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Oscillation and Nonoscillation of First Order Nonlinear Delay Differential Equations

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Abstract. Oscillation and nonoscillation criteria for the first order nonlinear delay differential equations of the form

$$\dot{x}(t) + p(t)|x(\tau(t))|^{\alpha} \operatorname{sgn}[x(\tau(t))] = 0, \quad t \geq t_0,$$

are established, where $\alpha > 1$.

1. Introduction

In this paper we shall study the oscillatory behaviour of the nonlinear functional differential equation

$$\dot{x}(t) + p(t)|x(\tau(t))|^{\alpha} \operatorname{sgn}[x(\tau(t))] = 0, \quad t \geq t_0, \tag{1}$$

where $\alpha > 1, p \in C([t_0,\infty),[0,\infty)), \tau \in C^1([t_0,\infty),[0,\infty))$, $\lim_{t\to\infty} \tau(t) = \infty$, $\tau(t) < t$, $t \geq t_0$. The existence of a positive solution of Eq. (1) is also treated.

By a solution of Eq. (1) we mean a continuous function $x(t)$ which satisfies Eq. (1) on the interval $[t_0,\infty)$.

Recently the original results have been published in [6] about oscillation and nonoscillation of nonlinear differential equation of the form

$$\dot{x}(t) + p(t)\prod_{j=1}^{m} |x(t-\tau_j)|^{\alpha_j} \operatorname{sgn}[x(t-\tau_1)] = 0, \quad t \geq t_0,$$

where $0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m$ are constants, $\sum_{j=1}^{m} \alpha_j > 1$, and its special case

$$\dot{x}(t) + p(t)|x(t-\tau)|^{\alpha} \operatorname{sgn}[x(t-\tau)] = 0, \quad t \geq t_0,$$

where $\alpha > 1, \tau > 0$.

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As far as we know there are no results for Eq. (1) on the subject of this paper. Due to inequalities

$$\begin{aligned} \exp\left(e^{\varepsilon t}\right)\exp\left(-e^{\lambda t}\right) &> \exp\left(-e^{\lambda t}\right), \quad 0<\varepsilon<\lambda,\ t>0, \\ e^{-\mu t}\exp\left(-e^{\mu t}\right) &< \exp\left(-e^{\mu t}\right), \quad \mu>0,\ t>0, \end{aligned}$$

Theorems 1 and 2 are extensions of the results in [6] for the equation

$$\dot{x}(t)+p(t)|x(t-\tau_1)|^{\alpha}\mathrm{sgn}[x(t-\tau_1)]=0, \quad \alpha>1,\ t\geq t_0.$$

The extension on the equation of the type

$$\dot{x}(t)+p(t)\prod_{j=1}^{m}|x(\tau_j(t))|^{\alpha_j}\,\mathrm{sgn}[x(\tau_1(t))]=0, \quad \sum_{j=1}^{m}\alpha_j>1,\ t\geq t_0,$$

is also possible. Oscillation and nonoscillation problem of cognate equations is also treated in [2–5].

2. Main Results

We shall need the following lemma.

Lemma 1. *Suppose that $\alpha>1, q\in C([t_0,\infty),[0,\infty)), q(t)\not\equiv 0$ on any interval and*

$$q(t)\leq p(t), \quad t\geq t_0.$$

If Eq. (1) has a nonoscillatory solution, then also equation

$$\dot{x}(t)+q(t)|x(\tau(t))|^{\alpha}sgn[x(\tau(t))]=0, \quad t\geq t_0, \tag{2}$$

has a nonoscillatory solution.

Proof. Assume that $v(t)$ is a nonoscillatory solution of Eq. (1) such that $v(\tau(t))>0$ on $[T,\infty), T>t_0$. Then

$$\dot{v}(t)=-p(t)[v(\tau(t))]^{\alpha}, \quad t\geq T,$$

and

$$\dot{v}(t)\leq 0 \quad \text{for } t\geq T,$$

i.e. $v(t)$ is decreasing on $[T,\infty)$. It follows from (1) that

$$v(t)\geq\int_t^{\infty}p(s)[v(\tau(s))]^{\alpha}\,ds, \quad t\geq T. \tag{3}$$

By $C_{loc}([T,\infty),R)$ we denote the space of continuous functions $x:[T,\infty)\to R$ endowed with the topology of local uniform convergence. We define the set $S\subset C_{loc}([T,\infty),R)$ of the functions x which satisfy the inequalities

$$0\leq x(t)\leq v(t) \quad \text{for } t\geq T.$$

The operator $F:S\to C_{loc}([T,\infty),R)$ is defined by

$$F(x)(t)=\begin{cases} \int_t^{\infty}q(s)[x(\tau(s))]^{\alpha}\,ds & \text{for} \quad t\geq t_1, \\ v(t)-v(t_1)+F(x)(t_1) & \text{for} \quad t\in[T,t_1), \end{cases}$$

where $t_1>T$ is such that $\tau(t)\geq T$ for $t\geq t_1$.

