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Topological Representations of Distributive Lattices and Brouwerian Logics.

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(Received December 9, 1936.)

In a series of papers, the writer has developed a theory of Boolean algebras dealing with their algebraic structure, their representation by algebras of classes, and their relations to general topology.[1]) It is the object of the present paper to outline an extension of the main features of this theory to the more general systems known variously as distributive lattices, C-lattices, or arithmetic structures.[2])

From certain points of view, the theory of distributive lattices is of secondary interest compared with that of Boolean algebras. Thus the theorem of Mac Neille,[3]) which states that every distributive lattice can be imbedded by a purely algebraic construction in a Boolean algebra, shows that distributive lattices are not significantly more general than Boolean algebras. In addition, the theory of distributive lattices gains in generality only at the sacrifice of a certain simplicity and symmetry, as we shall see below. Finally, certain parts of the theory do not have even the merit of novelty, the theorem that every distributive lattice can be isomorphically

[1]) Stone, Proceedings of the National Academy of Sciences, U. S. A., 20 (1934), pp. 197—202; ibid., 21 (1935), pp. 103—105; American Journal of Mathematics, 57 (1935), pp. 703—732; Transactions of the American Mathematical Society, 40 (1936), pp. 37—111; and an unpublished paper, „Applications of the Theory of Boolean Rings to General Topology", which has been submitted to the editors of the Transactions of the American Mathematical Society.

[2]) The term „distributive lattice", which we shall use here, was introduced by Mac Neille, Proceedings of the National Academy of Sciences, U. S. A., 22 (1936), pp. 45—50; „C-lattice" by Garrett Birkhoff, Proceedings of the Cambridge Philosophical Society, 29 (1933), pp. 441—464; „arithmetic structure" by Ore, Annals of Mathematics, (2) 36 (1935), pp. 406—437.

[3]) Mac Neille, Harvard doctoral dissertation, The Theory of Partially Ordered Sets, 1935, a summary of which appeared in Proceedings of the National Academy of Sciences, U. S. A., 22 (1936), pp. 44—50.

represented by an algebra of classes having been discovered and published by Garrett Birkhoff.[4])

In spite of these considerations, the actual development of the present theory serves a useful purpose in providing a mathematical background against which the theory of Boolean algebras can be more fully appreciated. It also offers a somewhat deeper algebraic analysis of the theorems already known.

The connections between Boolean algebras and classical logic, in the symbolic statement of Russell and Whitehead, are well known. We may describe them, without too scrupulous a regard for detail, by saying that every logical system of propositions can be represented by a Boolean algebra. Familiar evidence in support of this statement is found in the use of Leibnitz's diagrams to represent the logical relations between propositions. Turning to Brouwerian logic, we are naturally led to seek a similar representation. In the second part of the present paper, to which the theory of distributive lattices is a necessary preliminary, we shall construct such a representation for a Brouwerian system of propositions, relying upon the symbolic statement of Heyting.[5]) The most noteworthy feature of this representation is its topological character: whereas the Leibnitz diagrams for classical logic employ classes and the usual combinatorial operations upon them, the corresponding diagrams for Brouwerian logic employ also certain topological operations upon classes.

Part I. Distributive Lattices.

§ 1. Definition and Algebraic Properties. A distributive lattice (or C-lattice) has been defined[6]) as a system with double composition (we shall indicate the results of performing the two compositions by $a \vee b$ and $a \,.\, b$ or ab respectively) in which the following rules of operation are valid:

(1) $a \vee a = a, \quad aa = a;$
(2) $a \vee b = b \vee a, \quad ab = ba;$
(3) $a \vee (b \vee c) = (a \vee b) \vee c, \quad a(bc) = (ab)c;$
(4) $ab = a$ implies and is implied by $a \vee b = b;$
(5) $a(b \vee c) = ab \vee ac, \; a \vee bc = (a \vee b)\,(a \vee c).$

Since these rules of operation bear symmetrically on the two compositions, every distributive lattice has the following property of

[4]) Garrett Birkhoff, Proceedings of the Cambridge Philosophical Society, **29** (1933), pp. 441—464, Theorem 25.2.

[5]) Heyting, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse, 1930, pp. 42—56.

[6]) Garrett, Birkhoff, Proceedings of the Cambridge Philosophical Society, **29** (1933), pp. 441—464; especially pp. 442, 453.

duality: any rule of operation remains valid under interchange of the two fundamental operations.

To indicate that the elements a and b satisfy the equivalent relations (4), we shall write $a < b$ or $b > a$. The relations $<$ and $>$ have a number of simple properties which we recall without proof: $a < a$; $a < b$ and $b < c$ imply $a < c$; $a < b$ and $b < a$ imply $a = b$; $ab < a$; $a < a \vee b$; $a < c$ and $b < d$ imply $ab < cd$ and $a \vee b < c \vee d$.

If a distributive lattice contains an element a such that $a < x$ for every element x in the lattice, then this element is uniquely determined and will be called the zero-element or zero. We shall use the symbol 0 for a zero-element. Similarly, if a distributive lattice contains an element a such that $x < a$ for every element x in the lattice, then this element is uniquely determined and will be called the unit-element or unit. We shall use the symbol e for a unit-element. It is easily seen that a distributive lattice with only a finite number of elements $a_1, \ldots, a_n$ has both a zero and a unit: for the elements $a_1 \ldots a_n$ and $a_1 \vee \ldots \vee a_n$ respectively are such. In general, however, a distributive lattice has neither a zero nor a unit; but it is possible to adjoin a zero 0 by defining $0 \vee a = a \vee 0 = a$, $a0 = 0a = 0$, $0 \vee 0 = 0$, $00 = 0$, or to adjoin a unit e by defining $e \vee a = a \vee e = e$, $ae = ea = a$, $e \vee e = e$, $ee = e$.

The systems called generalized Boolean algebras[7]) are characterized among all distributive lattices by the existence of „relative complements": if $a < b$, then there exists an element c such that $a \vee c = b$, $ac < x$ for every element x in the lattice. Similarly, Boolean algebras are characterized by the existence of „complements": if a is an arbitrary element, then there exists an element a' such that $a \vee a' > x$, $aa' < x$ for every element x in the lattice.

From the preceding remarks, we see that a two-element distributive lattice consists of a zero and a unit alone. The two-element distributive lattices are therefore mutually isomorphic Boolean algebras, the typical example being the system consisting of two elemets 0 and e with the rules $0 = 0$, $e = e$, $0 \neq e$, $0 \vee 0 = 0$, $0 \vee e = e \vee 0 = e \vee e = e$, $0 . 0 = 0e = e0 = 0$, $ee = e$, $0 < 0$, $0 < e$, $e < e$, $0' = e$, $e' = 0$.

§ 2. Ideals. It is convenient to distinguish the following types of subsystem in a distributive lattice:

Definition 1. *A non-void subclass $\mathfrak{a}$ of a distributive lattice A is said to be a multiplicative ideal or μ-ideal if*

(1) *$a \,\varepsilon\, \mathfrak{a}$ and $b \,\varepsilon\, \mathfrak{a}$ imply $a \vee b \,\varepsilon\, \mathfrak{a}$;*
(2) *$a \,\varepsilon\, \mathfrak{a}$ and $b \,\varepsilon\, A$ imply $ab \,\varepsilon\, \mathfrak{a}$.*

[7]) Stone, American Journal of Mathematics, 57 (1935), pp. 703—732.

Dually, a non-void subclass $\mathfrak{a}$ is said to be an additive ideal or α-ideal if

(1) *$a \varepsilon \mathfrak{a}$ and $b \varepsilon A$ imply $a \vee b \varepsilon \mathfrak{a}$;*
(2) *$a \varepsilon \mathfrak{a}$ and $b \varepsilon \mathfrak{a}$ imply $ab \varepsilon \mathfrak{a}$.*

It is easily seen that the only class $\mathfrak{a}$ which is both a μ-ideal and an α-ideal is the class $\mathfrak{e}$ consisting of all elements of A: for $a \varepsilon \mathfrak{a}$ and $b \varepsilon A$ imply $a \vee b \varepsilon \mathfrak{a}$; and $a \vee b \varepsilon \mathfrak{a}$ and $b \varepsilon A$ then imply $b = (a \vee b)\, b \varepsilon \mathfrak{a}$.

If $\mathfrak{s}$ is any non-void subclass of a distributive lattice A, the class $\mathfrak{a}_\mu(\mathfrak{s})$ of all elements c such that $c < a_1 \vee \ldots \vee a_n$ for some elements $a_1, \ldots, a_n$ in $\mathfrak{s}$ is easily seen to be the least μ-ideal containing $\mathfrak{s}$: it is a μ-ideal, it contains $\mathfrak{s}$, and it is contained in every μ-ideal which contains $\mathfrak{s}$. We call $\mathfrak{a}_\mu(\mathfrak{s})$ the μ-ideal generated by $\mathfrak{s}$. Dually, the class $\mathfrak{a}_\alpha(\mathfrak{s})$ of all elements c such that $c > a_1 \ldots a_n$ for some elements $a_1, \ldots, a_n$ in $\mathfrak{s}$ is the least α-ideal containing $\mathfrak{s}$. We call $\mathfrak{a}_\alpha(\mathfrak{s})$ the α-ideal generated by $\mathfrak{s}$. When $\mathfrak{s}$ consists of a single element c, we write $\mathfrak{a}_\mu(c)$, $\mathfrak{a}_\alpha(c)$ for $\mathfrak{a}_\mu(\mathfrak{s})$, $\mathfrak{a}_\alpha(\mathfrak{s})$ respectively; and call these ideals the principal μ-ideal and principal α-ideal, respectively generated by c. Moreover, if $\mathfrak{s}$ is the union of a class $\mathfrak{A}$ of μ-ideals $\mathfrak{a}$, we see that $\mathfrak{a}_\mu(\mathfrak{s})$ is the class of all elements c such that $c = a_1 \vee \vee \ldots \vee a_n$ for some elements $a_1 \varepsilon \mathfrak{a}_1 \varepsilon \mathfrak{A}, \ldots, a_n \varepsilon \mathfrak{a}_n \varepsilon \mathfrak{A}$. Dually, if $\mathfrak{s}$ is the union of a class $\mathfrak{A}$ of α-ideals $\mathfrak{a}$, we see that $\mathfrak{a}_\alpha(\mathfrak{s})$ is the class of all elements c such that $c = a_1 \ldots a_n$ for some elements $a_1 \varepsilon \mathfrak{a}_1 \varepsilon \mathfrak{A}, \ldots, a_n \varepsilon \mathfrak{a}_n \varepsilon \mathfrak{A}$.

If $\mathfrak{A}$ is any non-void class of μ-ideals $\mathfrak{a}$, their sum $\underset{\mathfrak{a}\varepsilon\mathfrak{A}}{\mathsf{S}}\mathfrak{a}$ is the least μ-ideal containing every $\mathfrak{a}$ in $\mathfrak{A}$; and their product $\underset{\mathfrak{a}\varepsilon\mathfrak{A}}{\mathsf{P}}\mathfrak{a}$ is the greatest μ-ideal contained in every $\mathfrak{a}$ in $\mathfrak{A}$, if such exist. It is evident that $\underset{\mathfrak{a}\varepsilon\mathfrak{A}}{\mathsf{S}}\mathfrak{a}$ is the μ-ideal generated by the union of the classes $\mathfrak{a}$ in $\mathfrak{A}$ or, alternatively, the product of all μ-ideals containing every $\mathfrak{a}$ in $\mathfrak{A}$. Similarly, it is evident that $\underset{\mathfrak{a}\varepsilon\mathfrak{A}}{\mathsf{P}}\mathfrak{a}$ must coincide with the intersection of the classes $\mathfrak{a}$ in $\mathfrak{A}$; in particular, the product exists if and only if the μ-ideals $\mathfrak{a}$ have a common element. In case the distributive lattice A has a zero-element 0, then 0 belongs to every μ-ideal $\mathfrak{a}$ in A: for $a \varepsilon \mathfrak{a}$ and $0 \varepsilon A$ imply $0 = a \,.\, 0 \varepsilon \mathfrak{a}$. It follows that in this case the product of μ-ideals is defined without restriction. In any case, the product of a finite number of μ-ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ is defined: for $a_1 \varepsilon \mathfrak{a}_1, \ldots, a_n \varepsilon \mathfrak{a}_n$ imply $a_1 \ldots a_n \varepsilon \mathfrak{a}_1, \ldots, a_1 \ldots a_n \varepsilon \mathfrak{a}_n$. Indeed, the product consists of all elements c such that $c = a_1 \ldots a_n$ where $a_1 \varepsilon \mathfrak{a}_1, \ldots, a_n \varepsilon \mathfrak{a}_n$. If $\mathfrak{a}$ and $\mathfrak{b}$ are μ-ideals, it is convenient to denote their sum and product by $\mathfrak{a} \vee \mathfrak{b}$

and $\mathfrak{a}\mathfrak{b}$ respectively. Dually, we may define the sum and product of α-ideals: we have only to repeat the preceding statements replacing the term „μ-ideal" by the term „α-ideal", the element 0 by the unit-element e, the combination ab by the combination $a \vee b$. For the sum and product of two α-ideals $\mathfrak{a}$ and $\mathfrak{b}$ we shall write $\mathfrak{a} \vee \mathfrak{b}$ and $\mathfrak{a}\mathfrak{b}$ respectively.

