * y * z \rangle| \} ;$$

by the same technique, we can easily prove that $\lim_{\|y\| \to 0} a_2(y) = 0$.

The above mentioned results are preserved if we put L^1 in place of l^1. Indeed: $L^1 \subset l^1$ and if $x \in L^1$, $\|x\|_{l^1} \leqq \|x\|_{L^1}$. Consequently: $\|F'_x\|_{L^1} \leqq \|F'_x\|_{l^1}$ and $\|F''_x\|_{L^1} \leqq \|F''_x\|_{l^1}$.

2. Proof of Proposition 3.2.

We have to minimize $F(x)$ under the constraint

$$\begin{cases} x \in C & \text{closed convex subset of } L^1 \\ h(x) \leqq 0 & \text{where } h \text{ defined in (12) is an affine function;} \end{cases}$$

in particular $h'_x(y) = h(y)$, $h''_x(z, z) = 0$.

The functions F and h are twice differentiable into $C_a \cap L^1$, an open subset of L^1 such that $C \subset C_a \cap L_1$, so they are strictly differentiable.

The end of the proof lies on Pomerol's work. To apply that result we have to prove the condition, denoted (S) by the author, of existence of a Kuhn-Tucker vector written here as:

$$\forall (u, v) \in L^1 \times \mathbb{R}, \quad \exists \varepsilon > 0, \quad \exists x \in C \begin{cases} \text{i)} & -x \leqq \varepsilon u \\ \text{ii)} & \sum_{j=1}^{\infty} j x_j - m \leqq \varepsilon v \end{cases} \tag{A3}$$

i) Proof of (A3).

– If $m = 1$, $x = e_1 = (\delta_1^j)_{j \in \mathcal{N}}$ is the only one possible optimum.

– If $m > 1$, let $(u, v) \in L^1 \times \mathbb{R}$, $\theta \in]1, m[$,

$$S_1 = \sum_{u_n < 0} |u_n| < \infty, \quad \Sigma_1 = \sum_{u_n < 0} n|u_n| < \infty$$

and let $M = (M_n)_{n \in \mathcal{N}^*}$ be a probability distribution on $\mathcal{N}^*$ such that $\sum_{j=1}^{\infty} j M_j = \theta$. Define $x = (x_n)_{n \in \mathcal{N}^*}$ by

$$x_n = (1 - \lambda S_1) M_n + \lambda |u_n| \mathbf{1}_{\{u_n < 0\}}, \quad n = \mathcal{N}^*$$

where

$$\begin{cases} 0 < \lambda < \operatorname{Inf}\left(1/S_1, 1, \dfrac{n - \theta}{2|\Sigma_1 - S_1 \theta|}\right. & \text{if} \quad S_1 \neq 0 \\ 0 < \lambda < 1 & \text{if} \quad S_1 = 0 \end{cases}$$

It is easy to prove that $x \in C$.

Finally, let ε be such that:

$$\begin{cases} 0 < \varepsilon < \inf\left(\lambda, \dfrac{1}{|v|}, \dfrac{m - \theta}{3}\right) & \text{if} \quad v \neq 0 \\ 0 < \varepsilon < \lambda & \text{if} \quad v = 0 \end{cases}$$

For such an ε: $-x_n \leqq \varepsilon u_n$ for all $n \in \mathcal{N}^*$ and (A3, i)) is proved.

It remains to prove (A3, ii). By a good choice of λ and ε defined as above:

– If $v \geqq 0$ (A3, ii) is proved by noting that $\lambda < \dfrac{m - \theta}{2|\Sigma_1 - \theta S_1|}$.

– If $v < 0$ (A3, ii) is proved by noting that $0 < \varepsilon < \dfrac{m - \theta}{3|v|}$.

Consequently a necessary condition for $x \in \mathscr{P}_m$ to be an optimum is:

(A4): There exists $y = (r, q) \in L_+^\infty \times \mathscr{R}_+$ such that:

i) $\langle h(\bar{x}), y\rangle_U = 0$

ii) $\forall x \in C: 0 \leqq F'_{\bar{x}}(x - \bar{x}) + \langle h(x), y\rangle_U$

where $\langle , \rangle_U$ is the scalar product of the duality $(L^1 \times \mathscr{R}, L^\infty \times \mathscr{R})$. Taking the constraint in account, let $\bar{x} \in \mathscr{P}_m$,

(A4, i)) becomes: $-\sum_{j=1}^{\infty} jr_j\bar{x} + q(\sum_{j=1}^{\infty} j\bar{x}_j - m) = 0$

(A4, ii)) becomes: $\forall x \in C: 0 \leqq F'_{\bar{x}}(x - \bar{x}) + \langle h(x - \bar{x}), y\rangle_U$.

Substitute h by its expression, write $F'_{\bar{x}}(a)$ as $\langle\!\langle a, \nabla F_{\bar{x}}\rangle\!\rangle$ and $\nabla F_{\bar{x}}$ as $\sum_{j=1}^{\infty} (\nabla F_{\bar{x}}(j))\, e_j$. Consequently (A4, ii)) is equivalent to:

$$\forall x \in C: 0 \leqq \langle\!\langle x - \bar{x}, \sum_{j=1}^{\infty} (\nabla F_{\bar{x}}(j) - r_j + q)\, e_j\rangle\!\rangle$$

and $-\sum_{j=1}^{\infty} (\nabla F_{\bar{x}}(j) - r_j + q)\, e_j \in N_c(\bar{x})$ the normal cone in $\bar{x} \in C$. Consequently a necessary condition for $\bar{x} \in \mathscr{P}_m$ to be a solution of

$$\bar{x} = \arg \operatorname{Inf} \{F(x)/x \in C, h(x) \leqq 0\}$$

is: there exists $(r, q) \in L_+^\infty \times \mathscr{R}_+$ which completes the proof of (13).

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