If $x \in S$, then by (3) we have

$$0 \le F(x)(t) = \int_t^\infty q(s)[x(\tau(s))]^\alpha\, ds \le \int_t^\infty p(s)[v(\tau(s))]^\alpha\, ds \le v(t), \quad t \ge t_1.$$

Thus we get $F(S) \subset S$. We note that S is a nonempty closed convex subset of $C_{loc}([T,\infty),R)$ and the operator F is continuous. The functions belonging to the set $F(S)$ are equicontinuous on every compact subinterval of $[T,\infty)$. Then according to the Schauder-Tychonoff fixed point theorem (cf., e.g. [1, p. 231]), F has an element $x \in S$ such that $x = F(x)$. It is easy to see that x satisfies Eq. (2) on $[t_1,\infty)$.

Now we show that x is positive on $[t_1,\infty)$. Obviously $v(t) > v(t_1)$ on $[T,t_1)$, x is nonnegative on $[t_1,\infty)$, $x(t_1) = F(x)(t_1) > 0$ and moreover from Eq. (2) it follows that x is decreasing on $[t_1,\infty)$. Let $t_2 \in (t_1,\infty)$ be the first point in which $x(t_2) = 0$. Then by Eq. (2) we have

$$\dot{x}(t_3) = -q(t_3)[x(\tau(t_3))]^\alpha < 0, \quad t_3 \in [t_2,\infty).$$

By decreasing character of x we always have $x = 0$ on $[t_2,\infty)$, which gives $\dot{x}(t_3) = 0$. This contradiction proves that x has no zeros on $[t_2,\infty)$ and so x is positive on $[t_1,\infty)$. The proof is complete.

Theorem 1. *Suppose that* $\alpha > 1$, $0 < \dot{\tau}(t) \le 1$, $\liminf_{t\to\infty}[t-\tau(t)] \ge \sigma > 0$, *there exists* $\lambda > 0$ *such that*

$$\alpha e^{-\lambda\sigma} < 1 \tag{4}$$

and

$$\liminf_{t\to\infty}[p(t)\exp(e^{\varepsilon t})\exp(-e^{\lambda t})] > 0, \tag{5}$$

where $0 < \varepsilon < \lambda$. *Then every solution of Eq. (1) oscillates.*

Proof. With regard to conditions (4), (5), we may choose $\varepsilon < \beta < \lambda$ and $T > t_0$ such that

$$\alpha e^{-\beta\sigma} < 1 \tag{6}$$

and due to the condition (5) and the next inequality

$$e^{\lambda t} - e^{\varepsilon t} \ge \lambda t + (\alpha - 1)e^{\beta t}, \quad t \ge T,$$

we obtain

$$p(t) \ge \lambda e^{\lambda t}\exp[(\alpha-1)e^{\beta t}], \quad t \ge T.$$

Define

$$q(t) := \lambda e^{\lambda t}\exp[(\alpha-1)e^{\beta t}].$$

According to Lemma 1, if every solution of the equation

$$\dot{x}(t) + q(t)|x(\tau(t))|^\alpha \operatorname{sgn}[x(\tau(t))] = 0, \quad t \ge T, \tag{7}$$

oscillates, then also every solution of Eq. (1) oscillates.

Suppose to the contrary that (7) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (7). Then, with regard to the definition of $q(t)$ we obtain that

$$0 < x(\tau(t)) < 1 \quad \text{and} \quad \dot{x}(t) < 0 \quad \text{for } t \ge T_1 > T.$$

Set $y(t) = -\ln x(t)$, $t \geq T_1$. Then $y(t) > 0$ for $t \geq T_1$ and has a increasing nature. From (7) we get

$$\dot{y}(t) = q(t)\exp[y(t) - \alpha y(\tau(t))], \quad t \geq T_1. \tag{8}$$

By (6) we may choose $0 < \gamma < \beta$ and $0 < r < \sigma$ such that

$$\alpha e^{-\gamma r} < 1.$$

Now we can consider three possible cases.

