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A THEOREM ON THE LEBESGUE DIMENSION

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In a recent paper [1]*) of the present author, some results have been established concerning the relations between the inductive (Menger-Urysohn) dimension of a compact (= bicompact) space P and certain properties of the ring $C(P)$ consisting of all (bounded) continuous real-valued functions on P. In the present note I intend to give a characterization of the Lebesgue dimension (in a sense slightly different, for non-normal spaces, from the usual one) in terms of the ring $C(P)$, namely, to show that the Lebesgue dimension of P is equal to the analytic pseudodimension of P, to be defined in the sequel.

§ 1.

We first summarize some definitions and results given in [1]. — Space always means a Hausdorff topological space, mapping means a continuous transformation, function means a real-valued function. The letter P denotes a (non-void) completely regular space, R denotes a metric space.

Let C be a commutative ring (with a unity element) in which there is defined, for any $x \in C$ and any real number λ, the multiple $\lambda x \in C$ satisfying the usual axioms, and let C be, at the same time, a topological space such that the operations $x + y$, xy, λx are continuous. Then C will be called a (*real commutative*) *analytic ring* (*with a unity*). We shall say that a subring $C_1 \supset C$ is *algebraically closed* (in C) if (1) C_1 is an *analytic subring*, i. e. contains all λe where λ is real, e is the unity element of C, (2) $x \in C$ is contained in C_1 whenever $x^n + a_1 x^{n-1} + \ldots + a_n = 0$, $a_i \in C_1$; if, moreover, $\overline{C_1} = C_1$ (i. e. C_1 is a closed set) we shall say that C_1 is *analytically closed* (in C).

If P is a completely regular space, then $C(P)$ denotes the analytic ring consisting of all bounded continuous functions f in P (with the topology defined by the norm $|f| = \sup_{t \in P} |f(t)|$).

*) The number in brackets refer to the list at the end of the paper.

Lemma 1. *If $S \subset P$ is connected and $C_1 \subset C(P)$ consists of all $x \in C(P)$ which are constant on S, then C_1 is an analytically closed subring.*

See [1], Lemma 17.

Lemma 2. *If P is compact, $C_1 \subset C(P)$ is algebraically closed, then, for every $t_0 \in P$, the set of all $t \in P$ such that $x(t) = x(t_0)$, for any $x \in C_1$, is connected.*

See [1], Lemma 18.

Proposition 1. *Let P be compact and let $C_1 \subset C(P)$ be an analytic subring. Then $\overline{C_1}$ consists of all $x \in C(P)$ such that $x(t_1) = x(t_2)$ whenever $y(t_1) = y(t_2)$ for all $y \in C_1$.*

See [1], Theorem 2; cf. [2], Theorem 82, as well as [3], Theorem 4, and [4], Corollary 2.

It is clear that the intersection of an arbitrary system of analytically (algebraically) closed subrings of an analytic ring C is analytically (algebraically) closed. Consequently, there exists, for any $M \subset C$, the least analytically closed subring $C_1 \subset C$ containing M. We shall say that M is an *analytic base* of C_1 (in C), or that C_1 is *analytically generated*[1]) by M.

If C is an analytic ring, then the least power of an analytic base of C will be called the analytic dimension of C, denoted[2]) $\dim C$.

Proposition 2. *The analytically closed subring generated analytically by a set $M \subset C(P)$ consists of all functions $x \in C(P)$ which are constant on every connected set $S \subset P$ on which all functions $y \in M$ are constant.*

Proof. The set of all $x \in C(P)$ which have the above property is clearly a subring, contains M and is analytically closed by Lemma 1 (since the intersection of analytically closed subrings is analytically closed).

Let $C_1 \subset C(P)$ be an analytically closed subring containing M. For any $t \in P$, denote by $S(t)$ the set of all $t' \in P$ such that $y(t') = y(t)$ whenever $y \in C_1$. By Lemma 2, every $S(t)$ is connected. If $x \in C(P)$ has the property described in the proposition, then x is constant on every $S(t)$ and therefore, by Proposition 1, $x \in \overline{C_1} = C_1$.

We now state some further auxiliary definitions and lemmas referring, if necessary, for the proofs to [1].

