Radoslav M. Dimitrić

Trees, inverse systems, and valuated vector spaces

Acta Mathematica Universitatis Ostraviensis, Vol. 12 (2004), No. 1, 19--22

Trees, inverse systems, and valuated vector spaces

Radoslav M. Dimitrić

Abstract. We introduce (perhaps unexpected) correspondence between trees, inverse systems of sets or algebras and valuated vector spaces. We also define κ-inverse systems, Aronszajn and Kurepa inverse systems and use the correspondence to prove new results related to non-triviality of inverse limits of surjective inverse systems.

We first fix terminology. A strict partially ordered set $(T, \leq)$ is called a tree, if, for every $x \in T$, the set of predecessors $\{y \in T : y < x\} = (\longleftarrow, x)$ is well-ordered in the induced ordering. The *height of* $x \in T$, denoted by $ht(x,T)$, is the ordinal order equivalent to $(\longleftarrow, x)$. If α is an ordinal then $T_\alpha = Lev_\alpha(T) = \{x \in T : h(x,T) = \alpha\}$ is the *α-th level* of T. *The height of* $(T, \leq)$, denoted $ht(T)$, is the least ordinal τ such that $Lev_\tau(T) = \emptyset$. A *branch of* T is a maximal linearly ordered subset of T; B_T or simply B will denote the set of all branches of T. If b is a branch of T, then b is well ordered by the ordering of T. If $b \cap T_\alpha \neq \emptyset$ then it is a singleton and $b \cap T_\beta \neq \emptyset$ for all $\beta < \alpha$. If b is a branch then the *length of* b is the least ordinal λ such that $b \cap T_\lambda = \emptyset$. Note that this is the same as the ordinal that is order equivalent to b. If the length of a branch b is λ we shall refer to b as a λ-branch. T is called a *κ-tree*, if $ht(T) = \kappa$ and every level $|T_\alpha| < \kappa$. In this paper we shall be concerned with κ-trees and families of κ-branches, for a regular cardinal κ.

A κ-tree is called a) a *κ-Aronszajn* tree, if its every branch is of cardinality $< \kappa$, b) a *κ-Suslin tree* if its every branch and every antichain are of cardinality $< \kappa$ and c) a *κ-Kurepa tree* if it has at least κ^+ κ-branches. For more information on trees we refer to [Kunen, 1980], for instance. A κ-tree T is *well-pruned*, κ-tree, if $\forall x \in T\ \forall \alpha$, $(ht(x,T) < \alpha < \kappa) \Rightarrow \exists y \in Lev_\alpha(T)(x < y)$; to this, we will add also the condition $|T_0| = 1$, if need be. For a regular κ, every κ-tree has a well-pruned κ-sub-tree. [ibid, Lemma 5.11].

Received: February 18, 2005.

2000 *Mathematics Subject Classification:* 18A30, 16B70, 16D40, 15A78, 03E04, 03E65, 03E75. Secondary: 15A03, 16B50, 20K45.

Key words and phrases: Surjective inverse system, κ-inverse system, κ-Aronszajn inverse system, κ-Kurepa inverse system, tree, Aronszajn tree, Kurepa tree, Suslin tree, valuated vector space, inverse limit, Mittag-Leffler condition.

We call an inverse system (of sets, algebras, etc.) a *κ-inverse system*, if it is indexed by κ and if all the members are of cardinality $< \kappa$, i.e., if it is of the form $(F_\alpha, f_{\alpha\beta})_{\alpha<\kappa}$. and every $|F_\alpha| < \kappa$. We will call such a κ-inverse system an *$\aleph$-κ-inverse system* if $|\varprojlim_{\alpha<\kappa} F_\alpha| = \aleph$. However we single out two special cases and give them special names: The κ-inverse system is called a) *κ-Aronszajn inverse system*, if $\varprojlim_{\alpha<\kappa} F_\alpha$ is trivial, i.e. $= \emptyset, = 0$, etc.; it is called b) a *κ-Kurepa inverse system*, if $|\varprojlim_{\alpha<\kappa} F_\alpha| \geq \kappa^+$. Although these were not discussed by either Aronszajn or Kurepa, justification for the names may be seen in the sequel. The remaining unexplained terminology may be found in the references.