Case 1. $y(t) \leq \alpha e^{(\beta-\gamma)r} y(\tau(t))$ eventually holds. Let $T_2 > T_1$ be such that

$$y(t) \leq \alpha e^{(\beta-\gamma)r} y(\tau(t)), \quad t \geq T_2.$$

Then we get

$$\begin{aligned} \frac{y(t)}{e^{\beta t}} &\leq \frac{\alpha e^{(\beta-\gamma)r} y(\tau(t))}{e^{\beta t}} \leq \frac{\alpha e^{(\beta-\gamma)[t-\tau(t))]} y(\tau(t))}{e^{\beta t}} \\ &= \alpha e^{-\gamma[t-\tau(t)]} \frac{y(\tau(t))}{e^{\beta\tau(t)}} < \alpha e^{-\gamma r} \frac{y(\tau(t))}{e^{\beta\tau(t)}}, \qquad t \geq T_2. \end{aligned}$$

Define $z(t) = y(t)e^{-\beta t}$. Then for $t \geq T_2$ we have

$$z(t) < \alpha e^{-\gamma r} z(\tau(t)).$$

In view of this inequality, applying the result of [3, Lemma 2.1] we obtain

$$\lim_{t\to\infty} z(t) = 0.$$

According to above there exists a $T_3 > T_2$ such that

$$y(t) < e^{\beta t}, \quad t \geq T_3. \tag{9}$$

¿From (8) it follows

$$\dot{y}(t) \geq q(t)\exp[(1-\alpha)y(t)]$$

and using (9) we get

$$\dot{y}(t) \geq q(t)\exp[(1-\alpha)e^{\beta t}] = \lambda e^{\lambda t}, \quad t \geq T_3.$$

Integrating the last inequality we obtain

$$y(t) \geq y(T_3) + e^{\lambda t} - e^{\lambda T_3}, \quad t \geq T_3,$$

which contradicts (9).

Case 2. $y(t) > \alpha e^{(\beta-\gamma)r} y(\tau(t))$ eventually holds. Choose $T_4 > T_3$ such that

$$y(t) > \alpha e^{(\beta-\gamma)r} y(\tau(t)), \quad t \geq T_4.$$

Applying the above inequality in (8) we get

$$\dot{y}(t) > q(t)\exp[(1 - e^{(\gamma-\beta)r})y(t)], \quad t \geq T_4.$$

Set $c = 1 - e^{(\gamma-\beta)r}$. Then $c > 0$ and

$$\dot{y}(t)e^{-cy(t)} > q(t), \quad t \geq T_4.$$

Integrating this inequality we have

$$\int_{T_4}^{\infty} q(t)\,dt < \int_{T_4}^{\infty} \dot{y}(t)e^{-cy(t)}\,dt \leq \frac{1}{c}e^{-cy(T_4)} < \infty,$$

which is a contradiction to the definition of $q(t)$.

Case 3. $y(t) - \alpha e^{(\beta-\gamma)r}y(\tau(t))$ is oscillatory. Set

$$u(t) = y(t) - \alpha e^{(\beta-\gamma)r}y(\tau(t)).$$

Then $u(t)$ is oscillatory and there exists an increasing infinite sequence $\{t_n\}$ of real numbers with $T_4 < t_1 < t_2 < \cdots$ such that

$$u(t_n) = 0, \quad n = 1, 2, \ldots,$$

and

$$u(t) > 0 \quad \text{for } t \in (t_{2n-1}, t_{2n}), \quad n = 1, 2, \ldots .$$

Also there exists an increasing infinite sequence $\{\xi_n\}$, $\xi_n \in (t_{2n-1}, t_{2n})$ such that $u(\xi_n) = \max\{u(t) : t_{2n-1} \le t \le t_{2n}\}$ and $\dot{u}(\xi_n) = 0$, $n = 1, 2, \ldots$. It follows that

$$\begin{aligned} \dot{y}(t) &= q(t)\exp[u(t) + \alpha(e^{(\beta-\gamma)r} - 1)y(\tau(t))], \quad t \ge T_4, \qquad (10)\\ \dot{u}(\xi_n) &= \dot{y}(\xi_n) - \alpha e^{(\beta-\gamma)r}\,\dot{\tau}(\xi_n)\dot{y}(\tau(\xi_n)), \end{aligned}$$

and

$$\dot{y}(\xi_n) = \alpha e^{(\beta-\gamma)r}\,\dot{\tau}(\xi_n)\dot{y}(\tau(\xi_n)). \qquad (11)$$