If P is a space, R is a metric space, then $C(P, R)$ denotes the space consisting of all bounded mappings of P into R, with the metric $\varrho(f, g) = \sup_{t \in P} \varrho(f(t), g(t))$. E^n $(n = 1, 2, \ldots)$ denotes the n-dimensional Euclidean space, E^0 denotes the space containing a single point: instead of $C(P, E^1)$, $C(P)$ is written.

1) This notion is different from E. Hewitt's [4] notion of a „set of analytic generators".

2) Thus dim has, in this note, two different meanings: 1. the analytic dimension of an analytic ring, 2) the Lebesgue dimension of a space, to be defined below.

Let P be a space, and let $\mathfrak{A}$ be a finite open covering (abbreviated f. o. c.) of P; let $M \subset P$. If there exist M_i such that $\Sigma_1^p M_i = M$, $M_i\overline{M_j} = \emptyset$ (for $i \neq j$), and each M_i is contained in some $A \in \mathfrak{A}$, then we write $\delta(M) < \mathfrak{A}$. It is easy to see that M_i are open and closed in M. — If $\mathfrak{A}$, $\mathfrak{B}$ are f. o. coverings of P and every $A \in \mathfrak{A}$ is contained in some $B \in \mathfrak{B}$, then $\mathfrak{A} < \mathfrak{B}$ is written. — A set $M \subset P$ is said to have *property* $\Delta(R)$ in P, R being a metric space, if, for any $f \in C(P, R)$ any f. o. c. $\mathfrak{A}$ of P, and any $\varepsilon > 0$, there exists $g \in C(P, R)$ such that $\varrho(f, g) < \varepsilon$ and $\delta(Mg^{-1}(y)) < \mathfrak{A}$, for every $y \in R$.

Lemma 3. *If P is compact, $M \subset P$ is closed, $\mathfrak{A}$ is a f. o. c. of P, and $\delta(Mf^{-1}(y)) < \mathfrak{A}$, for any $y \in R$, then there exists a f. o. c. $\mathfrak{B}$ of $f(P) \subset R$ such that $\delta(f^{-1}(B)) < \mathfrak{A}$, for any $B \in \mathfrak{B}$.*

Proof. Since $\Pi M f^{-1}(\overline{G}) = Mf^{-1}(y)$, G running over all neighborhoods of $y \in R$, it is easy to see that there exists, for every $y \in f(P)$, an open neighborhood $G = G(y)$ such that $\delta(f^{-1}(G)) < \mathfrak{A}$. Since $f(P)$ is compact, $\{G(y)\}$ contains a finite subcovering.

Lemma 4. *Let $\mathfrak{A}$ be a f. o. c. of P and let $K \subset P$ be compact. Then $\delta(K) < \mathfrak{A}$ if and only if every connected $S \subset K$ is contained in some $A \in \mathfrak{A}$.*

Proof. The necessity being obvious suppose the condition to hold. For every $x \in K$, let $S(x)$ denote the intersection of all $H \subset K$ which are open and closed in K and contain the point x. Then $S(x)$ is connected; for otherwise $S(x) = S_1 + S_2$, S_i closed non-void, $S_1S_2 = \emptyset$, $x \in S_1$, and there exist open (in K) $G_i \subset K$ such that $G_i \supset S_i$, $G_1G_2 = \emptyset$; therefore, for appropriate H_j, open and closed in K, we have $x \in \Pi_1^p H_j \subset G_1 + G_2$, and $G_1\Pi_1^p H_j$ is easily seen to be open and closed in K from which a contradiction follows at once. Since $S(x)$ is connected, it is contained in some $A \in \mathfrak{A}$. There exists an open and closed (in K) set $H(x)$ such that $x \in S(x) \subset \subset H(x) \subset A$. Since K is compact, we have, for appropriate x_i, $K = \Sigma_1^b H_i$, each $H_i = H(x_i)$ being open and closed (in K) and contained in some $A \in \mathfrak{A}$. From this the assertion of the lemma follows at once.

Lemma 5. *If R is complete, then, for an arbitrary space P, $C(P, R)$ is complete.*

This is obvious. Cf. [1], Lemma 13.

Lemma 6. *If P is compact, $M \subset P$ is closed, $\mathfrak{A}$ is a f. o. c. of P, then the set of all $f \in C(P, R)$ such that $\delta(Mf^{-1}(y)) < \mathfrak{A}$, for any $y \in R$, is open.*

See [1], Lemma 7.