The First Fundamental Correspondence (FC1). There is a bijective correspondence between the classes of (κ-)trees and (κ-)inverse systems in such a way that the sets of κ-branches correspond bijectively to inverse limits of the κ-systems, as follows: If (T, T_α, B_κ) is a κ-tree with levels T_α and the set of its κ-branches B_κ, define a κ-inverse system $(T_\alpha, f_{\alpha\beta})$ and its inverse limit $L = \varprojlim T_\alpha$ by defining $f_{\alpha\beta} : T_\beta \longrightarrow T_\alpha$ via $f_{\alpha\beta}(t_\beta) = t_\alpha$, where $t_\alpha \in T_\alpha$ is the unique predecessor of $t_\beta \in T_\beta$. By this definition, we obtain a κ-inverse system. Given any branch $b \in B_\kappa$, it can be identified with the corresponding element of L, since both can be seen as subsets of $\prod_{\alpha<\kappa} T_\alpha$, and conversely. Given a κ-inverse system $(F_\alpha, f_{\alpha\beta})_{\alpha<\kappa}$, define $T_\alpha = F_\alpha \times \{\alpha\}$ to be the levels of a tree T and define $t_\alpha < t_\beta$, for $\alpha < \beta$, if $f_{\alpha\beta}(t_\beta) = t_\alpha$; thus we obtained a κ-tree. This is clearly a bijective correspondence between κ-trees and κ-inverse systems where branches correspond bijectively to appropriate elements of $\varprojlim$.

In this correspondence well-pruned trees correspond to surjective inverse systems.

Proposition 1 *There is no ω_0-Aronszajn inverse system.*

Proof. Given FC1, this is a simple consequence of the fact that there are no ω_0-Aronszajn trees. $\diamondsuit$

Theorem 2 *a. There is a κ-Aronszajn inverse system iff there is a κ-Aronszajn tree. There is an ω_1-Aronszajn inverse system. b. There is a κ-Kurepa inverse system iff there is a κ-Kurepa tree Thus, the statements on the existence of Aronszajn or Kurepa inverse systems are independent of ZFC.*

Proof. This is because the constructions with same names correspond to each other in the FC1 and the κ-branches in trees correspond to elements of the inverse limit in the inverse systems (consult also [ibid, Theorem 5.9]). $\diamondsuit$

The Second Fundamental Correspondence (FC2) We now expand our correspondence to the class of R modules (for some ring R, possibly a field, that we may restrict to $|R| \leq \kappa$). For a κ-tree (T, T_α, T_κ) define an inverse system of free R-modules as follows: $F_\alpha = \oplus_{i\in T_\alpha} Rx_i^\alpha$, $f_{\alpha\beta} : F_\beta \longrightarrow F_\alpha$ via $f_{\alpha\beta}(r_{i_1}x_{i_1}^\beta + \cdots + r_{i_n}x_{i_n}^\beta) = r_{j_1}x_{j_1}^\alpha + \cdots + r_{j_n}x_{j_n}^\alpha$, where $j_k \in T_\alpha$ is the unique predecessor of $i_k \in T_\beta$ (same j_k may correspond to different i's). It is routine to check that we obtain a κ-inverse system of free R-modules $(F_\alpha, f_{\alpha\beta})_{\alpha,\beta<\kappa}$. The reverse correspondence, from such an inverse system of free modules to the corresponding κ-tree is accomplished as above in the first fundamental correspondence.

