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A CONTRIBUTION TO EMBRACING THE BASIC CONCEPTIONS OF THE INTEGRAL GEOMETRY WITHIN THE SCOPE OF IDEAS OF LIE'S GROUP THEORY.

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Introduction: Since the publication of F. KLEIN's so called Erlangen Program in 1872 the conceptions of the group theory have proved extraordinarily useful in many branches of geometry. On the one hand, KLEIN recognized their value for a suitable classification of those partial branches of geometry that were already developed at that time; on the other hand, these new conceptions gave many suggestions for extensive generalizations, unthinkable without the basic ideas of group theory. Instead of being satisfied with the groups of classical geometry, KLEIN demanded, one should, starting from a highly arbitrary transformation group, develop for it a "geometry" of its representative space, i. e., from the algebraic-analytical standpoint, a "theory of invariants". There appeared, however, an essential difficulty, while this development was being worked out. Between the transformation groups of classical geometry and a general LIE transformation group there is an essential difference: The underlying domain of a classical group is a total space, i. e., a certain topological manifold, considered globally; the underlying domain of a general LIE transformation group is, on the contrary, a neighborhood in a space (e. g. EUCLIDEAN), generally not further defined in detail. While building up the geometry of transformation groups, G. PICK took these circumstances in to consideration, restricting it, according to the suggestions of the Erlangen Program, above all to a "local geometry", especially to the differential geometry. It is, on the whole, without importance that we are able to "continue analytically" a transformation group in some cases to such a degree that its underlying domain becomes eventually the whole of a manifold.

*) This paper is an extract of the author's 1938 doctor thesis. The author perished on February 12th, 1945 on one of the terrible death marches, organised by the Germans, when they were compelled to evacuate some concentracion camps. It was on the march from Falkenberg to Mauthausen. The thesis had been written under direction of prof. BETWALD, also murdered by the Nazis.

The present paper owes its origin to a stimulus similar to that giving rise to PICK's important papers (since 1906). If we consider, with KLEIN, the differential geometry as a "theory of invariants" for a certain transformation group, then we may regard accordingly the "integral geometry", inspired by CROFTON and H. POINCARÉ and developed in the last years by W. BLASCHKE and others, as a "theory of integral invariants" of a transformation group. The nature of the impulses that gave rise to integral geometry restricted the development, rather naturally, to the simplest geometrical groups, at least for the beginning, avoiding thereby the necessity of using the procedures of LIE's theory. (Cfr. to BLASCHKE |1|, |2|,*) for the full group of motions in a plane or in a three-dimensional space and to BERWALD |1| and BLASCHKE |3| for translation groups.) The present paper, however, intends a preliminar treatment of the development of an integral geometry, taking as a basis a general LIE transformation group. Even so simple an example as the affine group of a straight line shows to what extent one may be able to achieve results, which are geometrically not without interest.

There are above all two questions to be dealt with: 1. The derivation of integral invariants of the highest dimension for any LIE transformation group; we shall give only the results and refrain from proofs, which are founded on well known principles. 2. At what a set of geometrical forms is it possible to speak, with regard to a given transformation group, of a density that is an integral invariant of the highest dimension?

The first question has been treated in the literature about integral invariants proper, for instance in CARTAN's book |1|. This question was put in the same way, and answered, in a paper of N. TCHEBOTAREV |1|. The present paper stresses rather the precise treatment of some details. Special groups are treated by A. MÜLLER |1|. As to the density determination of parameter groups see E. CARTAN |2|.

It does not seem that the second question has been asked before in the present form. It is bound to lead to a confrontation of two kinds of geometrical objects (one kind of them can always be considered as points), which can be conceived as a generalization of the duality principle of the projective geometry.

1. The density function of a transformation group: — Let

$$X_\varrho = \sum_{\nu=1}^{n} \xi_{\nu\varrho}(x) \frac{\partial}{\partial x_\nu} \qquad (\varrho = 1, 2, \ldots, r), \tag{1}$$

be the symbols of the infinitesimal transformations of an r-parameter LIE transformation group, which operates in an open domain G_n of an n-dimensional EUCLIDEAN space R_n and is determined by the finite equations

$$y_\nu = f_\nu(x; u) = f_\nu(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r) \qquad (\nu = 1, 2, \ldots, n). \tag{2}$$

*) The bracketed numbers refer to the bibliography at the end of the paper.