Concerning the operations upon ideals, we now have the following result:

Theorem 1. *The relations*

(1) $\mathfrak{a} \vee \mathfrak{a} = \mathfrak{a}, \quad \mathfrak{a}\mathfrak{a} = \mathfrak{a}$;
(2) $\mathfrak{a} \vee \mathfrak{b} = \mathfrak{b} \vee \mathfrak{a}, \quad \mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$;
(3) $\mathfrak{a}\mathfrak{b} = \mathfrak{a}$ *if and only if* $\mathfrak{a} \vee \mathfrak{b} = \mathfrak{b}$ *or, equivalently,* $\mathfrak{a} \subset \mathfrak{b}$;
(4) $\mathfrak{a} \vee (\mathfrak{b} \vee \mathfrak{c}) = (\mathfrak{a} \vee \mathfrak{b}) \vee \mathfrak{c}, \ \mathfrak{a}(\mathfrak{b}\mathfrak{c}) = (\mathfrak{a}\mathfrak{b})\mathfrak{c}$;
(5) $\mathfrak{a}(\mathfrak{b} \vee \mathfrak{c}) = \mathfrak{a}\mathfrak{b} \vee \mathfrak{a}\mathfrak{c}, \ \mathfrak{a} \vee \mathfrak{b}\mathfrak{c} = (\mathfrak{a} \vee \mathfrak{b})\,(\mathfrak{a} \vee \mathfrak{c})$;
(6) *if* $\mathbf{B}$ *is a non-void class of non-void classes* $\mathfrak{B}$ *of ideals* $\mathfrak{a}$ *and if* $\mathfrak{C} = \sum_{\mathfrak{B} \varepsilon \mathbf{B}} \mathfrak{B}$, *then*

$$\underset{\mathfrak{B}\varepsilon\mathbf{B}}{\mathrm{S}}\Big(\underset{\mathfrak{a}\varepsilon\mathfrak{B}}{\mathrm{S}}\,\mathfrak{a}\Big) = \underset{\mathfrak{a}\varepsilon\mathfrak{C}}{\mathrm{S}}\,\mathfrak{a}, \quad \underset{\mathfrak{B}\varepsilon\mathbf{B}}{\mathrm{P}}\Big(\underset{\mathfrak{a}\varepsilon\mathfrak{B}}{\mathrm{P}}\,\mathfrak{a}\Big) = \underset{\mathfrak{a}\varepsilon\mathfrak{C}}{\mathrm{P}}\,\mathfrak{a},$$

both members of the second equation existing if either does;
(7) *if* $\mathfrak{b}$ *is an ideal and* $\mathfrak{B}$ *a non-void class of ideals, then*

$$\mathfrak{b}\,\underset{\mathfrak{a}\varepsilon\mathfrak{B}}{\mathrm{S}}\,\mathfrak{a} = \underset{\mathfrak{a}\varepsilon\mathfrak{B}}{\mathrm{S}}\,\mathfrak{a}\mathfrak{b};$$

hold in the class $\mathfrak{J}_\mu$ *of all* μ*-ideals in a distributive lattice and also in the class* $\mathfrak{J}_\alpha$ *of all* α*-ideals in a distributive lattice. Under the formation of finite sums and products* $\mathfrak{J}_\mu$ *and* $\mathfrak{J}_\alpha$ *are distributive lattices.*

The proof of the relations (1)—(7) in $\mathfrak{J}_\mu$ can be taken almost word for word from the proofs of Theorems 15 and 18 of our paper „The Theory of Representations for Boolean Algebras", Transactions of the American Mathematical Society, 40 (1936), pp. 37—111. The proof of the corresponding relations in $\mathfrak{J}_\alpha$ can then be obtained by appropriate dualization. Hence there is no need for us to go into detail.

We have also the following result:

Theorem 2. *The class* $\mathfrak{P}_\mu$ *of all principal* μ*-ideals in a distributive lattice* A *is isomorphic to* A *in accordance with the relations*

(1) $\mathfrak{a}_\mu(b) = \mathfrak{a}_\mu(c)$ *if and only if* $b = c$;
(2) $\mathfrak{a}_\mu(b \vee c) = \mathfrak{a}_\mu(b) \vee \mathfrak{a}_\mu(c), \ \mathfrak{a}_\mu(bc) = \mathfrak{a}_\mu(b)\,\mathfrak{a}_\mu(c)$.

Similarly, the class $\mathfrak{P}_\alpha$ *of all principal* α*-ideals is dually-isomorphic to* A *in accordance with the relations*

(1) $\mathfrak{a}_\alpha(b) = \mathfrak{a}_\alpha(c)$ *if and only if* $b = c$;

(2) $\mathfrak{a}_\alpha(b \vee c) = \mathfrak{a}_\alpha(b)\, \mathfrak{a}_\alpha(c), \mathfrak{a}_\alpha(bc) = \mathfrak{a}_\alpha(b) \vee \mathfrak{a}_\alpha(c)$.

It is evident that $\mathfrak{a}_\mu(b)$ is the class of all elements x such that $x < b$. Hence $\mathfrak{a}_\mu(b) = \mathfrak{a}_\mu(c)$ if only if $c < b$, $b < c$; that is, if and only if $b = c$. Since $x \,\varepsilon\, \mathfrak{a}_\mu(b) \vee \mathfrak{a}_\mu(c)$ if and only if $x < x_1 \vee x_2$ where $x_1 < b$ and $x_2 < c$, we see that $x \,\varepsilon\, \mathfrak{a}_\mu(b) \vee \mathfrak{a}_\mu(c)$ implies $x < b \vee c$, $x \,\varepsilon\, \mathfrak{a}_\mu(b \vee c)$, while $x \,\varepsilon\, \mathfrak{a}_\mu(b \vee c)$ implies $x = (b \vee c)\, x = (bx) \vee (cx)$, $bx < b$, $cx < c$ and hence $x \,\varepsilon\, \mathfrak{a}_\mu(b) \vee \mathfrak{a}_\mu(c)$. Hence $\mathfrak{a}_\mu(b \vee c) = \mathfrak{a}_\mu(b) \vee \mathfrak{a}_\mu(c)$. Since $x \,\varepsilon\, \mathfrak{a}_\mu(b)\, \mathfrak{a}_\mu(c)$ if and only if $x = x_1 x_2$ where $x_1 < b$ and $x_2 < c$, we see that $x \,\varepsilon\, \mathfrak{a}_\mu(b)\, \mathfrak{a}_\mu(c)$ implies $x < bc$, $x \,\varepsilon\, \mathfrak{a}_\mu(bc)$ while $x \,\varepsilon\, \mathfrak{a}_\mu(bc)$ implies $x = xx < bc$, $x < b$, $x < c$ and hence $x \,\varepsilon\, \mathfrak{a}_\mu(b)\, \mathfrak{a}_\mu(c)$. Hence $\mathfrak{a}_\mu(bc) = \mathfrak{a}_\mu(b)\, \mathfrak{a}_\mu(c)$. Thus $\mathfrak{P}_\mu$ is isomorphic to A. The discussion of $\mathfrak{P}_\alpha$ is now obtained by appropriate dualization.

§ 3. Prime and Divisorless Ideals. We shall now consider special types of μ-ideal and of α-ideal, introduced in two definitions.

Definition 2. *A μ-ideal $\mathfrak{p}$ is said to be prime if it is a proper subclass of the distributive lattice A and if $ab \,\varepsilon\, \mathfrak{p}$ implies $a \,\varepsilon\, \mathfrak{p}$ or $b \,\varepsilon\, \mathfrak{p}$. Similarly, an α-ideal $\mathfrak{q}$ is said to be prime if it is a proper subclass of A and if $a \vee b \,\varepsilon\, \mathfrak{q}$ implies $a \,\varepsilon\, \mathfrak{q}$ or $b \,\varepsilon\, \mathfrak{q}$.*

Definition 3. *A μ-ideal $\mathfrak{p}$ is said to be divisorless if it is a proper subclass of A and if, whenever $\mathfrak{a}$ is a μ-ideal, $\mathfrak{a} \supset \mathfrak{p}$ implies $\mathfrak{a} = \mathfrak{p}$ or $\mathfrak{a} = \mathfrak{e} = A$. Similarly an α-ideal $\mathfrak{q}$ is said to be divisorless if it is a proper subclass of A and if, whenever $\mathfrak{a}$ is an α-ideal, $\mathfrak{a} \supset \mathfrak{q}$ implies $\mathfrak{a} = \mathfrak{q}$ or $\mathfrak{a} = \mathfrak{e} = A$.*

The relations between these types of ideal are discussed in the following theorems.

Theorem 3. *Every divisorless ideal is prime.*

Let $\mathfrak{p}$ be a divisorless μ-ideal and let a and b be elements such that $ab \,\varepsilon\, \mathfrak{p}$. If a is not in $\mathfrak{p}$, then the μ-ideal $\mathfrak{a}_\mu(a) \vee \mathfrak{p}$ contains a and $\mathfrak{p}$ so that $\mathfrak{a}_\mu(a) \vee \mathfrak{p} = \mathfrak{e}$, $b \,\varepsilon\, \mathfrak{a}_\mu(a) \vee \mathfrak{p}$. From the latter relation we have $b < a \vee x$ where $x \,\varepsilon\, \mathfrak{p}$. Hence we see that $b = bb < ab \vee bx$, $ab \,\varepsilon\, \mathfrak{p}$, $bx \,\varepsilon\, \mathfrak{p}$. We conclude that $b \,\varepsilon\, \mathfrak{p}$. Thus the ideal $\mathfrak{p}$ is prime. The case where $\mathfrak{q}$ is a divisorless α-ideal is treated dually.

Theorem 4. *If A is partitioned into disjoint subclasses $\mathfrak{p}$ and $\mathfrak{q}$ then*

(1) *$\mathfrak{p}$ is a μ-ideal and $\mathfrak{q}$ an α-ideal only if both are prime;*

(2) *$\mathfrak{p}$ is a prime μ-ideal if and only if $\mathfrak{q}$ is a prime α-ideal.*

If $\mathfrak{p}$ is a μ-ideal and $\mathfrak{q}$ an α-ideal, we show that $\mathfrak{p}$ is prime in the following manner: if $ab \,\varepsilon\, \mathfrak{p}$, $a \,\varepsilon\, \mathfrak{q}$, $b \,\varepsilon\, \mathfrak{q}$, then $ab \,\varepsilon\, \mathfrak{q}$ by the ideal property of $\mathfrak{q}$; and we have a contradiction to our assumption that $\mathfrak{p}$ and $\mathfrak{q}$ are disjoint. Similarly, we show by a dual proof that $\mathfrak{q}$ is prime. If $\mathfrak{p}$ is a prime μ-ideal, then the class $\mathfrak{q}$ of elements which do not belong to $\mathfrak{p}$ is non-void. If $a \varepsilon \mathfrak{q}$ and $b \varepsilon A$, then the relation $a \vee b \varepsilon \mathfrak{p}$ would lead by the ideal property of $\mathfrak{p}$ to the contradiction $a = a(a \vee b) \varepsilon \mathfrak{p}$; hence $\mathfrak{q}$ contains $a \vee b$ whenever it contains a. The fact that $\mathfrak{p}$ is prime shows that $a \varepsilon \mathfrak{q}$, $b \varepsilon \mathfrak{q}$ imply $ab \varepsilon \mathfrak{p}$ and hence $ab \varepsilon \mathfrak{q}$. Thus $\mathfrak{q}$ is an α-ideal; and, by (1), $\mathfrak{q}$ must be prime. Dually, when $\mathfrak{q}$ is a prime α-ideal, the class $\mathfrak{p}$ of all elements which do not belong to $\mathfrak{q}$ is a prime μ-ideal.