Definition. The *order* of a finite collection $\mathfrak{M}$ of sets is the largest integer n such that there are $n + 1$ sets from $\mathfrak{M}$ with a non-void intersection. Given a (non-void) *normal* space P, the least cardinal number $\mathfrak{m}$ such that, for any f. o. c. $\mathfrak{A}$ of P, there exist a f. o. c. $\mathfrak{B} < \mathfrak{A}$ of order

$\leq \mathfrak{m}$ is called the *Lebesgue dimension* of P, denoted $\dim P$. Clearly, $0 \leq \dim P \leq \aleph_0$; for $S = \emptyset$, we put $\dim S = -1$.

We now proceed to establish the following proposition from which our main theorem will easily follow.

Proposition 3. *The following properties of a compact space P are equivalent (for $n = 0, 1, 2, \ldots$): (1) $\dim P \leq n$; (2) property $\Delta(E^n)$ (in P); (3) every countable $M \subset P$ is contained in an analytically closed subring $C_1 \subset C(P)$ analytically generated by a set $N \subset C(P)$ of power $\leq n$; (4) property (3) with arbitrary finite, instead of countable, M.*

Proof. The proposition is easily seen to hold for $n = 0$ (observe that a compact space P is 0-dimensional if and only if no connected $S \subset P$ contains more than one point and apply Lemma 4). Therefore we may suppose $n \geq 1$. — I. (1) implies (2). — Let $\mathfrak{A}$ be a f. o. c. of P, $f \in C(P, E^n)$, $\varepsilon > 0$. There exists a f. o. c. $\mathfrak{G} < \mathfrak{A}$ of order $\leq n$ such that, for each $G \in \mathfrak{G}$, $f(G)$ is of diameter $< \frac{1}{2}\varepsilon$. Let $\mathfrak{G}$ consist of sets $G_1, \ldots G_p$. By a well known theorem on normal spaces (P is compact, hence normal) there exist open H_i such that $\overline{H_i} \subset G_i$, $\sum_1^p H_i = P$. By Urysohn-Tietze Extension Theorem there exist $g_i \in C(P)$ such that $0 \leq g_i(x) \leq 1$, for any $x \in P$, $g_i(x) = 0$, for $x \in P - G_i$, $G_i(x) = 1$, for $x \in \overline{H_i}$. Choose points $z_i \in E^n$ such that (1) the distance $\varrho(z_i, f(G_i))$ is $< \frac{1}{2}\varepsilon$, (2) every hyperplane in E^n contains n points z_i at most. Put, for every $x \in P$, $\gamma(x) = (\Sigma_1^p g_i(x))^{-1}$ (this is possible, for every x lies in some H_i which implies $g_i(x) = 1$), and put $g(x) = \gamma(x) \,.\, \Sigma_1^p g_i(x) z_i \in E^n$, points z_i being considered, of course, as vectors. Evidently, $g \in C(P, E^n)$. For any $x \in P$, $g_i(x) \neq 0$ only if $x \in G_i$; since, for $x \in G_i$, $f(x) \in f(G_i)$, $\varrho(z_i, f(x)) < \varepsilon$, we have $\varrho(g(x), f(x)) = \varrho(\Sigma_1^p \gamma(x) g_i(x) z_i,\ \Sigma_1^p \gamma(x) g_i(x) f(x)) \leq \Sigma_1^p \gamma(x) g_i(x) \,.\, \varrho(z_i, f(x)) < \varepsilon$. Hence, $\varrho(f, g) \leq \varepsilon$.

For an arbitrary $y \in g(P)$, denote by A_y the set of all $\lambda = (\lambda_1, \ldots \lambda_p) \in E^p$ such that $\Sigma_1^p \lambda_i z_i = y$, and for some $x \in P$, $\lambda_i = \gamma(x) g_i(x)$ $(i = 1, \ldots, p)$. The set A_y is finite, for otherwise there would exist (since, for any $x \in P$, $g_k(x) = 0$ for all k except $n + 1$ at most) points $z_{i1}, \ldots, z_{ir}$, $r \leq n + 1$, such that $y = \Sigma_1^r \lambda_{ik} z_{ik}$ for infinitely many r-uples $(\lambda_{i1}, \ldots, \lambda_{ir})$ which is impossible ($z_{i1}, \ldots, z_{ir}$ are independent). Since, for any given $\lambda = (\lambda_1, \ldots, \lambda_p) \in E^p$, the set of all $x \in P$ such that $g_i(x) = \lambda_i$ $(i = 1, \ldots, p)$ is clearly contained in some G_j, we have, consequently, $\delta(g^{-1}(y)) < \mathfrak{G}$, $\delta(g^{-1}(y)) < \mathfrak{A}$. Hence P has property $\Delta(E^n)$.