The points of this space are given by their coordinates in a coordinate system S_n. Besides, let B denote an n-dimensional closed domain, lying totally in the operational domain G_n of the group and therefore also in the domain of definition of the functions

$$\xi_{\nu\varrho} = \xi_{\nu\varrho}(x) = \xi_{\nu\varrho}(x_1, \ldots, x_n) \quad (\nu = 1, \ldots, n; \varrho = 1, \ldots, r), \tag{3}$$

the boundary of B being a closed $(n - 1)$-dimensional hypersurface. In this closed domain let the function

$$F = F(x) = F(x_1, \ldots, x_n) \tag{4}$$

be continuous and continuously differentiable. Let us now consider the integral

$$\int\int \ldots \int_B F(x_1, x_2, \ldots, x_n)[\mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_n]. \tag{5}$$

If we apply a transformation of the group (2) to the coordinate system S_n, the integral (5) is transformed into the following function of the parameters $u_1, u_2, \ldots, u_r$:

$$\begin{aligned} J_B(u) &= \int\int \ldots \int_B F(y_1, \ldots, y_n)[\mathrm{d}y_1\, \mathrm{d}y_2 \ldots \mathrm{d}y_n] \\ &= \int\int \ldots \int_B F(y_1, \ldots, y_n) \left|\frac{\partial y_\mu}{\partial x_\nu}\right| [\mathrm{d}x_1\, \mathrm{d}x_1 \ldots \mathrm{d}x_n]. \end{aligned} \tag{6}$$

We call the integral (5) an *integral invariant* of the group (2), if the function $J_B(u)$ is independent of $u_1, u_2, \ldots, u_r$, whatever the choice of B within G. Under this condition, the integrated function F, the *density function* of the integral invariant, must satisfy the functional equations

$$F(y_1, \ldots, y_n) \left|\frac{\partial y_\mu}{\partial x_\nu}\right| = F(x_1, \ldots, x_n). \tag{7}$$

In order to determine the density function F of a transformation group (2) it is now sufficient to consider the functional equation (7) as applied to an infinitesimal transformation

$$y_\nu = x_\nu + \xi_\nu \varepsilon, \quad \xi_\nu = \sum_{\varrho=1}^{r} e_\varrho \xi_{\nu\varrho}, \tag{8}$$

In such a case the Jacobian $\left|\frac{\partial y_\mu}{\partial x_\nu}\right|$ has the value

$$\left|\frac{\partial(x_\mu + \xi_\mu \varepsilon)}{\partial x_\nu}\right| = \left|E + \left(\frac{\partial \xi_\mu}{\partial x_\nu}\right)\varepsilon\right| = 1 + \sum_{\nu=1}^{n} \frac{\partial \xi_\nu}{\partial x_\nu}\varepsilon, \tag{9}$$

(where E is the unit matrix), whence, together with

$$\chi_\varrho = \chi(X_\varrho) = \sum_{\nu=1}^{n} \frac{\partial \xi_{\nu\varrho}}{\partial x_\nu} \quad (\varrho = 1, 2, \ldots, r) \tag{10}$$

follows the system of partial differential equations for the determination of the density function F:

$$X_\varrho F + \chi_\varrho F = 0 \quad (\varrho = 1, 2, \ldots, r). \tag{11}$$

The function χ_ϱ, which appears in these equations, will be called the *divergence* of the ϱ-th infinitesimal transformation of the group (2).