Combining (10) and (11) we can find that

$$\begin{aligned} &q(\xi_n)\exp[u(\xi_n) + \alpha(e^{(\beta-\gamma)r} - 1)y(\tau(\xi_n))] = \alpha e^{(\beta-\gamma)r}\,\dot{\tau}(\xi_n)\dot{y}(\tau(\xi_n))\\ &= \alpha e^{(\beta-\gamma)r}\,\dot{\tau}(\xi_n)q(\tau(\xi_n))\exp[u(\tau(\xi_n)) + \alpha(e^{(\beta-\gamma)r} - 1)y(\tau(\tau(\xi_n)))]. \end{aligned}$$

Using

$$\alpha e^{-\gamma r} < 1, \quad e^{\beta r} < e^{\alpha r}, \quad q(\tau(\xi_n)) = \lambda e^{\lambda\tau(\xi_n)}\exp[(\alpha-1)e^{\beta\tau(\xi_n)}]$$

and $\xi_n \ge \sigma + \tau(\xi_n) > r + \tau(\xi_n)$, we obtain

$$\begin{aligned} &q(\xi_n)\exp[u(\xi_n) + \alpha(e^{(\beta-\gamma)r} - 1)y(\tau(\xi_n))]\\ &< \lambda e^{\lambda r}e^{\lambda\tau(\xi_n)}\exp[(\alpha-1)e^{\beta\tau(\xi_n)} + u(\tau(\xi_n)) + \alpha(e^{(\beta-\gamma)r} - 1)y(\tau(\tau(\xi_n)))]\\ &< \lambda e^{\lambda\xi_n}\exp[(\alpha-1)e^{\beta\tau(\xi_n)} + u(\tau(\xi_n)) + \alpha(e^{(\beta-\gamma)r} - 1)y(\tau(\tau(\xi_n)))]. \end{aligned}$$

The above inequality implies that

$$\begin{aligned} &\exp[u(\xi_n) + \alpha(e^{(\beta-\gamma)r} - 1)y(\tau(\xi_n))]\\ &< \exp[-(\alpha-1)e^{\beta\xi_n} + (\alpha-1)e^{\beta\tau(\xi_n)} + u(\tau(\xi_n)) + \alpha(e^{(\beta-\gamma)r} - 1)y(\tau(\tau(\xi_n)))]. \end{aligned}$$

So we have

$$\begin{aligned} &u(\xi_n) + \alpha(e^{(\beta-\gamma)r} - 1)y(\tau(\xi_n)) < u(\tau(\xi_n)) + \alpha(e^{(\beta-\gamma)r} - 1)y(\tau(\tau(\xi_n)))\\ &-(\alpha-1)(1 - e^{-\beta r})e^{\beta\xi_n}, \quad n = 1, 2, \ldots . \qquad (12) \end{aligned}$$

If $\limsup_{t\to\infty} u(t) = \limsup_{n\to\infty} u(\xi_n) = \infty$, then there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $u(\xi_{n_k}) = \max\{u(t) : T_4 \le t \le \xi_{n_k}\}$, $k = 1, 2, \ldots$. Then it follows from (12) that

$$\begin{aligned} 0 &< \alpha(e^{(\beta-\gamma)r} - 1)[y(\tau(\xi_{n_k})) - y(\tau(\tau(\xi_{n_k})))]\\ &< -(\alpha-1)(1 - e^{-\beta r})e^{\beta\xi_{n_k}} < 0, \quad k = 1, 2, \ldots, \end{aligned}$$

which is a contradiction. If $\limsup_{t\to\infty} u(t) = \limsup_{n\to\infty} u(\xi_n) < \infty$, then (12) implies that

$$\begin{aligned} 0 &< \limsup_{n\to\infty}\{u(\xi_n) + \alpha(e^{(\beta-\gamma)r} - 1)[y(\tau(\xi_n)) - y(\tau(\tau(\xi_n)))]\} \\ &\leq \limsup_{n\to\infty}[u(\tau(\xi_n)) - (\alpha - 1)(1 - e^{-\beta r})e^{\beta\xi_n}] = -\infty. \end{aligned}$$

This is a contradiction. The proof is complete.