II. (2) implies (3). — Let P have property $\Delta(E^n)$ (in P). Let $f_i \in C(P)$ $(i = 1, 2, \ldots)$. It is easy to see that there exists, for $m = 1, 2, \ldots$, a f. o. c. $\mathfrak{A}_m$ of P such that the diameter $d(f_k(A))$ is $< m^{-1}$ whenever $A \in \mathfrak{A}_m$, $k \leq m$ (to find such a f. o. c., we have only to choose f. o. coverings $\mathfrak{B}_k$ of $f_k(P)$ such that $d(B) < m^{-1}$ whenever $B \in \mathfrak{B}_k$ and to take for $\mathfrak{A}_m$ the collection of all $\prod_{k=1}^m f_k^{-1}(B_k)$, $B_k \in \mathfrak{B}_k$). By Lemmas 5 and 6, and

Baire's Theorem, property $\Delta(E^n)$ implies that there exists $g \in C(P, E^n)$ such that $\delta(g^{-1}(y)) < \mathfrak{A}_m$ $(m = 1, 2, \ldots)$, for any $y \in E^n$. Put, for $x \in P$, $k = 1, \ldots, n$, $g_k(x) = k$-th coordinate of $g(x)$; then $g_k \in C(P)$. Let $S \subset P$ be connected and let every g_k be constant on S; then, for some y, $S \subset g^{-1}(y)$ and therefore, for $m = 1, 2, \ldots$, S is contained in some $A \in \mathfrak{A}_m$. Hence $d(f_k(S)) < m^{-1}$ $(k, m = 1, 2, \ldots; k \leq m)$, $d(f_k(S)) = 0$ $(k = 1, 2, \ldots)$, every f_k is constant on S. Hence, by Proposition 2, all the functions f_k are contained in the subring analytically generated by $g_1, \ldots, g_n$. Thus P has property (3).

III. (3) implies (4) (trivially). IV. (4) implies (1). — Suppose that (4) holds. Let $\mathfrak{G} = \{G_i\}$ $(i = 1, \ldots, p)$ be a f. o. c. of P. There exist open sets H_i such that $\overline{H_i} \subset G_i, \sum_1^p H_i = P$, and continuous functions $f_i \in C(P)$ $(i = 1, \ldots, p)$ such that $0 \leq f_i(x) \leq 1$, for any $x \in P$, $f_i(x) = 1$, for $x \in \overline{H_i}$, $f_i(x) = 0$, for $x \in P - G_i$. Since (4) holds, there exist $g_j \in C(P)$ $(j = 1, \ldots, n)$ such that every f_i is contained in the ring $C_1 \subset C(P)$ generated analytically by the functions g_j. By Proposition 2, every f_i is constant on every connected $S \subset P$ on which each g_j is constant. Put, for any $x \in P$, $g(x) = (g_1(x), \ldots, g_n(x)) \in E^n$; then $g \in C(P, E^n)$. Every (non-void) connected $S \subset P$ which is contained in some $g^{-1}(y)$ is clearly contained in some $G_i \in \mathfrak{G}$, for otherwise we would have, for appropriate $u_i \in S$ $(i = 1, \ldots, p)$, $u_i \in P - G_i$, $f_i(u_i) = 0$ (since every f_i is constant on S), $f_i(x) = 0$ whenever $x \in S$ $(i = 1, \ldots, p)$, hence $S \subset \Pi_1^p(P - H_i)$ which is impossible. Therefore, by Lemma 4, $\delta(g^{-1}(y)) < \mathfrak{G}$, for any $y \in E^n$, which, by Lemma 3, implies that there exists a f. o. c. $\mathfrak{B}$ of $g(P)$ such that, for each $B \in \mathfrak{B}$, $\delta(g^{-1}(B)) < \mathfrak{B}$. Since $T = g(P)$ is n-dimensional at most, there exists a f. o. c. $\mathfrak{L} < \mathfrak{B}$ of order $\leq n$. Let $\mathfrak{L} = \{U_1, \ldots \ldots, U_r\}$. Since $\delta(g^{-1}(U_i)) < \mathfrak{G}$, for each U_i, there exist $V_{ij} \subset P$ $(i = 1, \ldots \ldots, r; j = 1, \ldots, k_i)$ such that $V_{ih} \cdot \overline{V_{ij}} = \emptyset$ (for $h \neq j$), $\Sigma_1^{k_i} V_{ij} = g^{-1}(U_i)$, every V_{ij} is contained in some $G \in \mathfrak{G}$. It is easy to see that the collection of all V_{ij} is a f. o. c. (of P) of order $\leq n$. This completes the proof.