Theorem 5. *If f is a single-valued function defined over a distributive lattice A with values in a two-element distributive lattice consisting of the elements 0 and e, then the correspondence $a \rightarrow f(a)$ is a homomorphism if and only if the classes $\mathfrak{p}$ and $\mathfrak{q}$ specified by the respective equations $f(a) = 0$ and $f(a) = e$ are respectively a prime μ-ideal and a prime α-ideal.*

First, let f define a homomorphism. Then $\mathfrak{p}$ and $\mathfrak{q}$ are non-void — in other words, f assumes both the values 0 and e. Now $a \varepsilon \mathfrak{p}$ and $b \varepsilon \mathfrak{p}$ imply $f(a \vee b) = f(a) \vee f(b) = 0 \vee 0 = 0$ and hence $a \vee b \varepsilon \mathfrak{p}$; and $a \varepsilon \mathfrak{p}$ and $b \varepsilon A$ imply $f(ab) = f(a)\, f(b) = 0\, f(b) = 0$ and hence $ab \varepsilon \mathfrak{p}$. Also $ab \varepsilon \mathfrak{p}$ implies $f(a)\, f(b) = f(ab) = 0$, hence $f(a) = 0$ or $f(b) = 0$, and hence $a \varepsilon \mathfrak{p}$ or $b \varepsilon \mathfrak{p}$. Thus $\mathfrak{p}$ is a prime μ-ideal. Theorem 4 now shows that $\mathfrak{q}$ is a prime α-ideal.

Next, let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals, multiplicative and additive respectively. Since the associated function f assumes both values 0 and e, we show that it defines a homomorphism if we show that $f(a \vee b) = f(a) \vee f(b)$, $f(ab) = f(a)\, f(b)$. Since $f(a \vee b) = e$ if and only if $a \vee b \varepsilon \mathfrak{q}$; since $a \vee b \varepsilon \mathfrak{q}$ if and only if one of the relations $a \varepsilon \mathfrak{q}$, $b \varepsilon \mathfrak{q}$ is valid; since $a \varepsilon \mathfrak{q}$ is equivalent to $f(a) = e$, $b \varepsilon \mathfrak{q}$ to $f(b) = e$; and since, finally, $f(a) \vee f(b) = e$ if and only if one of the relations $f(a) = e$, $f(b) = e$ is valid — we see that $f(a \vee b) = f(a) \vee f(b)$ in all cases. Since $f(ab) = 0$ if and only if $ab \varepsilon \mathfrak{p}$; since $ab \varepsilon \mathfrak{p}$ if and only if one of the relations $a \varepsilon \mathfrak{p}$, $b \varepsilon \mathfrak{p}$ is valid; since $a \varepsilon \mathfrak{p}$ is equivalent to $f(a) = 0$, $b \varepsilon \mathfrak{p}$ to $f(b) = 0$; and since, finally, $f(a)\, f(b) = 0$ if and only if one of the relations $f(a) = 0$, $f(b) = 0$ is valid — we see that $f(ab) = f(a)\, f(b)$ in all cases.

The most important aspect of the theory of prime ideals is the proof of their existence in an arbitrary distributive lattice with two or more elements. We shall now state a suitable existence theorem and give for it two proofs of somewhat different character.

Theorem 6. *If the distributive lattice A contains a μ-ideal $\mathfrak{a}$ and an α-ideal $\mathfrak{b}$ which are disjoint, then there exists a partition of A into a prime μ-ideal $\mathfrak{p}$ and a prime α-ideal $\mathfrak{q}$ such that $\mathfrak{a} \subset \mathfrak{p}$, $\mathfrak{b} \subset \mathfrak{q}$.*

Our first proof is essentially due to Garrett Birkhoff[8]): we merely rephrase the original demonstration in the language of

[8]) Garrett Birkhoff, Proceedings of the Cambridge Philosophical Society, 29 (1933), pp. 441—464, Theorem 21.1.

ideals. The essential step in the proof of the theorem is this: if $\mathfrak{a}$ and $\mathfrak{b}$ are given as stated and if they do not together exhaust A, we show that there exist a μ-ideal $\mathfrak{a}^*$ and an α-ideal $\mathfrak{b}^*$ where $\mathfrak{a}^* \supset \mathfrak{a}$, $\mathfrak{b}^* \supset \mathfrak{b}$, and $\mathfrak{a}^* \neq \mathfrak{a}$ or $\mathfrak{b}^* \neq \mathfrak{b}$. Once we have justified this step, an obvious transfinite induction enables us to form a partition of A into a μ-ideal $\mathfrak{p}$ and an α-ideal $\mathfrak{q}$ such that $\mathfrak{a} \subset \mathfrak{p}$, $\mathfrak{b} \subset \mathfrak{q}$; and Theorem 3 shows that both $\mathfrak{p}$ and $\mathfrak{q}$ are prime. It is unnecessary for us to describe the inductive construction in detail. If $\mathfrak{a}$ and $\mathfrak{b}$ together exhaust A, we take $\mathfrak{a} = \mathfrak{p}$, $\mathfrak{b} = \mathfrak{q}$.

If $\mathfrak{a}$ and $\mathfrak{b}$ do not exhaust A, we find the indicated ideals $\mathfrak{a}^*$ and $\mathfrak{b}^*$ in the following way. We select an arbitrary element c in A which belongs neither to $\mathfrak{a}$ nor to $\mathfrak{b}$. Then there exists no pair of elements a, b such that $a \,\varepsilon\, \mathfrak{a}$, $b \,\varepsilon\, \mathfrak{b}$, $c \vee a \,\varepsilon\, \mathfrak{b}$, and $cb \,\varepsilon\, \mathfrak{a}$: for otherwise the element $bc \vee ba = b(c \vee a)$ would belong both to $\mathfrak{a}$ and to $\mathfrak{b}$. Hence $c \vee a$ belongs to $\mathfrak{b}$ for no a in $\mathfrak{a}$ or cb belongs to $\mathfrak{a}$ for no b in $\mathfrak{b}$. If $c \vee a$ belongs to $\mathfrak{b}$ for no a in $\mathfrak{a}$, we put $\mathfrak{a}^* = \mathfrak{a}_\mu(c) \vee \mathfrak{a}$, $\mathfrak{b}^* = \mathfrak{b}$. Since $\mathfrak{a}^* \supset \mathfrak{a}$, $\mathfrak{a}^* \neq \mathfrak{a}$, $\mathfrak{b}^* \supset \mathfrak{b}$, we have only to prove that $\mathfrak{a}^*$ and $\mathfrak{b}^*$ are disjoint. Now an element b in $\mathfrak{b} = \mathfrak{b}^*$ belongs to $\mathfrak{a}^*$ if and only if $b < c \vee a$ for some element a in $\mathfrak{a}$; but the relations $b \,\varepsilon\, \mathfrak{b}$, $b < c \vee a$ imply $c \vee a \,\varepsilon\, \mathfrak{b}$, contrary to hypothesis. Our discussion is thus complete. On the other hand if cb belongs to $\mathfrak{a}$ for no b in $\mathfrak{b}$, we put $\mathfrak{a}^* = \mathfrak{a}$, $\mathfrak{b}^* = \mathfrak{a}_\alpha(c) \vee \mathfrak{b}$. Since $\mathfrak{a}^* \supset \mathfrak{a}$, $\mathfrak{b}^* \supset \mathfrak{b}$, $\mathfrak{b}^* \neq \mathfrak{b}$, we have only to show that $\mathfrak{a}^*$ and $\mathfrak{b}^*$ are disjoint. Now an element a in $\mathfrak{a} = \mathfrak{a}^*$ belongs to $\mathfrak{b}^*$ if and only if $a > cb$ for some b in $\mathfrak{b}$; but then $cb \,\varepsilon\, \mathfrak{a}$, contrary to hypothesis. With this our first proof is complete.

Our second proof is based upon Theorem 5: we construct a function f defined over A with the properties

(1) $f(a \vee b) = f(a) \vee f(b), \quad f(ab) = f(a)\, f(b);$
(2) $f(a) = 0$ in $\mathfrak{a}$, $f(a) = e$ in $\mathfrak{b}$.

We can then take $\mathfrak{p}$ as the class specified by the equation $f(a) = 0$, $\mathfrak{q}$ as the class specified by the equation $f(a) = e$. The construction is based upon transfinite induction and is similar to one (for Boolean algebras) already in the literature.[9]) We shall suppress the obvious details of the inductive process. For present purposes, it is convenient for us to replace (1) by the equivalent property

(1′) $a_1 \ldots a_m < b_1 \vee \ldots \vee b_n$ implies $f(a_1) \ldots f(a_m) < f(b_1) \vee \ldots \vee f(b_n)$ for $m \geq 1$, $n \geq 1$.

As to the equivalence of (1) and (1′) we make the following remarks. First, if (1) holds, then $a_1 \ldots a_m < b_1 \vee \ldots \vee b_n$ implies $(a_1 \ldots a_m)(b_1 \vee \ldots \vee b_n) = a_1 \ldots a_m$, hence $(f(a_1) \ldots f(a_m))\,(f(b_1) \vee$

[9]) von Neumann and Stone, Fundamenta Mathematicae, 25 (1935), pp. 353—378, Theorem 14.

$\vee \ldots \vee f(b_n)) = f(a_1) \ldots f(a_m)$, and hence $f(a_1) \ldots f(a_m) < f(b_1) \vee \vee \ldots \vee f(b_n)$; in other words, (1) implies (1′). On the other hand, if (1′) holds, then the relations

$$ab < a,\ ab < b,\ ab < ab,\ a < a \vee b,$$
$$b < a \vee b,\ a \vee b < a \vee b$$

imply the respectively corresponding relations

$$f(ab) < f(a),\quad f(ab) < f(b),\quad f(a)\,f(b) < f(ab),$$
$$f(a) < f(a \vee b),\quad f(b) < f(a \vee b),\quad f(a \vee b) < f(a) \vee f(b),$$

from which we infer the relations (1).

The basis of the inductive construction is the following result: if in any well-ordering of A, the function f has been defined for all predecessors of a given element x in such a way that (1′) holds whenever the elements involved are predecessors of x, then the value $f(x)$ can be so determined that (1′) holds for x and all its predecessors. In proving this result, we have only to examine the limitations imposed upon our choice of $f(x)$. On eliminating all those conditions which are satisfied by hypothesis or by virtue of algebraic identities, we find that the only conditions which are not automatically met are

(3) $a_1 \ldots a_m x < b_1 \vee \ldots \vee b_n$ implies $f(a_1) \ldots f(a_m)\, f(x) < < f(b_1) \vee \ldots \vee f(b_n)$ for $m \geqq 0,\ n \geqq 1$;
(4) $c_1 \ldots c_p < x \vee d_1 \vee \ldots \vee d_q$ implies $f(c_1) \ldots f(c_p) < f(x) \vee \vee f(d_1) \vee \ldots \vee f(d_q)$ for $p \geqq 1,\ q \geqq 0$;

-- where the letters a, b, c, d denote predecessors of x. Now (3) restricts our choice of $f(x)$ only if we can find predecessors $a_1^*, \ldots, a_m^*,\ b_1^*, \ldots, b_n^*$ of x such that $a_1^* \ldots a_m^* x < b_1^* \vee \ldots \vee b_n^*$, $f(a_1^*) = \ldots = f(a_m^*) = e$, $f(b_1^*) = \ldots = f(b_n^*) = 0$. In this case we can satisfy (3) by putting $f(x) = 0$ and in no other way. At the same time, this choice for $f(x)$ leads to the satisfaction of (4). In fact, $c_1 \ldots c_p < x \vee d_1 \vee \ldots \vee d_q$ implies

$$a_1^* \ldots a_m^* c_1 \ldots c_p < b_1^* \vee \ldots \vee b_n^* \vee d_1 \vee \ldots \vee d_q.$$

By hypothesis we therefore have $f(a_1^*) \ldots f(a_m^*)\, f(c_1) \ldots f(c_p) < < f(b_1^*) \vee \ldots \vee f(b_n^*) \vee f(d_1) \vee \ldots \vee f(d_q)$. Remembering that $f(a_1^*) = \ldots = f(a_m^*) = e$, $f(b_1^*) = \ldots = f(b_n^*) = f(x) = 0$, we conclude that $f(c_1) \ldots f(c_p) < f(x) \vee f(d_1) \vee \ldots \vee f(d_q)$. Similarly (4) restricts our choice of $f(x)$ only if we can find predecessors $c_1^*, \ldots, c_p^*, d_1^*, \ldots, d_q^*$ of x such that $c_1^* \ldots c_p^* < x \vee d_1^* \vee \vee \ldots \vee d_q^*$, $f(c_1^*) = \ldots = f(c_p^*) = e$, $f(d_1^*) = \ldots = f(d_q^*) = 0$. In this case we can satisfy (4) by putting $f(x) = e$ and in no other way. At the same time, this choice for $f(x)$ leads to the satisfaction of (3). In fact, $a_1 \ldots a_m x < b_1 \vee \ldots \vee b_n$ implies

$$a_1 \ldots a_m c_1{}^* \ldots c_p{}^* < b_1 \vee \ldots \vee b_n \vee d_1{}^* \vee \ldots \vee d_q{}^*.$$

By hypothesis, we therefore have

$$f(a_1) \ldots f(a_m)\, f(c_1{}^*) \ldots f(c_p{}^*) < f(b_1) \vee \ldots \vee f(b_n) \vee f(d_1{}^*) \vee \ldots \vee \vee f(d_q{}^*).$$

We can then conclude that $f(a_1) \ldots f(a_m)\, f(x) < f(b_1) \vee \ldots \vee f(b_n)$.