It can easily be proved by direct computation that the divergences χ_ϱ satisfy the partial differential equations

$$X_\varrho \chi_\sigma - X_\sigma \chi_\varrho + \sum_{\tau=1}^{r} c_{\varrho\sigma}^{\tau} \chi_\tau = 0 \quad (\varrho, \sigma = 1, 2, \ldots, r), \tag{12}$$

if $c_{\varrho\sigma}^{\tau}$ denote the r^3 structure constants of the group (2). Consequently the system of differential equations (11) is *complete.*

It does not involve any restriction, if we admit for the time being only such domains B, where the required function $F(x) > 0$. Introducing now, instead of F, the new unknown function

$$G = \lg F, \tag{13}$$

we receive, equivalently to (11), the inhomogeneous system of differential equations

$$X_\varrho G + \chi_\varrho = 0. \tag{14}$$

There follows immediately: *If for a group* (2) *there are two different density functions F_1, F_2, the difference of the corresponding two functions $G_1 = \lg F_1$, $G_2 = \lg F_2$ is an invariant of the group.* We conclude:

Theorem I. *For a simply transitive transformation group* (2) *there is always, but for a multiplicative constant, one-uniquely defined density function F.*

In order to decide about the existence of a density function for multiply transitive groups (2) we proceed as follows. Owing to the independence of the r parameters $u_1, \ldots, u_r$ the infinitesimal transformations of the group are certainly linearly independent, considering constant coefficients only. Nevertheless, there can be functions

$$\varphi_\varrho = \varphi_\varrho(x_1, \ldots, x_n) \quad (\varrho = 1, 2, \ldots, r), \tag{15}$$

not all of which are identically zero and for which

$$\varphi_1 X_1 + \varphi_2 X_2 + \ldots + \varphi_r X_r = 0 \tag{16}$$

holds true. Let us assume, however, the symbols $X_1, X_2, \ldots, X_q$ to be linearly independent, considering variable coefficients as well. Let us assume, further,

$$r \geqq n \text{ and } q = n. \tag{17}$$

We shall now consider the matrix

$$M(x) = \begin{vmatrix} \chi_1, & \xi_{11}, & \xi_{21}, & \dots, & \xi_{n1} \\ \chi_2, & \xi_{12}, & \xi_{22}, & \dots, & \xi_{n2} \\ \vdots & \vdots & \vdots & & \vdots \\ \chi_n, & \xi_{1n}, & \xi_{2n}, & \dots, & \xi_{nn} \\ \vdots & \vdots & \vdots & & \vdots \\ \chi_r, & \xi_{1r}, & \xi_{2r}, & \dots, & \xi_{nr} \end{vmatrix} \tag{18}$$

Its rank s is evidently equal to n or $n + 1$, and we obtain

Theorem 2. *For a transitive transformation group* (2) *there is, but for a multiplicative constant, a uniquely defined density function, if, and only if, the rank of the matrix $M(x)$ is n.*

2. Example: We consider the "doubled" affine group of a straight line:

$$y_1 = (1 + u_1)\, x_1 + u_2, \quad y_2 = (1 + u_2)\, x_2 + u_2. \tag{19}$$

We see at once that the group is simply transitive in every domain G_2 of the x_1, x_2-plane that does not cover the straight line $x_1 = x_2$. The differential equations (11) take the form

$$x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} + 2F = 0, \quad \frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_2} = 0. \tag{20}$$

The solution is, but for a constant factor,

$$F = \frac{1}{(x_1 - x_2)^2}. \tag{21}$$

A two-dimensional domain B is here defined by two intervals

$$\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle, \quad (a_1 < b_1 < a_2 < b_2), \tag{22}$$

and the integral invariant of the group can be explicitly represented as a function of the endpoints of both intervals. We get

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{dx_1\, dx_2}{(x_1 - x_2)^2} = \tfrac{1}{2} \lg \mathrm{Ar}(a_1, b_1; a_2, b_2), \tag{23}$$

where Ar is the anharmonic ratio of the four points within the bracket. This result may be said to reduce the entire content of the "integral geometry of the group (19)" to known facts, i. e., the properties of the anharmonic ratio of four points on a straight line. This can be interpreted geometrically, considering the group (19) as the affine group of pairs of points on a straight line.