Definition. Let C be an analytic ring. The least cardinal number $\mathfrak{m}$ such that every countable $M \subset C$ is contained in a subring generated analytically by a set of power $\leq \mathfrak{m}$ is called the *analytic pseudodimension* of C, denoted psdim$C$.

Remarks. (1) Proposition 3 implies that, for $C = C(P)$, P compact, „finite" may be substituted for „countable" in the above definition. — (2) Evidently, (a) psdim$C \leq$ dim$C$, (b) psdim$C \leq \aleph_0$, (c) psdim$C =$ dim$C$ whenever dim$C \leq \aleph_0$.

Proposition 4. dim$P =$ psdim$C(P)$, *for any compact P.*

This follows at once from Proposition 3.

Remark. By Proposition 4 and the preceding remark (2), dim$P =$ $=$ dim$C(P)$ whenever P is compact, dim$C(P) \leq \aleph_0$. The main theorem of [1] asserts that the inductive (Menger-Urysohn) dimension of a compact space P is equal to $\dim C(P)$ whenever $\dim C(P) \leqq \aleph_0$. Thus $\dim C(P) \leqq \aleph_0$ implies, for a compact P, $\dim P = \mathrm{ind} P$, $\mathrm{ind} P$ denoting the inductive dimension.

As a matter of fact, the main theorem of [1] is virtually contained in the above Proposition 4. For it is easy to show that $\mathrm{ind} P \leqq \dim C(P)$ (cf. [1], Theorems 1 and 3). On the other hand, it is known (see [5]) that, for a compact P, $\dim P \leqq \mathrm{ind} P$; hence, by Proposition 4, $\mathrm{psdim} C(P) \leqq \leqq \mathrm{ind}(P)$ and therefore $\mathrm{ind} P = \dim P$ whenever $\dim C(P) = \mathrm{psdim} C(P)$ which is equivalent to $\dim C(P) \leqq \aleph_0$.

§ 2.

We are now going to extend the equality $\dim P = \mathrm{psdim} C(P)$ to arbitrary completely regular spaces, after defining the Lebesgue dimension of non-normal completely regular spaces in an adequate way.

Let P be completely regular. It is well known (see e. g. [6]) that there exists an (essentially unique) compact space βP, called the β-extension of P, such that (1) $P \subset \beta P$, $\overline{P} = \beta P$; (2) every $f \in C(P)$ admits of an extension $F \in C(\beta P)$.

It is clear that the correspondence between a function $f \in C(P)$ and its extension $F \in C(\beta P)$ is one-to one and preserves algebraic operations as well as closures of sets (in fact, even distances). Therefore, analytic rings $C(P)$ and $C(\beta P)$ enjoy the same properties and may be considered as identical.

Lemma 7. *If P is normal, then, for arbitrary closed (in P) sets $F_k \subset P$, the closure of $\Pi_1^m F_i$ in βP is equal to the intersection of closures of F_i in βP.*

Remark. Lemma 7 and the following Proposition 5 are essentially due to H. Wallman [7] (observe that, if P is normal, Wallman's extension ωP and β-extension coincide).