We now suppose that A is well-ordered in such a manner that the first element following those of $\mathfrak{a}$ and $\mathfrak{b}$ is the first element belonging neither to $\mathfrak{a}$ nor to $\mathfrak{b}$ — in other words, in such a manner that the class obtained by uniting $\mathfrak{a}$ and $\mathfrak{b}$ is an initial segment of the well-ordering. If each of the elements $a_1, \ldots, a_m, b_1, \ldots, b_n$ belongs to $\mathfrak{a}$ or to $\mathfrak{b}$, then the relation $a_1 \ldots a_m < b_1 \vee \ldots \vee b_n$ implies that some one of $a_1, \ldots, a_m$ belongs to $\mathfrak{a}$ or somme one of $b_1, \ldots, b_n$ to $\mathfrak{b}$. Thus if we put $f = 0$ in $\mathfrak{a}$ and $f = e$ in $\mathfrak{b}$, the condition (1′) is satisfied for all elements belonging to $\mathfrak{a}$ or $\mathfrak{b}$. Proceeding inductively we can define f for all elements in A, beginning with the first element following those of $\mathfrak{a}$ and $\mathfrak{b}$, in such a manner that (1′) is satisfied. The function f so obtained has the properties (1) and (2), so that our proof is complete.

From our existence theorem we can immediately deduce a number of important consequences. First we have

Theorem 7. *Every distributive lattice A containing two or more elements can be partitioned into a prime μ-ideal $\mathfrak{p}$ and a prime α-ideal $\mathfrak{q}$.*

If A has two or more elements, it contains elements a and b such that the relation $a > b$ is false. If we put $\mathfrak{a} = \mathfrak{a}_\mu(a)$ and $\mathfrak{b} = \mathfrak{a}_\alpha(b)$, then $\mathfrak{a}$ and $\mathfrak{b}$ have no common element: for $c \varepsilon \mathfrak{a}$, $c \varepsilon \mathfrak{b}$ would imply $c < a$, $b < c$ and hence $a < b$, contrary to hypothesis. Applying Theorem 6 to the ideals $\mathfrak{a}$ and $\mathfrak{b}$, we obtain the desired partition of A.

We have further

Theorem 8. *If $\mathfrak{a}$ and $\mathfrak{b}$ are distinct μ-ideals, then there exists a prime μ-ideal $\mathfrak{p}$ which contains one but not both of $\mathfrak{a}$ and $\mathfrak{b}$. Dually, if $\mathfrak{a}$ and $\mathfrak{b}$ are distinct α-ideals, then there exists a prime α-ideal $\mathfrak{q}$ which contains one but not noth of $\mathfrak{a}$ and $\mathfrak{b}$.*

When $\mathfrak{a}$ and $\mathfrak{b}$ are distinct μ-ideals, we may suppose the notation so chosen that $\mathfrak{b}$ contains an element b not in $\mathfrak{a}$. The principal α-ideal $\mathfrak{a}_\alpha(b)$ then has no element in common with $\mathfrak{a}$: for $a \varepsilon \mathfrak{a}$ and $a \varepsilon \mathfrak{a}_\alpha(b)$ would imply $b < a$, $b \varepsilon \mathfrak{a}$, contrary to hypothesis. We can thus apply Theorem 6 to the ideals $\mathfrak{a}$ and $\mathfrak{a}_\alpha(b)$, obtaining a prime μ-ideal which contains $\mathfrak{a}$ but not $\mathfrak{a}_\alpha(b)$, b, or $\mathfrak{b}$. The first statement is thereby proved. The second, dual statement may be left to the reader.

From Theorem 8 we now draw the following consequence:

Theorem 9. *If the μ-ideal $\mathfrak{a}$ is distinct from $\mathfrak{e} = A$, then $\mathfrak{a}$ is the product of the prime μ-ideals which contain it. Dually, if the α-ideal $\mathfrak{a}$ is distinct from $\mathfrak{e} = A$, then $\mathfrak{a}$ is the product of the prime α-ideals which contain it.*

Since the μ-ideal $\mathfrak{a}$ is contained in, but distinct from, the μ-ideal $\mathfrak{e}$, there exists a prime μ-ideal which contains $\mathfrak{a}$ (but not $\mathfrak{e}$). The product of the prime μ-ideals which contain $\mathfrak{a}$ is therefore defined. It is a μ-ideal $\mathfrak{b}$ containing $\mathfrak{a}$. If $\mathfrak{a} \neq \mathfrak{b}$, there exists a prime μ-ideal containing $\mathfrak{a}$ but not $\mathfrak{b}$. Since this is impossible, we have $\mathfrak{a} = \mathfrak{b}$, as we wished to prove. The second, dual part of the theorem may be left to the reader.

A fundamental theorem about prime ideals is the following:

Theorem 10. *If a prime μ-ideal $\mathfrak{p}$ is connected with the μ-ideals $\mathfrak{a}$ and $\mathfrak{b}$ by the relation $\mathfrak{p} \supset \mathfrak{a}\mathfrak{b}$, then $\mathfrak{p} \supset \mathfrak{a}$ or $\mathfrak{p} \supset \mathfrak{b}$. Dually, if a prime α-ideal $\mathfrak{q}$ is connected with the α-ideals $\mathfrak{a}$ and $\mathfrak{b}$ by the relation $\mathfrak{q} \supset \mathfrak{a}\mathfrak{b}$, then $\mathfrak{q} \supset \mathfrak{a}$ or $\mathfrak{q} \supset \mathfrak{b}$.*

If $\mathfrak{p} \supset \mathfrak{a}\mathfrak{b}$ and both relations $\mathfrak{p} \supset \mathfrak{a}$, $\mathfrak{p} \supset \mathfrak{b}$ are false, we reach a contradiction as follows: the elements not in $\mathfrak{p}$ constitute a prime α-ideal $\mathfrak{q}$ in accordance with Theorem 4; by hypothesis there exist elements a and b such that $a \,\varepsilon\, \mathfrak{a}$, $b \,\varepsilon\, \mathfrak{b}$, $a \,\varepsilon\, \mathfrak{q}$, $b \,\varepsilon\, \mathfrak{q}$; and hence the element ab belongs both to $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ and to $\mathfrak{q}$. The first part of the theorem is thus established. The second, dual part may be left to the reader.

§ 4. Adjunction of Zero and Unit Elements. In § 1 we have observed that it is always possible to adjoin a zero (or, dually, a unit) to a distributive lattice. In the next section we shall find it convenient to have precise information concerning the effect of such an adjunction upon the ideal structure of a distributive lattice. We therefore state the following result:

Theorem 11. *The adjunction of a zero-element 0 to a distributive lattice A in the manner described in § 1 alters the ideal structure of A only by the adjunction of a prime μ-ideal contained in every μ-ideal and the adjunction of a prime α-ideal containing all α-ideals other than the maximal one. More precisely, the ideal structure of the distributive lattice A^* obtained from A by the indicated adjunction is characterized as follows: every μ-ideal $\mathfrak{a}$ in A becomes through the adjunction of 0 a μ-ideal $\mathfrak{a}^*$ in A^*, $\mathfrak{a}^*$ being prime (principal) if and only if $\mathfrak{a}$ is prime (principal); every α-ideal $\mathfrak{a}$ in A is an α-ideal $\mathfrak{a}^*$ in A^*, $\mathfrak{a}^*$ being prime (principal) in A^* if and only if $\mathfrak{a} = A$ or $\mathfrak{a}$ is prime in A (if and only if $\mathfrak{a}$ is principal in A); and the only ideals in A^* not so obtained are the prime principal μ-ideal generated by 0 and the principal α-ideal coinciding with A^*.*

If $\mathfrak{a}$ is a μ-ideal in A, the adjunction of 0 results in a non-void subset $\mathfrak{a}^*$ of A^*; and it is readily verified that $\mathfrak{a}^*$ has the algebraic properties of a μ-ideal in A^*. Similarly, if $\mathfrak{a}$ is an α-ideal in A, it is a non-void subset $\mathfrak{a}^*$ of A^* which has the algebraic properties of an α-ideal in A^*. If A is partitioned into a μ-ideal $\mathfrak{p}$ and an α-ideal $\mathfrak{q}$, both necessarily prime by Theorem 4, then the corresponding classes $\mathfrak{p}^*$ and $\mathfrak{q}^*$ constitute a partition of A^* into a μ-ideal and an α-ideal; thus $\mathfrak{p}^*$ and $\mathfrak{q}^*$ are both prime. We may therefore state that the ideal $\mathfrak{a}^*$ obtained from $\mathfrak{a}$ in the indicated manner is prime whenever $\mathfrak{a}$ is, in accordance with Theorem 4. Is is easily seen that $\mathfrak{a}^*$ is principal whenever $\mathfrak{a}$ is. The principal μ-ideal generated in A^* by the element 0 consists of the element 0 alone; moreover, since it is evident that $ab = 0$ if and only if $a = 0$ or $b = 0$, this ideal is prime. It follows that the elements of A constitute a prime α-ideal in A^*, in accordance with Theorem 4. The elements of A^* constitute an α-ideal in A^*, which is maximal in the sense that it contains every other α-ideal in A^*; it is the principal α-ideal generated by 0.

If $\mathfrak{a}^*$ is a μ-ideal in A^* which contains some element other than 0, which necessarily belongs to A, the suppression of 0 results in a non-void subset $\mathfrak{a}$ of A; and it is readily verified that $\mathfrak{a}$ has the algebraic properties of a μ-ideal in A. Similarly, if $\mathfrak{a}^*$ is an α-ideal in A^* which does not contain the element 0, it is a non-void subset $\mathfrak{a}$ of A which has the algebraic properties of an α-ideal. If A^* is partitioned into a μ-ideal $\mathfrak{p}^*$ containing some element other than 0 and an μ-ideal $\mathfrak{q}^*$, both necessarily prime by Theorem 4, then the corresponding classes $\mathfrak{p}$ and $\mathfrak{q}$ constitute a partition of A into a μ-ideal and an α-ideal; thus $\mathfrak{p}$ and $\mathfrak{q}$ are both prime in accordance with Theorem 4. We may therefore state that the ideal $\mathfrak{a}$ obtained from $\mathfrak{a}^*$ in the indicated manner is prime whenever $\mathfrak{a}^*$ is, in accordance with Theorem 4. It is easily seen that $\mathfrak{a}$ is principal whenever $\mathfrak{a}^*$ is.

The two preceding paragraphs show that the ideal structure of A^* is characterized in terms of that of A in the manner stated in the theorem. Thus, if we regard the ideal structure of A abstractly, we find that the only alterations brought about by the adjunction of the zero-element are: first, the introduction of the prime μ-ideal corresponding to the μ-ideal generated in A^* by 0; second, the introduction of the prime α-ideal corresponding to the α-ideal A in A^*. The first of these ideals is contained in every μ-ideal, the second contains every α-ideal other than the maximal one, corresponding to A^*.

§ 5. Representations of Distributive Lattices. We shall now turn to the construction of isomorphic representations of distributive lattices by algebras of classes. We have

Theorem 12. *The system $\mathfrak{I}_\mu$ of all μ-ideals in a distributive lattice A is represented isomorphically by an algebra of subclasses* *of the class $\mathfrak{E}$ of all prime μ-ideals in A, in accordance with the following relations: if $\mathfrak{E}(\mathfrak{a})$ is the class of all prime μ-ideals $\mathfrak{p}$ which do not contain the μ-ideal $\mathfrak{a}$, then*

$$(1)\ \mathfrak{E}\left(\mathop{\mathsf{S}}_{\mathfrak{a}\in\mathfrak{A}} \mathfrak{a}\right) = \sum_{\mathfrak{a}\in\mathfrak{A}} \mathfrak{E}(\mathfrak{a});$$

$$(2)\ \mathfrak{E}(\mathfrak{a}\mathfrak{b}) = \mathfrak{E}(\mathfrak{a})\,\mathfrak{E}(\mathfrak{b});$$

$$(3)\ \mathfrak{E}(\mathfrak{a}) = \mathfrak{E}(\mathfrak{b}) \textit{ if and only if } \mathfrak{a} = \mathfrak{b}.$$

In case A has only one element, $\mathfrak{E}$ is void; but the representation described is evidently valid even in this case.