3. A group theoretical principle of duality: One of the fundamental ideas of integral geometry consists in allotting a measure not only to sets of points but also to sets of other geometrical entities, i. e., straight lines, circles, conic sections, etc. This idea originates in the theory of geometrical probabilities (see H. Poincaré's book [1]). The systematical determination of the content of straight line sets brought about, besides the transformation group of Euclidean motions, also other transformation groups, which, in abstracto, are identical with a motion group in point coordinates, differ, however, from it as transformation group. In order to use for these relations a special term we shall speak about the different "*realization forms*" of the motion group. We confront its original realization, in point coordinates, with its realization in straight line coordinates, provided the group is built in R_2, i. e., in the EUCLIDEAN plane. This confrontation is based on the duality: *point* $\longleftrightarrow$ *straight line*. We shall now try to transmit this idea to the treatment of any transformation group. The most important question, which arises at once, is this: *Assuming two realizations of a transformation group, which are, in the underlying space of its first realization, the geometrical forms g that agree with a duality*

$$\textit{point} \longleftrightarrow g \tag{24}$$

on account of a confrontation of both realizations?

With regard to the simple example, mentioned above (motion group in point and straight line coordinates), see the paper of G. POLYA [1]. We shall return to this matter a little later, changing somewhat the symbolism.

In order to discuss this whole process in general we start from two different realizations of the transformation group (2):

$$y_\nu = f_\nu(x; u), \quad \bar{y}_\nu = \bar{f}_\nu(\bar{x}; u) \quad (\nu = 1, 2, \ldots, n). \tag{25}$$

For the sake of generality let us assume that these two isomorph transformation groups are not similar, though they have the same parameter groups and, therefore, the same structure constants $c^\tau_{\varrho\sigma}$; their infinitesimal transformations, i. e., the symbols

$$X_\varrho = \sum_{\nu=1}^{n} \xi_{\nu\varrho} \frac{\partial}{\partial x_\nu}, \quad \bar{X}_\varrho = \sum_{\nu=1}^{n} \bar{\xi}_{\nu\varrho} \frac{\partial}{\partial \bar{x}_\nu}, \quad (\varrho = 1, 2, \ldots, n), \tag{26}$$

respectively, satisfy therefore LIE's bracket-relations with the constants $c^\tau_{\varrho\sigma}$:

$$(X_\varrho X_\sigma) = \sum_{\tau=1}^{r} c^\tau_{\varrho\sigma} X_\tau, \quad (\bar{X}_\varrho \bar{X}_\sigma) = \sum_{\tau=1}^{r} c^\tau_{\varrho\sigma} \bar{X}_\tau. \tag{27}$$

We shall call a *simultaneous invariant* of two realizations (25) of the group (2) a function of $2n$ variables

$$g = g(x; \bar{x}) = g(x_1, x_2, \ldots, x_n; \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n), \tag{28}$$

which is defined (continuous and differentiable), with regard to the first n variables $x_1, \ldots, x_n$ in the underlying domain of the first realization (2), and, with regard to the variables $\bar{x}_1, \ldots, \bar{x}_n$, in the underlying domain of the second realization; it shall, moreover, have the property that, under these conditions and on account of the equations (25), the expression

$$g(y; \bar{y}) \tag{29}$$

is *independent* of $u_1, u_2, \ldots, u_r$. We agree to regard the $2n$ variables $x_1, \ldots, x_n; \bar{x}_1, \ldots, \bar{x}_n$ as mutually independent; we can therefore consider the equations (25), within a $2n$-dimensional underlying space of the variables $x_1, x_2, \ldots, x_n; \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ as a representation of a transformation group, isomorphic to (2). The infinitesimal transformations of this group are given by the symbols

$$Y_\varrho = X_\varrho + \overline{X}_\varrho \quad (\varrho = 1, 2, \ldots, r), \tag{30}$$

and a simultaneous invariant of both realizations (25) may be characterized as an invariant of a $2n$-dimensional group, composed of both in the way just described, i. e., as the solution of the following system of linear, homogeneous partial differential equations

$$X_\varrho g + \overline{X}_\varrho g = 0 \quad (\varrho = 1, 2, \ldots, r). \tag{31}$$