Proof. It is sufficient to prove $\overline{F_1 F_2} = \overline{F}_1 \overline{F}_2$. Obviously, $\overline{F}_1 \overline{F}_2 \supset \supset \overline{F_1 F_2}$. Suppose $b \in \overline{F}_1 \overline{F}_2 - \overline{F_1 F_2}$. Choose an open (in βP) set G such that $b \in G$, $\overline{G} F_1 F_2 = \emptyset$, and put $A_k = \overline{G} F_k$. Then $A_1 A_2 = \emptyset$, $b \in \overline{A}_k$. There exists, by Urysohn's Lemma, a function $f \in C(P)$ such that $f(x) = k$ for $x \in A_k$. Since f admits of an extension $F \in C(\beta P)$, we have a contradiction (namely, $F(b) = k$ for $k = 1, 2$).

Proposition 5. *For a normal P, $\dim P = \dim \beta P$.*

Proof. I. Suppose $\dim \beta P \leqq n$. Let $\mathfrak{G} = \{G_1, \ldots, G_m\}$ be a f. o. c. of P. Put $U_i = \beta P - \overline{P - G_i}$. Lemma 7 implies $\Sigma_1^m U_i = \beta P$. There exists a f. o. c. $\{H_j\}$ of βP, of order $\leqq n$, such that each H_j is contained in some U_i. Clearly, $\{PH_j\} < \mathfrak{G}$. II. Suppose $\dim P \leqq n$. Let $\mathfrak{G} = \{G_1, \ldots, G_r\}$ be a f. o. c. of βP. Let H_i be open in βP, $\overline{H}_i \supset G$, $\Sigma_1^r H_i = \beta P$.

There exists a f. o. c. $\mathfrak{V} = \{V_j\}$ $(j = 1, \ldots, s)$ of P, of order $\leqq n$, such that each V_j is contained in some H_i. Put $U_j = \beta P - \overline{P - V_j}$. Then Lemma 7 implies $\Sigma_1^s U_j = \beta P - \Pi_1^s \overline{P - V_j} = \beta P$. If $\Pi V_j = \emptyset$, j running over a given set of integers $1, \ldots, s$, then $\Sigma \overline{P - V_j} = \beta P$, $\Pi(\beta P - \overline{P - V_j}) = \emptyset$, $\Pi U_j = \emptyset$. Hence $\{U_j\}$ is of order $\leqq n$. If $V_j \subset H_i$, then

$$\beta P - \overline{P - V_j} \subset \beta P - \overline{P - H_i} \subset \overline{H_i} \subset G_i.$$

Hence $\{U_i\} < \mathfrak{G}$ which proves the proposition.

Propositions 4 and 5 imply (since $C(P)$ and $C(\beta P)$ may be considered as identical):

Proposition 6. *For a normal space* P, $\dim P = \operatorname{psdim} C(P)$.

Definition. If P is completely regular then the Lebesgue dimension of the compact (hence, normal) space βP will be called the *Lebesgue dimension* of P, denoted $\dim P$.

Proposition 7. *The above definition coincides, for a normal* P, *with the usual one* (this note, p. 81—82).

This follows at once from Proposition 5.

Remark. It is possible to replace the above definition by an equivalent one not making use of the β-extension. This may be done e. g. by restricting the considerations to normal (Tukey [8]) f. o. coverings or, which is the same, to f. o. coverings possessing refinements of the form $\{f^{-1}(G_i)\}$ where f is a mapping of P into E^r, $\{G_i\}$ is a f. o. c. of $f(P)$.

We now state our main theorem.

Theorem 1. *For any completely regular space* P, $\dim P = \operatorname{psdim} C(P)$.

This follows immediately from Propositions 4 and 5 and includes Proposition 6 as a special case (cf. Proposition 7).

We now have to show that the above generalized definition of the Lebesgue dimension is „reasonable“ which essentially means that the inequality $\dim M \leqq \dim P$, for $M \subset P$, and the Sum Theorem obtain, under some reasonable assumptions. This will be shown below (Theorem 2).

Definition. A subset M of a completely regular space P will be called *normally closed* if it is closed and every $f \in C(M)$ admits of an extension $F \in C(P)$.

Proposition 8. *If* P *is completely regular,* $M \subset P$ *is normally closed, then* $\dim M \leqq \dim P$.