Property (1) is an obvious consequence of the fact that a prime μ-ideal $\mathfrak{p}$ contains $\mathop{\mathsf{S}}_{\mathfrak{a}\in\mathfrak{A}} \mathfrak{a}$ if and only if it contains every $\mathfrak{a}$ in $\mathfrak{A}$. Property (2) is a restatement of Theorem 10. Property (3) is similarly a restatement of Theorem 8.

Theorem 13. *If $\mathfrak{E}(a)$ is the class $\mathfrak{E}(\mathfrak{a}_\mu(a))$, then the correspondence $a \to \mathfrak{E}(a)$ defines an isomorphic representation of the distributive lattice A in accordance with the relations*

$$(1)\ \mathfrak{E}(a \vee b) = \mathfrak{E}(a) \cup \mathfrak{E}(b);$$

$$(2)\ \mathfrak{E}(ab) = \mathfrak{E}(a)\,\mathfrak{E}(b);$$

$$(3)\ \mathfrak{E}(a) = \mathfrak{E}(b) \text{ if and only if } a = b.$$

Furthermore, if $\mathfrak{a}$ is any μ-ideal, then

$$(4)\ \mathfrak{E}(\mathfrak{a}) = \sum_{a\in\mathfrak{a}} \mathfrak{E}(a).$$

Properties (1), (2), (3) follow from the corresponding properties of Theorem 12 with the help of Theorem 2. Property (4) is an immediate consequence of Theorem 12 (1) and the obvious relation $\mathfrak{a} = \mathop{\mathsf{S}}_{a\in\mathfrak{a}} \mathfrak{a}_\mu(a)$.

From Theorem 11, we can immediately read off the effect of the adjunction of a zero-element upon the representation described in Theorems 12 and 13. We have:

Theorem 14. *If the distributive lattice A^* is obtained from A by the adjunction of a zero-element, then its class-representation can be obtained from that of A by adjoining a single point to $\mathfrak{E}$ and each of the classes $\mathfrak{E}(\mathfrak{a})$ and including the void class with the classes so obtained.*

In view of the simple relations disclosed in Theorems 11 and 14, we may henceforth confine our attention to distributive lattices with zeros: we are able to adjoin a zero to any distributive lattice which has no zero; and we do not thereby make any complicated alterations in the ideal structure or its class-representation. On the other hand, if we attempt to consider the general case, the theorems which follow become needlessly complicated through the intrusion of slight but unavoidable exceptions. For these reasons we make the restriction to the indicated special case.

We now proceed to the introduction of topological concepts.

Theorem 15. *If A has a zero, the class E of all sets $\mathfrak{E}(\mathfrak{a})$ described in Theorem 12 may be taken as the class of all open sets in a certain uniquely determined topology for $\mathfrak{E}$. In this topology $\mathfrak{E}$ is a T_0-space in which the sets $\mathfrak{E}(a)$ constitute a basis. The sets $\mathfrak{E}(a)$ are characterized topologically as the relatively bicompact*[10]*) open sets in $\mathfrak{E}$. The space $\mathfrak{E}$ has the following topological properties: if B is the class of all relatively bicompact open sets in $\mathfrak{E}$, then*

(1) B *is a basis for* $\mathfrak{E}$;
(2) B *is a multiplicative class — that is, contains the intersection of any two of its members;*
(3) *if $\mathfrak{F}$ is any closed set and if C is any subclass of B such that $\mathfrak{E}_1 \varepsilon \mathsf{C}, \ldots, \mathfrak{E}_n \varepsilon \mathsf{C}$ imply $\mathfrak{F}\mathfrak{E}_1 \ldots \mathfrak{E}_n \neq \mathfrak{O}$, then $\mathfrak{F}\prod_{\mathfrak{E}\varepsilon\mathsf{C}} \mathfrak{E} \neq \mathfrak{O}$.*

Since E contains the void set $\mathfrak{O} = \mathfrak{E}(0)$, the set $\mathfrak{E} = \mathfrak{E}(A)$, the union of any subclass of its sets (by Theorem 12 (1)), and the intersection of any finite subclass of its sets (by Theorem 12 (2)), there exists a topology in $\mathfrak{E}$ such that E is the class of all open sets; and in this topology $\mathfrak{E}$ is a T-space.[11]) We can show that $\mathfrak{E}$ is in fact a T_0-space: if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are distinct points in $\mathfrak{E}$, we consider them as prime μ-ideals in A and note that at least one of them does not contain the other; we thus infer that $\mathfrak{E}(\mathfrak{p}_1)$ does not contain $\mathfrak{p}_1$, $\mathfrak{E}(\mathfrak{p}_2)$ does not contain $\mathfrak{p}_2$, but that $\mathfrak{E}(\mathfrak{p}_1)$ contains $\mathfrak{p}_2$ or $\mathfrak{E}(\mathfrak{p}_2)$ contains $\mathfrak{p}_1$; and we therefore conclude that one of the two points $\mathfrak{p}_1$ and $\mathfrak{p}_2$ is contained in an open set which does not contain the other. Theorem 13 (4) shows that the sets $\mathfrak{E}(a)$ constitute a basis for $\mathfrak{E}$. To prove that every set $\mathfrak{E}(a)$ is relatively bicompact as well as open, we proceed as follows: if a family of open sets $\mathfrak{E}(\mathfrak{a})$, $\mathfrak{a} \varepsilon \mathfrak{A}$, covers $\mathfrak{E}(a)$, $\mathfrak{E}(a) \subset \sum_{\mathfrak{a}\varepsilon\mathfrak{A}} \mathfrak{E}(\mathfrak{a})$, then Theorem 12 (1) shows that $a \varepsilon \underset{\mathfrak{a}\varepsilon\mathfrak{A}}{\mathsf{S}}\, \mathfrak{a}$; hence, as we have seen in § 2, the relations $a = a_1 \vee \ldots \vee a_n$, $a_1 \varepsilon \mathfrak{a}_1 \varepsilon \mathfrak{A}, \ldots, a_n \varepsilon \mathfrak{a}_n \varepsilon \mathfrak{A}$ can be satisfied; and it follows from Theorem 13 that

$$\mathfrak{E}(a) = \mathfrak{E}(a_1) \cup \ldots \cup \mathfrak{E}(a_n) \subset \mathfrak{E}(\mathfrak{a}_1) \cup \ldots \cup \mathfrak{E}(\mathfrak{a}_n),$$

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[10]) We shall say that a subset of a topological space is relatively bicompact if, whenever it is covered by a family of open sets, it is also covered by some finite subfamily.

[11]) Alexandroff and Hopf, Topologie I (Berlin, 1935), Kap. I, § 2, Satz X, p. 44.

as we wished to prove. On the other hand, if the open set $\mathfrak{E}(\mathfrak{a})$ is relatively bicompact, the relation (4) of Theorem 13 becomes $\mathfrak{E}(\mathfrak{a}) = \mathfrak{E}(a_1) \cup \ldots \cup \mathfrak{E}(a_n)$ for suitable elements $a_1, \ldots, a_n$ in $\mathfrak{a}$; and thus $\mathfrak{E}(\mathfrak{a}) = \mathfrak{E}(a_1 \vee \ldots \vee a_n)$ in accordance with Theorem 13 (1). From what we have already proved, we see that the class $\mathbf{B}$ of all relatively bicompact open sets in $\mathfrak{E}$ has the properties (1) and (2) asserted in the present theorem. To establish property (3) we proceed as follows. Let $\mathfrak{F}$ and $\mathbf{C}$ be as described there. Then $\mathfrak{F}$ is certainly a non-void closed set. Its open complement we denote by $\mathfrak{E}(\mathfrak{a})$. By hypothesis, the relations $\mathfrak{C}_1 \varepsilon \mathbf{C}, \ldots, \mathfrak{C}_n \varepsilon \mathbf{C}$, $\mathfrak{C}_1 \ldots \mathfrak{C}_n \subset \subset \mathfrak{E}(\mathfrak{a})$ are incompatible; in other words, if $\mathfrak{c}$ is the class of all elements c in A such that $\mathfrak{E}(c) = \mathfrak{C} \varepsilon \mathbf{C}$, the relations $c_1 \varepsilon \mathfrak{c}, \ldots, c_n \varepsilon \mathfrak{c}$, $c_1 \ldots c_n \varepsilon \mathfrak{a}$ are incompatible, by virtue of Theorem 13 (2). If $\mathfrak{b}$ is the α-ideal generated by $\mathfrak{c}$ — namely, the class of all elements b such that $c_1 \ldots c_n < b$ for $c_1, \ldots, c_n$ in $\mathfrak{c}$ — then $\mathfrak{b}$ has no element in common with $\mathfrak{a}$. According to Theorem 6, there exists a prime μ-ideal $\mathfrak{p}$ which contains $\mathfrak{a}$ but is disjoint from $\mathfrak{b}$ (and hence also from $\mathfrak{c}$). It follows that $\mathfrak{p}$ belongs to $\mathfrak{F} = \mathfrak{E}'(\mathfrak{a})$ and to $\mathfrak{E}(c)$, $c \varepsilon \mathfrak{c}$. Hence $\mathfrak{p}$ belongs to $\mathfrak{F} \prod_{\mathfrak{C} \varepsilon \mathbf{C}} \mathfrak{C}$, as we wished to prove.

We shall now invert the results obtained in Theorems 12, 13 and 15 — that is to say, we shall show that the topological properties set forth in Theorem 15 characterize the representations of distributive lattices obtained in Theorems 12 and 13.

Theorem 16. *Let $\mathfrak{S}$ be a T_0-space in which the class $\mathbf{B}$ of relatively bicompact open sets has the properties (1)—(3) of the preceding theorem. Then the sets in $\mathbf{B}$ constitute a distributive lattice A under the operations of forming finite unions and intersections. The algebra of all open sets in $\mathfrak{S}$ is isomorphic to that of the μ-ideals in A under the correspondence $\mathfrak{a} \to \mathfrak{S}(\mathfrak{a}) = \sum_{a \varepsilon \mathfrak{a}} a$. If $\mathfrak{p}$ is any prime μ-ideal, there exists a unique point $\mathfrak{s}$ in $\mathfrak{S}$ such that $\mathfrak{S}(\mathfrak{p}) = \{\mathfrak{s}\}^{-\prime}$; and the resulting correspondence $\mathfrak{p} \to \mathfrak{s}$ defines a topological equivalence between the given space $\mathfrak{S}$ and the space $\mathfrak{E}$ constructed from A in the manner described in Theorems 12 and 15.*

If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are relatively bicompact open sets, then so is their union: if a family of open sets covers $\mathfrak{B}_1 \cup \mathfrak{B}_2$, then there exist finite subfamilies which cover $\mathfrak{B}_1$ and $\mathfrak{B}_2$ separating, so that the union of these two subfamilies is a subfamily covering $\mathfrak{B}_1 \cup \mathfrak{B}_2$; hence $\mathfrak{B}_1 \cup \mathfrak{B}_2$, which is known to be open, is relatively bicompact as well. The intersection of relatively bicompact open sets $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is, of course, open and is here relatively bicompact by assumption. Thus the sets in $\mathbf{B}$ constitute a distributive lattice A, as stated in the theorem. For convenience in comparing the present analysis with previous results we shall denote the elements of this lattice by the letters $a, b, c, \ldots$

If now $\mathfrak{a}$ is any μ-ideal in A, we may associate with it the open set $\mathfrak{S}(\mathfrak{a}) = \sum_{a \varepsilon \mathfrak{a}} a$ in $\mathfrak{S}$. We can show at once that $\mathfrak{S}(\mathfrak{a}) \subset \mathfrak{S}(\mathfrak{b})$ implies $\mathfrak{a} \subset \mathfrak{b}$: if a is any element in $\mathfrak{a}$, we have $a = \mathfrak{S}(\mathfrak{a}_\mu(a)) \subset \subset \mathfrak{S}(\mathfrak{a}) \subset \mathfrak{S}(\mathfrak{b}) = \sum_{b \varepsilon \mathfrak{b}} b$; hence there exist elements $b_1, \ldots, b_n$ in $\mathfrak{b}$ such that $a < b_1 \vee \ldots \vee b_n$ in accordance with the relative bicompactness of a; and we thus see that $a \varepsilon \mathfrak{a}$ implies $a \varepsilon \mathfrak{b}$. It follows that $\mathfrak{S}(\mathfrak{a}) = \mathfrak{S}(\mathfrak{b})$ if and only if $\mathfrak{a} = \mathfrak{b}$. If $\mathfrak{G}$ is an arbitrary open set in $\mathfrak{S}$, the class of all sets in **B** which are contained in $\mathfrak{G}$ is obviously a μ-ideal $\mathfrak{a}$ in A such that $\mathfrak{G} = \mathfrak{S}(\mathfrak{a})$ in accordance with our assumption that **B** is a basis for $\mathfrak{S}$.