Theorem 2. *Suppose that* $\alpha > 1, \sigma > 0$,

$$\limsup_{t\to\infty}[t - \tau(t)] \leq \sigma, \tag{13}$$

there exists $\mu > 0$ *such that*

$$\alpha e^{-\mu\sigma} > 1 \tag{14}$$

and

$$\limsup_{t\to\infty}[p(t)e^{-\mu t}\exp(-e^{\mu t})] < \infty. \tag{15}$$

Then Eq. (1) has an eventually positive solution.

Proof. According to conditions (13), (14), (15) we may choose $\omega > \mu, r > \sigma$ and $T > t_0$ such that

$$\alpha e^{-\omega r} > 1$$

and

$$\begin{aligned} p(t) &\leq \omega e^{\omega t}\exp[(\alpha e^{-\omega r} - 1)e^{\omega t}] \\ &\leq \omega e^{\omega t}\exp[(\alpha e^{-\omega[t-\tau(t)]} - 1)e^{\omega t} \\ &= \omega e^{\omega t}\exp[\alpha e^{\omega\tau(t)} - e^{\omega t}], \quad t \geq T. \end{aligned} \tag{16}$$

We define the set $S \subset C_{loc}([T,\infty), R)$ of functions x which satisfy the inequalities

$$0 \leq x(t) \leq v(t) \quad \text{for } t \geq T,$$

where

$$v(t) = \exp(-e^{\omega t}), \quad t \geq T.$$

The operator $F : S \to C_{loc}([T,\infty), R)$ is defined by

$$F(x)(t) = \begin{cases} \int_t^\infty p(s)x^\alpha(\tau(s))\,ds & \text{for } t \geq T_1, \\ v(t) - v(T_1) + F(x)(T_1) & \text{for } t \in [T, T_1), \end{cases}$$

where $T_1 > T$ is such that $\tau(t) \geq T$ for $t \geq T_1$.

If $x \in S$, then by virtue of (16) we get

$$\begin{aligned} 0 &\leq F(x)(t) = \int_t^\infty p(s)x^\alpha(\tau(s))\,ds \leq \int_t^\infty p(s)v^\alpha(\tau(s))\,ds \\ &= \int_t^\infty p(s)\exp[-\alpha e^{\omega\tau(s)}]\,ds \leq \int_t^\infty \omega e^{\omega s}\exp(-e^{\omega s})\,ds \\ &= \exp(-e^{\omega t}) = v(t), \quad t \geq T_1. \end{aligned}$$

Thus $F(S) \subset S$. Now we can proceed as in the proof of Lemma 1. So Eq. (1) has a nonoscillatory solution. The proof is complete.

Corollary 1. *Suppose that* $\alpha > 1, \sigma > 0$.

(i) Let $\liminf_{t\to\infty}[t - \tau(t)] \geq \sigma,\ 0 < \dot{\tau}(t) \leq 1$ *and there exists* $\lambda > \sigma^{-1}\ln\alpha$ *such that*

$$\liminf_{t\to\infty}[p(t)\exp(-e^{\lambda t})] > 0. \tag{17}$$

Then every solution of Eq. (1) oscillates.

(ii) Let $\limsup_{t\to\infty}[t - \tau(t)] \leq \sigma$ *and there exists* $\mu < \sigma^{-1}\ln\alpha$ *such that*

$$\limsup_{t\to\infty}[p(t)\exp(-e^{\mu t})] < \infty. \tag{18}$$

Then Eq. (1) has an eventually positive solution.

Proof. The conditions (17) and (18) imply that (5) and (15) hold and we can apply Theorem 1 and 2.

Applying Corollary 1 to equation

$$\dot{x}(t) + p(t)|x(t-\tau)|^{\alpha}\operatorname{sgn}[x(t-\tau)] = 0, \quad t \geq t_0, \tag{19}$$

where $\tau > 0$, we obtain the result in [6, Corollary 1].

Corollary 2. *Suppose that* $\alpha > 1$. *Then the following conclusions hold:*

(i) If there exists $\lambda > \tau^{-1}\ln\alpha$ *such that (17) holds, then every solution of (19) oscillates.*

(ii) If $p(t) \not\equiv 0$ *on any interval of length* τ *and there exists* $\mu < \tau^{-1}\ln\alpha$ *such that (18) holds, then (19) has an eventually positive solution.*

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