Proof. Clearly, $\overline{M} \subset \beta P$ is compact, and every $f \in C(M)$ may be extended over βP. Hence $\overline{M} = \beta M$. Now let $\mathfrak{G} = \{G_i\}$ be a f. o. c. of $\overline{M}$. There exist open (in βP) sets H_i such that $\overline{M} H_i = G_i$. The sets H_i together with $\beta P - \overline{M}$ cover βP. Therefore, supposing $\dim P \leqq n$, there exists a f. o. c. $\mathfrak{V} = \{V_j\}$ of βP, of order $\leq n$, such that each V_j is contained either in $\beta P - \overline{M}$ or in some H_i. The f. o. c. $\mathfrak{U} = \{\overline{M}V_j\}$ of $\overline{M} = \beta M$ is of order $\leq n$; $\mathfrak{U} < \mathfrak{G}$.

Proposition 9. *If P is normal, $A_k \subset P$ are closed, $P = \Sigma_1^\infty A_k$, $\dim A_k \leq n$, then $\dim P \leq n$.*

This well known result is due to E. Čech [9]; cf. E. Hemmingsen [10].

Proposition 10. *If P is regular and every open covering of P contains a countable subcovering, then P is normal.*

Remark. This result is due to E. Čech (unpublished). The idea of the proof is due to A. Tychonoff [11].

Proof. Let $A \subset P$, $B \subset P$ be closed, $AB = \emptyset$. For every $x \in P$ choose an open set $G(x)$ such that $x \in G(x)$ and either $A\overline{G(x)}$ or $B\overline{G(x)}$ is void. The covering $\{G(x)\}$ contains a countable subcovering $\{G_n\}$. Denote by F_n $(n = 1, 2, \ldots)$ the sum of $\overline{G_k}$, $k \leq n$, such that $A\overline{G_k} = \emptyset$, and put $G = \Sigma(G_n - F_n)$. Since $\Sigma_1^\infty G_n = P$, $F_nA = \emptyset$ $(n = 1, 2, \ldots)$, we have $A \subset G$. If $x \in B$, then $x \in G_m$, for some m, and clearly $G_m(G_n - F_n) = \emptyset$ $(n = m, m + 1, \ldots)$ whereas, for $n \leq m$, we have either $G_n - F_n = \emptyset$ or $\overline{G_n}B = \emptyset$; therefore x non $\in \overline{G}$. Hence $\overline{G}B = \emptyset$ which proves the normality of P.

Lemma 8. *If there exist, in a space P, compact sets $K_n \subset P$ such that $P = \Sigma_1^\infty K_n$, then every open covering $\mathfrak{G}$ of P contains a countable subcovering.*

Proof. Since K_n is compact, $\mathfrak{G}$ contains G_{ni} such that $\Sigma_{i=1}^{p_n} G_{ni} \supset K_n$. The collection of all G_{ni} covers P.

Theorem 2. *If P is completely regular, $P = \Sigma_1^\infty A_n$, A_n are normally closed in P, then* $\dim P = \sup \dim A_n$.

Proof. Denote by B_n the closure of A_n in βP and put $B = \Sigma B_n$. Since A_n are normally closed, $B_n = \beta A$, and therefore $\dim B_n = \dim A_n$. Clearly $\beta P = \beta B$ which implies $\dim B = \dim \beta P = \dim P$. Now apply Lemma 8 and Propositions 10 and 8.

Remark. It is sufficient to suppose, in Theorem 2, instead of A_n being normally closed only that every $f \in C(A_n)$ admits of an extension $F \in C(P)$.

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Věta o Lebesgueově dimensi

(Obsah předešlého článku).

Hlavním výsledkem článku je věta: Je-li P úplně regulární prostor, pak $\dim P = \operatorname{psdim} C(P)$. Při tom je $\dim P$ definována jako Lebesgueova dimense Čechova obalu βP (takže pro normální prostor P se shoduje s Lebesgueovou dimensí, definovanou obvyklým způsobem pomocí konečných otevřených pokrytí prostoru P), $\operatorname{psdim} C(P)$ je pak nejmenší kardinální číslo $\mathfrak{m}$ takové, že každá spočetná $M \subset C(P)$ je obsažena v jistém analyticky uzavřeném podokruhu C_1 okruhu $C(P)$ (jenž se skládá z omezených spojitých funkcí v P), vytvořeném nejvýše $\mathfrak{m}$ funkcemi z $C(P)$.