We can now investigate the nature of the set $\mathfrak{S}(\mathfrak{p})$ when $\mathfrak{p}$ is a prime μ-ideal. The class $\mathfrak{q}$ of all elements in A which do not belong to $\mathfrak{p}$ is a prime α-ideal in accordance with Theorem 4. Thus the closed set $\mathfrak{F} = \mathfrak{S}'(\mathfrak{p})$ and the subclass $\mathfrak{q}$ of A and **B** have the property that $a_1 \varepsilon \mathfrak{q}, \ldots, a_n \varepsilon \mathfrak{q}$ imply $\mathfrak{F} a_1 \ldots a_n \neq \mathfrak{O}$: for $a_1 \ldots a_n \subset \subset \mathfrak{F}' = \mathfrak{S}(\mathfrak{p})$ would imply $a_1 \ldots a_n \varepsilon \mathfrak{p}$ in contradiction to the known relation $a_1 \ldots a_n \varepsilon \mathfrak{q}$. By hypothesis, we can therefore conclude that $\mathfrak{F}$ contains a point $\mathfrak{s}$ common to all the members of $\mathfrak{q}$. We thus see that the partition of **B** into the ideals $\mathfrak{p}$ and $\mathfrak{q}$ coincides with the partition of **B** into the class of all its members which do not contain $\mathfrak{s}$ and the class of all its members which do contain $\mathfrak{s}$. Since **B** is a basis for $\mathfrak{S}$, we infer that $\mathfrak{S}(\mathfrak{p}) = \{\mathfrak{s}\}^{-\prime}$. On the other hand, if $\mathfrak{s}$ is an arbitrary point in $\mathfrak{S}$, we can form a partition of **B** into the class $\mathfrak{p}$ of all its members which do not contain $\mathfrak{s}$ and the class $\mathfrak{q}$ of all its members which do contain $\mathfrak{s}$. Since **B** is a basis for the T_0-space $\mathfrak{S}$, neither class is void; and it is easily verified that $\mathfrak{p}$ is a μ-ideal, $\mathfrak{q}$ an α-ideal. Theorem 4 shows that both $\mathfrak{p}$ and $\mathfrak{q}$ are prime. It is evident that $\mathfrak{S}(\mathfrak{p}) = \{\mathfrak{s}\}^{-\prime}$. If $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime μ-ideals and if $\mathfrak{S}(\mathfrak{p}_1) = \{\mathfrak{s}_1\}^{-\prime}$, $\mathfrak{S}(\mathfrak{p}_2) = \{\mathfrak{s}_2\}^{-\prime}$, then the relations $\mathfrak{p}_1 = \mathfrak{p}_2$ and $\mathfrak{s}_1 = \mathfrak{s}_2$ are equivalent: for the relations $\mathfrak{p}_1 = \mathfrak{p}_2$, $\mathfrak{S}(\mathfrak{p}_1) = \mathfrak{S}(\mathfrak{p}_2)$, $\{\mathfrak{s}_1\}^- = \{\mathfrak{s}_2\}^-$ are equivalent; and in the T_0-space $\mathfrak{S}$, the third relation here is equivalent to $\mathfrak{s}_1 = \mathfrak{s}_2$. Thus the correspondence $\mathfrak{p} \to \mathfrak{s}$ resulting from the relation $\mathfrak{S}(\mathfrak{p}) = \{\mathfrak{s}\}^{-\prime}$ is a biunivocal correspondence between $\mathfrak{E}$ and $\mathfrak{S}$. Furthermore, if $\mathfrak{p} \to \mathfrak{s}$ in this correspondence, we see that $\mathfrak{p} \varepsilon \mathfrak{E}(a)$ if and only if $\mathfrak{s} \varepsilon a$: for $\mathfrak{p} \varepsilon \mathfrak{E}(a)$ is equivalent to a non $\varepsilon\, \mathfrak{p}$; and the latter relation is equivalent to $\mathfrak{s} \varepsilon a$ by virtue of the preceding analysis of the relation $\mathfrak{S}(\mathfrak{p}) = \{\mathfrak{s}\}^{-\prime}$. Thus the correspondence between $\mathfrak{E}$ and $\mathfrak{S}$ carries the basis of all sets $\mathfrak{E}(a)$ into the basis B of all sets a. It is therefore a topological equivalence between $\mathfrak{E}$ and $\mathfrak{S}$.

It is now evident that this correspondence carries the open sets $\mathfrak{E}(\mathfrak{a})$ and $\mathfrak{S}(\mathfrak{a})$, associated with a μ-ideal $\mathfrak{a}$ in A, into one another. The remaining statements of the present theorem then follow from the properties of the sets $\mathfrak{E}(\mathfrak{a})$ established in Theorem 12.

We may remark in closing that the void class $\mathfrak{O}$ belongs to B and is the zero-element of the distributive lattice A.

The reader who wishes to investigate what occurs in Theorems 15 and 16 if the condition of the existence of a zero-element be relaxed should now have no difficulty in doing so. The study of a distributive lattice without zero is carried out most simply by adjoining a zero-element as previously described and later removing the appropriate point from the representative space constructed for the enlarged lattice, as we have already suggested in Theorem 14.

It is interesting to consider certain specializations of the results of Theorem 15. We first have

Theorem 17. *The space $\mathfrak{E}$ of Theorem 15 is a T-space if and only if every prime μ-ideal in the distributive lattice A is divisorless.*

From Theorems 15 and 16 it is evident that $\mathfrak{E}(\mathfrak{p}) = \{\mathfrak{p}\}^{-\prime}$ whenever $\mathfrak{p}$ is a prime μ-ideal. In order that $\mathfrak{E}$ be a T_1-space it is necessary and sufficient that $\{\mathfrak{p}\}^- = \{\mathfrak{p}\}$ for every point $\mathfrak{p}$ in $\mathfrak{E}$. Thus, if $\mathfrak{E}$ is a T_1-space, we have $\mathfrak{E}(\mathfrak{p}) = \{\mathfrak{p}\}'$ for every prime μ-ideal $\mathfrak{p}$. Consequently a μ-ideal $\mathfrak{a}$ which contains $\mathfrak{p}$ must have the property that $\mathfrak{E}(\mathfrak{a}) = \mathfrak{E}(\mathfrak{p})$ or $\mathfrak{E}(\mathfrak{a}) = \mathfrak{E}$. Hence $\mathfrak{a} \supset \mathfrak{p}$ implies $\mathfrak{a} = \mathfrak{p}$ or $\mathfrak{a} = \mathfrak{e} = A$. Consequently, every prime μ-ideal in A is divisorless. On the other hand, if $\mathfrak{E}$ is not a T_1-space, we can find prime μ-ideals $\mathfrak{p}_1, \mathfrak{p}_2$ such that $\mathfrak{p}_1 \neq \mathfrak{p}_2$, $\mathfrak{p}_1 \,\varepsilon\, \{\mathfrak{p}_2\}^-$. It follows that $\{\mathfrak{p}_1\}^- \subset \{\mathfrak{p}_2\}^-$, $\mathfrak{E}(\mathfrak{p}_1) \supset \mathfrak{E}(\mathfrak{p}_2)$, $\mathfrak{p}_1 \supset \mathfrak{p}_2$. Thus the prime μ-ideal $\mathfrak{p}_2$ is not divisorless. The theorem is thereby established. We recall that it was proved in Theorem 3 that every divisorless ideal is prime.

A more interesting result is the following:

Theorem 18. *The space $\mathfrak{E}$ of Theorem 15 is an H-space if and only if the distributive lattice A is a generalized Boolean algebra.*

In an H-space every relatively bicompact subset is closed.[12]) Thus if $\mathfrak{E}$ is an H-space and if $a < b$, the sets $\mathfrak{E}(a)$ and $\mathfrak{E}'(a)\,\mathfrak{E}(b)$, contained in $\mathfrak{E}(b)$, are open and closed in $\mathfrak{E}$ and hence open and closed relative to $\mathfrak{E}(b)$. Since $\mathfrak{E}'(a)\,\mathfrak{E}(b)$ is closed relative to the relatively bicompact set $\mathfrak{E}(b)$, it is also relatively bicompact. Consequently, $\mathfrak{E}'(a)\,\mathfrak{E}(b) = \mathfrak{E}(c)$ where $c \,\varepsilon\, A$. We have thus proved that, whenever $a < b$, the system of equations $x \vee a = b$, $xa = 0$ has

[12]) Alexandroff and Hopf, Topologie I (Berlin, 1935), Kap. II, § 1, Satz XI, p. 91.

a solution in A, namely, the element c just constructed. Thus A is a generalized Boolean algebra. On the other hand, if A is a generalized Boolean algebra and $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are distinct prime μ-ideals in A, we may suppose, since $\mathfrak{E}$ is a T_0-space, that there exist elements a_1 and b_2 in A such that $\mathfrak{p}_1 \varepsilon \mathfrak{E}(a_1)$, $\mathfrak{p}_2 \varepsilon \mathfrak{E}(b_2)$ and $\mathfrak{p}_1$ non $\varepsilon \mathfrak{E}(b_2)$ or $\mathfrak{p}_2$ non $\varepsilon \mathfrak{E}(a_1)$. By proper choice of notation, we may arrange that $\mathfrak{p}_2$ non $\varepsilon \mathfrak{E}(a_1)$. Then, if a_2 is a solution of the system of equations $x \vee a_1b_2 = b_2$, $x(a_1b_2) = 0$ where $a_1b_2 < b_2$, we see that $\mathfrak{p}_1 \varepsilon \mathfrak{E}(a_1)$, $\mathfrak{p}_2$ non $\varepsilon \mathfrak{E}(a_2)$, $\mathfrak{E}(a_1)\,\mathfrak{E}(a_2) = \mathfrak{O}$. Hence $\mathfrak{E}$ is an H-space.

We may remark that, when $\mathfrak{E}$ is an H-space, the conditions (2) and (3) of Theorem 15 are redundant.[13]) In other words, the representative topological spaces for generalized Boolean algebras are characterized as those H-spaces in which B is a basis.

Theorem 18 shows how greatly simplified the theory of distributive lattices becomes in the case of generalized Boolean algebras and emphasizes the special importance of the latter from a topological point of view.

§ 6. Applications to General Topology. In concluding our discussion of distributive lattices, we may make a few remarks on the application of the representation theory to the study of general topological spaces.

If B is any basis in a T_0-space $\mathfrak{S}$, we can form the class A of all sets obtained from those of B by the application of the operations of forming finite unions and intersections. Then A is a basis for $\mathfrak{S}$ and is also a distributive lattice. The construction of the representative space $\mathfrak{E}$ of Theorems 12 and 15 for this lattice therefore provides us with a representation of $\mathfrak{S}$ in $\mathfrak{E}$. The precise sense in which $\mathfrak{S}$ is represented in $\mathfrak{E}$ will not be considered here. We may observe, however, that such a representation is most interesting in the cases where $\mathfrak{E}$ is of sharply defined topological nature. Now to require that $\mathfrak{E}$ be an H-space is to require also that A be a generalized Boolean algebra. In general, the latter requirement is impossible of fulfillment — unless, in constructing A, we admit the formation of complements and relinquish the condition that A be a basis. In our unpublished work on the applications of Boolean algebras to topology, we have elaborated the program thus suggested. The alternative program of keeping A a basis allowing and $\mathfrak{E}$ to be other than an H-space seems to be of less immediate interest although it is technically similar and can hardly present peculiar difficulties.

[13]) Alexandroff and Hopf, Topologie I (Berlin, 1935), Kap. II, § 1, Satz XI, p. 91; Alexandroff and Urysohn, Mémoire sur les espaces topologiques compacts, Verhandelingen der Koninklijke Akademie der Wetenschappen te Amsterdam, Deel XIV, No. 1 (1929), p. 12 and preceding discussion.