It follows: Since g is a simultaneous invariant of two realizations, the same holds true for (g + const) and for any arbitrary (differentiable) function of g. We shall, however, regard such simultaneous invariants, created from g, as not essentially differing from g. In order to get a survey of all possible simultaneous invariants of two group realizations, we have to integrate the differential equations (31). The totality of the solutions depends on the rank of the coefficient matrix of the system (31). Supposing the linear independence of exactly q among the X_ϱ and of exactly $\bar{q}$ among the $\overline{X}_\varrho$, the rank is equal to

$$m \leqq q + \bar{q}; \tag{32}$$

if

$$m = q + \bar{q}, \text{ then } q + \bar{q} \leqq r. \tag{33}$$

Two group realizations that can be composed into a transitive group in R_n are therefore of no importance for our consideration, for there is no not-constant simultaneous invariant. Of great interest are, however, cases that offer essentially one, and only one, simultaneous invariant. This condition is fulfilled, as we shall see in section 4, with the motion group in point and straight line coordinates, and this creates the unique position of the straight line, among the geometrical objects, for measures of the integral geometry.

Let $g(x; \bar{x})$ denote a simultaneous invariant of both realizations (25). We assume this function not to be constant, but to become 0 for certain real values of the variables $x_\nu, \bar{x}_\nu$. (In a given case, this can always be achieved by addition to g of a suitable constant.) Let us consider the equation

$$g(x; \bar{x}) = 0. \tag{34}$$

In the space R_n of the points x, transformed by the group in the first realization (2), it represents an $(n - 1)$-dimensional hypersurface, the position of which depends on the parameters $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$; more precisely: To every $\bar{x}$ within a certain range of $\overline{R}_n$, equation (34) coordinates a certain $(n - 1)$-dimensional hypersurface, from a certain n-parameter family of such forms. The second realization (25) of our group "describes" how the hypersurfaces of this n-parameter family change into one another, if we transform, according to the law of the first realization, the points of R_n, which contains the surfaces of the family. We call mutually *dual*, with regard to the transformation group (2), the points and those hypersurfaces (34) of R_n, in the same sense as we call dual, with regard to the group of all EUCLIDEAN motions, the points and straight lines of a plane. There are as many duality principles to every transformation group as there are different non-similar realizations.

In the following section we are going to outline briefly the working of these statements in the case of motion groups in two and three dimensions.

4. Application to motion groups: The *plane* motion group in point coordinates is given by

$$y_1 = u_1 + x_1 \cos\alpha - x_2 \sin\alpha, \quad y_2 = u_2 + x_1 \sin\alpha + x_2 \cos\alpha, \tag{35}$$

where the third group parameter (the angle of rotation) is denoted by α. Let the equation of the straight line in the x_1x_2-plane be

$$g(x_1x_2;\, p_1p_2) = \frac{x_1}{p_1} + \frac{x_2}{p_2} - 1 = 0, \tag{36}$$

where p_1, p_2 are the straight line coordinates. (We write now p_1, p_2 instead of $\bar{x}_1, \bar{x}_2$ and, correspondingly, q_1, q_2 instead of $\bar{y}_1, \bar{y}_2$.) If we transform, according to (35), the straight line equation (36), and reform it, in the new coordinates y_1, y_2, again into the normal shape (36), we arrive easily at the following transformation law of the straight line coordinates

$$\begin{aligned} q_1 &= u_1 + \frac{p_1p_2 + u_2(p_1 \cos\alpha + p_2 \sin\alpha)}{-p_1 \sin\alpha + p_2 \cos\alpha} \\ q_2 &= u_2 + \frac{p_1p_2 + u_1(-p_1 \sin\alpha + p_2 \cos\alpha)}{p_1 \cos\alpha + p_2 \sin\alpha}; \end{aligned} \tag{37}$$

this is the motion group in the second form of realization, in straight line coordinates p_1, p_2.