Part II. Brouwerian Logics.

§ 1. Descriptive Introduction. In order to describe our general approach to the analysis of logical systems, we must attempt a brief characterization of the construction of such systems. The customary procedure in setting up a formal or symbolic logic of propositions may be summarized not too inaccurately as follows: (1) propositions are regarded as abstract entities which can be affected or combined by the application of certain postulated operations; (2) propositions are to be grouped into „asserted" and „unasserted" propositions, the grouping to be subject to subsequently stated conditions; (3) certain combinations of propositions are required postulationally to be „asserted" in all groupings; (4) certain assignments of certain sets of propositions as respectively „asserted" or „unasserted" are postulated (or, alternatively, excluded) in all groupings. In (3) are provided the „formal" rules of the logical system, in (4) the „informal". The admissible groupings under (2), (3), and (4) may be called „evaluations" of the system of propositions in the given logic. In studying such a logic of propositions from a mathematical point of view, we confine our attention to a class of propositions which has the property that it contains all the propositions obtained from any of its members by application of the operations postulated in (1). Associated with this class, we consider the class of all possible „evaluations" under (2), (3), (4). It is then natural to represent any proposition in the chosen class by the class of all those „evaluations" in which it is „asserted". In this manner, propositions, operations upon them, and even the construction of „evaluations" are represented in terms of abstract classes. As we have already suggested in the introduction to this paper, this kind of mathematical representation is illustrated by the familiar Leibnitz diagrams.

We propose to consider the Brouwerian logic of propositions from the point of view set forth in the preceding paragraph. A symbolic statement of this logic has been given by Heyting,[14]) with the significant warning, „In principle, it is impossible to set up a formal system which would be equivalent to intuitionist mathematics . . .". We shall follow Heyting's statement, with a few modifications which will be mainly notational in character.

We begin therefore with a class A of objects, $a, b, c, \ldots$, called propositions, and four operations upon them which can be applied within A — three binary operations which we indicate by $a \vee b$, ab, and $a \rightarrow b$ (to be read as the propositions „a or b", „a and b", and „a implies b", respectively); and a unary operation which we

[14]) Heyting, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse, 1930, pp. 42—56.

indicate by a^* (to be read as the proposition „not a"). Heyting writes $a \wedge b$ for ab, $a \supset b$ for $a \rightarrow b$, and $\neg a$ for a^*. We consider the class $\mathfrak{E}$ of all partitions $\mathfrak{p}$ of A into disjoint nonvoid classes $\mathfrak{a}$ and $\mathfrak{u}$ (called, respectively, the class of „asserted" propositions and the class of „unasserted" propositions) subject to „formal" and „informal" rules given below. The admissible partitions $\mathfrak{p}$ will be called „evaluations" of A. If a is any proposition we denote by $\mathfrak{E}(a)$ the class of all evaluations $\mathfrak{p}$ such that a is „asserted". If $\mathfrak{E}(a) = \mathfrak{E}$, we write $\vdash a$; if $\mathfrak{E}(a) = \mathfrak{E}$ is assumed as a postulate we write $\vdash\vdash a$. The „formal" rules as given by Heyting then take the form

(2.1) $\vdash\vdash a \rightarrow aa$,
(2.11) $\vdash\vdash ab \rightarrow ba$,
(2.12) $\vdash\vdash (a \rightarrow b) \rightarrow (ac \rightarrow bc)$,
(2.13) $\vdash\vdash (a \rightarrow b)\,(b \rightarrow c) \rightarrow (a \rightarrow c)$,
(2.14) $\vdash\vdash b \rightarrow (a \rightarrow b)$,
(2.15) $\vdash\vdash a(a \rightarrow b) \rightarrow b$,
(3.1) $\vdash\vdash a \rightarrow a \vee b$,
(3.11) $\vdash\vdash a \vee b \rightarrow b \vee a$,
(3.12) $\vdash\vdash (a \rightarrow c)\,(b \rightarrow c) \rightarrow (a \vee b \rightarrow c)$,
(4.1) $\vdash\vdash a^* \rightarrow (a \rightarrow b)$,
(4.11) $\vdash\vdash (a \rightarrow b)\,(a \rightarrow b^*) \rightarrow a^*$.

We have followed Heyting's numbering for convenience in referring to his paper. The „informal" rules which we shall postulate go beyond those of Heyting. They can be stated as follows:

(1.2) $\mathfrak{E}(a)\,\mathfrak{E}(b) \subset \mathfrak{E}(ab)$;
(1.3) $\mathfrak{E}(a)\,\mathfrak{E}(a \rightarrow b) \subset \mathfrak{E}(b)$;
(1.7) $\mathfrak{E}(a \vee b) \subset \mathfrak{E}(a) \cup \mathfrak{E}(b)$;
(1.8) if $\mathfrak{E}(a) \subset \mathfrak{E}(b)$, then $\mathfrak{E}(a \rightarrow b) = \mathfrak{E}$.

It is clear that (1.2) „if a is asserted and b is asserted, then ab is asserted" and (1.3) „if a is asserted and ‚a implies b' is asserted, then b is asserted" correspond to the like-numbered „informal" rules of Heyting. On the other hand, (1.7) „if ‚a or b' is asserted, then a is asserted or b is asserted" and (1.8) „if b is asserted whenever a is asserted, then ‚a implies b' is asserted in all evaluations" are „informal" rules which have no application in working out consequences of the „formal" rules by finite procedures: (1.7) merely presents an alternative and (1.8) requires the construction of all evaluations before it can be used. We may regard (1.7) and (1.8) as bringing the postulated operations $\vee$ and $\rightarrow$ closer to their colloquial meanings. It is clear that (1.8) is also a postulate of „completeness" in that it describes the possibility of inferring b from a under all evaluations by the statement that ‚a implies b' is asserted in all evaluations.

From Heyting's paper we may now collect certain consequences of (1.2), (1.3), (2.1) — (4.11) which we shall use below. With his numbering and our notation they are

(2.2) $\vdash ab \to a,$
(2.282) $\vdash a(ab \to c) \to (b \to c),$
(4.4) $\vdash a^*a \to b,$
(4.5) $\vdash (ba \to a^*) \to (b \to a^*).$

§ 2. Representations. We turn now to the analysis of the representation of the system A by means of the classes $\mathfrak{C}(a)$. We have

Theorem 1. *$\mathfrak{C}(a) \subset \mathfrak{C}(b)$ if and only if $\mathfrak{C}(a \to b) = \mathfrak{C}$ — that is, if and only if $\vdash a \to b$.*

This result follows immediately from (1.3) and (1.8).

Theorem 2. $\mathfrak{C}(ab) = \mathfrak{C}(a)\,\mathfrak{C}(b)$.

From (2.2) and Theorem 1, we have $\mathfrak{C}(ab) \subset \mathfrak{C}(a)$, $\mathfrak{C}(a) \subset \mathfrak{C}(b)$. From (2.11) and Theorem 1, we have $\mathfrak{C}(ab) \subset \mathfrak{C}(ba)$. Hence $\mathfrak{C}(ab) \subset \subset \mathfrak{C}(a)\,\mathfrak{C}(b)$. Combining this result with (1.2), we obtain the theorem.

Theorem 3. $\mathfrak{C}(a \vee b) = \mathfrak{C}(a) \cup \mathfrak{C}(b)$.

The proof, similar to that of Theorem 2, depends in an obvious way upon (1.7), (3.1), (3.11), and Theorem 1.

Theorem 4. *If $\mathfrak{O}$ is the void class, then*

(1) $\mathfrak{C}(a^*)\,\mathfrak{C}(a) = \mathfrak{O}$;
(2) $\mathfrak{C}(b)\,\mathfrak{C}(a) = \mathfrak{O}$ *implies* $\mathfrak{C}(b) \subset \mathfrak{C}(a^*)$.

Theorems 1 and 2 show that (4.4) is equivalent to $\mathfrak{C}(a^*)\,\mathfrak{C}(a) \subset \subset \mathfrak{C}(b)$. By hypothesis every evaluation $\mathfrak{p}$ assigns some proposition b to the class of unasserted propositions. Hence a^* and a cannot both be asserted propositions, whatever the evaluation $\mathfrak{p}$. In other words, the class $\mathfrak{C}(a^*)\,\mathfrak{C}(a)$ is void. On the other hand, $\mathfrak{C}(b)\,\mathfrak{C}(a) = \mathfrak{O}$ implies $\mathfrak{C}(b)\,\mathfrak{C}(a) \subset \mathfrak{C}(a^*)$. Applying Theorems 1 and 2 to the latter relation, we obtain $\mathfrak{C}(ba \to a^*) = \mathfrak{C}$. Applying Theorem 1 to (4.5), we likewise obtain $\mathfrak{C}(ba \to a^*) \subset \mathfrak{C}(b \to a^*)$. It follows that $\mathfrak{C}(b \to \to a^*) = \mathfrak{C}$. Another application of Theorem 1 now yields the desired relation $\mathfrak{C}(b) \subset \mathfrak{C}(a^*)$.

Theorem 5. *The class $\mathfrak{C}(a \to b)$ has the following properties:*

(1) $\mathfrak{C}(a \to b) \subset \mathfrak{C}'(a) \cup \mathfrak{C}(b)$;
(2) $\mathfrak{C}(c) \subset \mathfrak{C}'(a) \cup \mathfrak{C}(b)$ *implies* $\mathfrak{C}(c) \subset \mathfrak{C}(a \to b)$.

By (2.15) and Theorem 1, we have $\mathfrak{C}(a)\,\mathfrak{C}(a \to b) \subset \mathfrak{C}(b)$. Hence we have $\mathfrak{C}(a \to b) = \mathfrak{C}'(a)\,\mathfrak{C}(a \to b) \cup \mathfrak{C}(a)\,\mathfrak{C}(a \to b) \subset \subset \mathfrak{C}'(a) \cup \mathfrak{C}(b)$. On the other hand, $\mathfrak{C}(c) \subset \mathfrak{C}'(a) \cup \mathfrak{C}(b)$ implies $\mathfrak{C}(c)\,\mathfrak{C}(a) \subset \mathfrak{C}(b)$; and by Theorems 1 and 2 the latter relation implies $\mathfrak{C}(ca \to b) = \mathfrak{C}$. Applying Theorems 1 and 2 to (2.282) (with a, b, c replaced by c, a, b respectively), we have $\mathfrak{E}(c)\,\mathfrak{E}(ca \to b) \subset \mathfrak{E}(a \to b)$. Hence the desired relation $\mathfrak{E}(c) \subset \mathfrak{E}(a \to b)$ must be valid.

We now introduce topological considerations, as suggested by the results reached in Theorems 4 and 5.

Theorem 6. *If each class $\mathfrak{E}(a)$, $a \varepsilon A$, in $\mathfrak{E}$ is assigned as a neighborhood of every point $\mathfrak{p}$ which it contains, then $\mathfrak{E}$ becomes a T_0-space. The totality of classes $\mathfrak{E}(a)$ is a basis* A *which is also a distributive lattice under the operations of forming finite unions and intersections. $\mathfrak{E}(a^*)$ is the interior of the class $\mathfrak{E}'(a)$, $\mathfrak{E}(a \to b)$ the interior of the class $\mathfrak{E}'(a) \cup \mathfrak{E}(b)$. In terms of the closure operation for $\mathfrak{E}$, the relations $\mathfrak{E}(a^*) = \mathfrak{E}^{-\prime}(a)$, $\mathfrak{E}(a \to b) = [\mathfrak{E}(a)\,\mathfrak{E}'(b)]^{-\prime}$ are therefore valid.*

If $\mathfrak{p}$ is any evaluation of A, then there is at least one proposition a which is „asserted" under $\mathfrak{p}$ — for instance, all the propositions listed under (2.1) — (4.11) are such. Thus every point $\mathfrak{p}$ in $\mathfrak{E}$ has at least one neighborhood $\mathfrak{E}(a)$. If $\mathfrak{E}(a)$ and $\mathfrak{E}(b)$ are neighborhoods of $\mathfrak{p}$, then $\mathfrak{E}(ab) = \mathfrak{E}(a)\,\mathfrak{E}(b)$ is also a neighborhood of $\mathfrak{p}$ contained in both. If $\mathfrak{q}$ is any point belonging to a neighborhood $\mathfrak{E}(a)$ of $\mathfrak{p}$, then $\mathfrak{E}(a)$ is a neighborhood of $\mathfrak{q}$ contained in $\mathfrak{E}(a)$. Finally, if $\mathfrak{p}$ and $\mathfrak{q}$ are distinct evaluations of A, there is some proposition a which is asserted under one and not under the other. Hence $\mathfrak{E}(a)$ is a neighborhood of one of the two points $\mathfrak{p}$ and $\mathfrak{q}$ which does not contain the other. These properties of the indicated neighborhood-system show that its imposition upon $\mathfrak{E}$ converts $\mathfrak{E}$ into a T_0-space.[15]) From Theorems 2 and 3 it is obvious that A is a distributive lattice as stated. From Theorems 4 and 5 the characterizations of the classes $\mathfrak{E}(a^*)$ and $\mathfrak{E}(a \to b)$ are also obvious.

We can now show that the properties of T_0-spaces lead to no contradiction between Theorem 6 and the „formal rules" with which we started. We have

Theorem 7. *Let $\mathfrak{S}$ be a T_0-space; and let A be any class of open sets $a, b, c, \dots$ in $\mathfrak{S}$ such that A contains $a \vee b = a \cup b$, ab, $a \to b = (ab')^{-\prime}$, and $a^* = a^{-\prime}$ together with a and b. Then the following combinations of arbitrary sets in A coincide with $\mathfrak{S}$: $a \to aa$, $ab \to ba$, $(a \to b) \to (ac \to bc)$, $(a \to b)\,(b \to c) \to (a \to c)$, $b \to (a \to b)$, $a(a \to b) \to b$, $a \to a \vee b$, $a \vee b \to b \vee a$, $(a \to c)\,(b \to c) \to (a \vee b \to c)$, $a^* \to (a \to b)$, $(a \to b)\,(a \to b^*) \to a^*$. Furthermore, the inclusion relation $a(a \to b) \subset b$ is valid; and the inclusion relation $a \subset b$ implies $a \to b = \mathfrak{S}$. Consequently, if $\mathfrak{z}$ is any point in $\mathfrak{S}$, the partition of A into the class $\mathfrak{a}$ of all sets a containing $\mathfrak{z}$ and the class $\mathfrak{u}$ of all sets a not containing $\mathfrak{z}$, has all the formal properties demanded of an evaluation of a Brouwerian system of propositions.*

[15]) Alexandroff and Hopf, Topologie I (Berlin, 1935), Kap. I, § 4, pp. 58—59.

We begin by proving that $a \subset b$ if and only if $a \to b = \mathfrak{S}$: we know that $a \subset b$ is equivalent to $ab' = \mathfrak{O}$; since $\mathfrak{S}$ is a T_0-space, $ab' = \mathfrak{O}$ is equivalent to $(ab')^- = \mathfrak{O}$; and the latter relation is equivalent to $a \to b = (ab')^{-\prime} = \mathfrak{S}$. It is then immediately evident that the sets $a \to aa$, $ab \to ba$, $a \to a \vee b$, and $a \vee b \to b \vee a$ coincide with $\mathfrak{S}$. To prove that $(a \to b) \to (ac \to bc) = \mathfrak{S}$, we start from the relation $(ac)(bc)' = ab'c \subset ab'$. We then see that $[(ac)(bc)']^- \subset \subset (ab')^-$, $(ab')^{-\prime} \subset [(ac)(bc)']^{-\prime}$, and $(a \to b) \subset (ac \to bc)$. The desired relation follows at once. To prove that $(a \to b)(b \to c) \to \to (a \to c) = \mathfrak{S}$, we start from the relations $ac' = a(b' \vee b)c' \subset \subset ab' \vee bc'$. We then have $(ab')^{-\prime}(bc')^{-\prime} = (ab' \vee bc')^{-\prime} \subset (ac')^{-\prime}$ and hence $(a \to b)(b \to c) \subset (a \to c)$. The desired relation follows immediately. In showing that $b \to (a \to b) = \mathfrak{S}$, we first recall that b is an open set, $b'^- = b'$. We therefore have $b' = b'^- \supset (ab')^-$, $b \subset (ab')'^{-\prime} = (a \to b)$, and hence $b \to (a \to b) = \mathfrak{S}$. To prove that $a(a \to b) \to b = \mathfrak{S}$, we begin with the relation $ab' \subset (ab')^-$. We then have $a(ab')^{-\prime} \subset a(ab')' = ab \subset b$ or $a(a \to b) \subset b$. The desired relation results at once. To prove that $(a \to c)(b \to c) \to (a \vee b \to c) = = \mathfrak{S}$, we have only to use the relations $a \vee b \to c = [(a \vee b)c']^{-\prime} = = (ac')^{-\prime}(bc')^{-\prime} = (a \to c)(b \to c)$ and thus infer the desired relation directly. To prove that $a^* \to (a \to b) = \mathfrak{S}$, we start from the relation $a \supset (ab')$. We then have $a^* = a^{-\prime} \subset (ab')^{-\prime}$ or $a^* \subset (a \to b)$. The desired relation follows directly. Finally to prove that $(a \to \to b)(a \to b^*) \to a^* = \mathfrak{S}$, we need merely observe the relations $a^* = a^{-\prime} = [a(b' \vee b^-)]^{-\prime} = (ab')^{-\prime}(ab^-)^{-\prime} = (a \to b)(a \to b^*)$. In the course of the discussion we have already seen that the relations $a(a \to b) \subset b$ and $a(a \to b) \to b = \mathfrak{S}$ are equivalent and both true.

The remaining statements of the theorem are evident.

We are now in a position to obtain further information about the representation of Brouwerian logics described in Theorem 6. We have

Theorem 8. *The space $\mathfrak{E}$ of Theorem 6 is abstractly identical with the representation-space of the distributive lattice* **A** *as described in Part I, Theorems 12, 13, and 15.*

We shall show that the evaluations $\mathfrak{p}$ of A correspond biunivocally to the prime μ-ideals **P** in **A** in such a manner that $\mathfrak{p} \,\varepsilon\, \mathfrak{E}(a)$ if and only if $\mathfrak{E}(a)$ non ε **P**. The theorem is then evident.

If $\mathfrak{p}$ is an evaluation of A, the distributive lattice **A** can be partitioned into the class **P** of all its members which do not contain $\mathfrak{p}$ and the class **Q** of all its members which do contain $\mathfrak{p}$. It is readily verified that **P** and **Q** are disjoint μ-ideal and α-ideal respectively. Hence both are prime by Part I, Theorem 4. Obviously $\mathfrak{p} \,\varepsilon\, \mathfrak{E}(a)$ if and only if $\mathfrak{E}(a) \,\varepsilon\,$ **Q** — that is, if and only if $\mathfrak{E}(a)$ non ε **P**.

If $\mathbf{P}$ is a prime μ-ideal in $\mathbf{A}$, the members of $\mathbf{A}$ which do not belong to $\mathbf{P}$ constitute a prime α-ideal $\mathfrak{Q}$, in accordance with Part I, Theorem 4. A partition $\mathfrak{p}$ of A into disjoint non-void subclasses $\mathfrak{a}$ and $\mathfrak{u}$ can now be defined by putting $a \,\varepsilon\, \mathfrak{a}$ or $a \,\varepsilon\, \mathfrak{u}$ according as $\mathfrak{E}(a) \,\varepsilon\, \mathfrak{Q}$ or $\mathfrak{E}(a) \,\varepsilon\, \mathbf{P}$. By application of Theorem 7 we can now verify that the partition $\mathfrak{p}$ is an evaluation of A. First, let d be any of the composite propositions occurring in the „formal" rules (2.1)–(4.11). By Theorems 6 and 7 we have $\mathfrak{E}(d) = \mathfrak{E} \,\varepsilon\, \mathfrak{Q}$, $d \,\varepsilon\, \mathfrak{a}$. The reasoning may be sufficiently illustrated by the consideration of one of the rules, say (2.13). By Theorem 6 we have

$$\begin{aligned}\mathfrak{E}(d) &= \mathfrak{E}((a \to b)\,(b \to c) \to (a \to c)) \\ &= [\mathfrak{E}((a \to b)\,(b \to c))\,\mathfrak{E}'(a \to c)]^{-\prime} \\ &= [\mathfrak{E}(a \to b)\,\mathfrak{E}(b \to c)\,\mathfrak{E}'(a \to c)]^{-\prime} \\ &= \{[\mathfrak{E}(a)\,\mathfrak{E}'(b)]^{-\prime}\,[\mathfrak{E}(b)\,\mathfrak{E}'(c)]^{-\prime}\,[\mathfrak{E}(a)\,\mathfrak{E}'(c)]^{-}\}^{-\prime}.\end{aligned}$$

In the notation of Theorem 7, this relation becomes

$$\mathfrak{E}(d) = [\mathfrak{E}(a) \to \mathfrak{E}(b)]\,[\mathfrak{E}(b) \to \mathfrak{E}(c)] \to [\mathfrak{E}(a) \to \mathfrak{E}(c)].$$

Theorem 7 then asserts that $\mathfrak{E}(d) = \mathfrak{E}$. Next, we may show that the „informal" rules are satisfied. In the case of (1.2), for instance we have the following reasoning: if $a \,\varepsilon\, \mathfrak{a}$ and $b \,\varepsilon\, \mathfrak{a}$ then $\mathfrak{E}(a)$ and $\mathfrak{E}(b)$ are in $\mathfrak{Q}$; hence $\mathfrak{E}(ab) = \mathfrak{E}(a)\,\mathfrak{E}(b) \,\varepsilon\, \mathfrak{Q}$; and it follows that $ab \,\varepsilon\, \mathfrak{a}$. Similarly, for (1.7), we see that, if $a \vee b \,\varepsilon\, \mathfrak{a}$, then $a \,\varepsilon\, \mathfrak{a}$ or $b \,\varepsilon\, \mathfrak{a}$. The verification of (1.3) is obtained as follows: if $a \,\varepsilon\, \mathfrak{a}$ and $a \to b \,\varepsilon\, \mathfrak{a}$, then $\mathfrak{E}(a)$ and $\mathfrak{E}(a \to b) = [\mathfrak{E}(a)\,\mathfrak{E}'(b)]^{-\prime}$ belong to $\mathfrak{Q}$; hence $\mathfrak{E}(a)\,\mathfrak{E}(a \to b)$ belongs to $\mathfrak{Q}$; since $\mathfrak{E}(a)\,\mathfrak{E}(a \to b) \subset \mathfrak{E}(b)$ in accordance with Theorem 7, $\mathfrak{E}(b)$ also belongs to $\mathfrak{Q}$; and hence $b \,\varepsilon\, \mathfrak{a}$. The „informal" rule (1.8) does not impose any condition upon a single evaluation. By definition, the evaluation $\mathfrak{p}$ and the prime μ-ideal $\mathbf{P}$ are so related that $\mathfrak{p} \,\varepsilon\, \mathfrak{E}(a)$ if and only if $\mathfrak{E}(a)$ non $\varepsilon\, \mathbf{P}$.

The proof of the theorem is thereby completed.

We may close with a few remarks on the construction of evaluations of a Brouwerian system of propositions. If A is such a system, we begin by introducing a relation of equivalence $\equiv$ defined by putting $a \equiv b$ if and only if $\vdash (a \to b)\,(b \to a)$. It is evident that $a \equiv b$ if and only if $\mathfrak{E}(a) = \mathfrak{E}(b)$ in the representation of Theorem 6. Consequently we see directly (and can prove in detail, if we wish) that, after the introduction of this equivalence, A becomes a distributive lattice with zero and unit in terms of the operations $a \vee b$ and ab, every proposition appearing in the „formal" rules being equivalent to the unit. The evaluations of A can then be obtained by partitioning A into a prime μ-ideal and a prime α-ideal: for the distributive lattice A is isomorphic to the lattice $\mathbf{A}$ of Theorem 6; and the proof of Theorem 8 reveals that the evaluations of A are to be obtained by such partitions of **A**. The construction of such partitions has been described in Part I, Theorems 6 und. 7

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Topologická interpretace distributivních mříží a brouwerovské logiky.

(Obsah předešlého článku.)

Distributivní mříž je systém, ve kterém jsou definovány dvě binární operace $a \vee b$, ab vyhovující všem axiomům Booleovy algebry až na to, že se nepředpokládá existence komplementu. Hlavní výsledek je, že každá distributivní mříž se dá realisovati systémem všech otevřených bikompaktních podmnožin topologického prostoru podrobeného jednoduchým podmínkám. Ve druhé části je nastíněno znázornění brouwerovské logiky distributivní mříží, které odpovídá znázornění klasické logiky Booleovou algebrou; podstatný rozdíl je v tom, že při znázornění brouwerovské logiky jsou třídy vedle kombinatorických operací podrobeny ještě operacím topologickým.