If we compute the infinitesimal transformations for either realization, (35) and (37), we can derive the most general simultaneous invariant, according to the scheme in the preceding section, from the following system of partial differential equations in four independent variables x_1, x_2, p_1, p_2:

$$\begin{aligned} \frac{\partial g}{\partial x_1} \qquad\qquad &+ \frac{\partial g}{\partial p_1} + \frac{p_2}{p_1}\frac{\partial g}{\partial p_2} = 0, \\ \frac{\partial g}{\partial x_2} &+ \frac{p_1}{p_2}\frac{\partial g}{\partial p_1} + \frac{\partial g}{\partial p_2} = 0, \\ x_2 \frac{\partial g}{\partial x_1} - x_1 \frac{\partial g}{\partial g_2} &- \frac{p_1^2}{p_1}\frac{\partial g}{\partial p_1} + \frac{p_2^2}{p_1}\frac{\partial g}{\partial p_2} = 0. \end{aligned} \tag{38}$$

The rank of the matrix of this system is 3, and there is, therefore, exactly $4 - 3 = 1$ simultaneous invariant. By explicit integration we can easily ascertain that g is nothing else but an arbitrary function of the left hand term of the straight line equation (36). It follows

Theorem 3. *The straight lines are essentially the only plane curves that are dual to the points of the plane, with reference to the motion group in point and straight line coordinates.*

Incidentally, it is not difficult to compute the density function in straight line coordinates, as a solution of a system of differential equations, formed according to (11); we obtain, uniquely but for a constant factor,

$$F(p_1, p_2) = \frac{p_1 p_2}{(\sqrt{p_1^2 + p_2^2})^3}. \tag{39}$$

We can proceed in the same way with the spatial motion group

$$y = Ax + u,$$

where A is an orthogonal matrix. If we write the general equation of a plane in the form

$$p'x - 1 = \sum_{\nu=1}^{n} p_\nu x_\nu - 1 = 0, \tag{41}$$

we derive the simple law of transformation

$$q = \frac{Ap}{1 + u'Ap} \tag{42}$$

for the plane, respectively hyperplane, coordinates. This represents also the realization of the motion group in plane coordinates $p_1, p_2, \ldots, p_n$. For $n = 3$, the result is again an easily integrable system of partial differential equations for the determination of the most general simultaneous invariant of both realizations (40) and (42). The most general solution is an arbitrary function of the expression $p'x - 1$, and there follows again

Theorem 4. *The planes are essentially the only surfaces of a three-dimensional space that are dual to the points of the space, with reference to the motion group in point and plane coordinates.*

It is not difficult to derive a corresponding theorem for an arbitrary number n of dimensions, and to prove it.

The density function of the motion group in plane coordinates $p_1, p_2, \ldots, p_n$ is

$$F = \frac{1}{(p'p)^2} = \frac{1}{(p_1^2 + p_2^2 + \ldots + p_n^2)^2}. \tag{43}$$

5. Final remarks: We shall not proceed in deriving the straight line density function in a three-dimensional space, because this would involve some considerations of different kind, which would not easily fit into the development, stated hitherto. But the case is of great interest, because there evolves a "self-duality", which has not appeared up to now. A generalization for a widely arbitrary transformation group would involve a fundamental question, which would form the main part of the investigation: Which are the k-dimensional forms in an n-dimensional space R_n, $(k < n)$, for which, with reference to a given transformation group (2), we can find a family of $(n - k - 1)$-dimensional forms that are dual to them? Evidently, this duality

$$g_k \longleftrightarrow \bar{g}_{n-k-1} \tag{44}$$

must, for a motion group, include the duality (*straight line* $\longleftrightarrow$ *straight line*). Generally it will be found, however, that not every dimensional number k is admissible. It depends on the choice of the transformation group (2), whether or not k admits an unambigous duality (44).

A further problem may be challenged by the computation of the densities of g_k and g_{n-k-1}, according to the method of LIE's theory (as we did in the case of $k = 0$), and by the study of the geometrical properties of the integral invariants derived at.

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Základní pojmy integrální geometrie z hlediska theorie Lie-ových grup.

(Obsah předešlého článku.)

Autor vychází z Kleinova pojetí geometrie, takže problémy integrální geometrie převádí na studium integrálních invariantů vůči nějaké grupě transformací. Až dosud byly základem těchto úvah jen nejjednodušší grupy (na příklad grupa translací); v této práci snaží se autor připraviti základ k vybudování integrální geometrie na podkladě obecných Lie-ových grup transformací v prostorech vícerozměrných. Zvláště porovnává různé druhy geometrických útvarů z hlediska integrálních invariantů, což vede k jakési analogii principu duality v projektivní geometrii.