The benefits of the fundamental correspondence are manifold. We specify here some (once the correspondence is established, the proofs become straightforward):

Theorem 3 *For a non-empty upward directed set I, the following are equivalent:*

(1) *I has a maximal element, or it contains a countable cofinal sequence.*
(2) *For every surjective I-inverse system of non-trivial free Abelian groups (modules), its inverse limit is likewise non-trivial.*
(3) *For every surjective I-inverse system $\{M_\alpha, f_{\alpha\beta} : \alpha, \beta \in I\}$ of non-trivial Abelian groups (modules), its inverse limit is likewise non-trivial.*
(4) *Every surjective I-inverse system $\{X_\alpha : f_{\alpha\beta}\}$ of non-empty sets has a non-empty inverse limit.*
(5) *For every surjective map $\mathbf{g} = (g_\alpha)_{\alpha \in I} : \{E_\alpha : \epsilon_{\alpha\beta}\} \longrightarrow \{S_\alpha : \sigma_{\alpha\beta}\}$, of I-inverse systems of sets, such that all $\epsilon_{\alpha\beta}$ are surjective and $\sigma_{\alpha\beta}$ are injective, the induced inverse limit map $\varprojlim \mathbf{g} : \varprojlim \mathbf{E} \longrightarrow \varprojlim \mathbf{S}$ is likewise surjective.*
(6) *Every I-inverse system of non-empty sets $\{X_\alpha : f_{\alpha\beta}\}$ that satisfies the ML condition has a non-empty inverse limit.*
(7) *For every group G, every surjective I-inverse system of non-empty transitive G-sets has a non-empty inverse limit.*

Proof. (3)$\Rightarrow$(2): as a more general statement implying more special.

(2)$\Rightarrow$(1): If I has no maximal element assume, on the contrary, that it is of uncountable cofinality $cfI = \kappa$; then it has a subset (an initial segment) J of type ω_1. We noted beforehand that there is a (well-pruned) ω_1-Aronszajn tree, which is equivalent to the existence of an ω_1-Aronszajn inverse system of non-zero free groups (hence with trivial inverse limit $\lim_{\omega_1} F_\alpha = 0$). Expand this surjective inverse system to a κ-surjective inverse system as follows: For any $\beta \in \kappa$, $\beta \geq \omega_1$, let $F_\beta = \oplus_{\alpha<\omega_1} F_\alpha$ and let $f_{\beta_1\beta_2} : F_{\beta_2} \longrightarrow F_{\beta_1}$ be the identity if $\beta_2 \geq \beta_1 \geq \omega_1$, and for $\alpha_1 < \omega_1 < \beta_2$ define $f_{\alpha_1\beta_2} : F_{\beta_2} \longrightarrow F_{\alpha_1}$ to be the corresponding canonical projection. $(F_\beta, f_{\alpha\beta})_{\alpha,\beta\in\kappa}$ is a surjective inverse system with the trivial inverse limit, for otherwise, $\varprojlim_{\omega_1} F_\alpha$ would not be trivial (if an element of $\varprojlim_\kappa$ had non-zero (equal) coordinates above level ω_1 they would translate, via the projections into non-zero elements below level ω_1, which would be a contradiction). We now have $\varprojlim_{\alpha<\kappa} F_\alpha = \varprojlim_{\alpha<\omega_1} F_\alpha = 0$; this is a contradiction, since $cfI = \kappa$ and the inverse limit over I is supposed to be non-trivial, by the assumption.

(1)$\Rightarrow$(3): This is a consequence of König's lemma, since in an ω_0-tree, there is always an ω_0-branch. The equivalences (1)$\leftrightarrow$ (4),(5),(6),(7) have already been established in [Dimitric, 2004, Theorems 1,2]. $\Diamond$

The fundamental correspondence leads to an interesting relationship between κ-trees and valuated vector spaces. Recall that (V, v) is a valuated vector space (over the field F), if the valuation $v : V \longrightarrow Ord \cup \{\infty\}$ is a function that satisfies: $v(x) = \infty$ iff $x = 0$, $v(rx) = v(x)$, for scalars $r \neq 0$, and $v(x+y) \geq \min(v(x), v(y))$. Given a limit ordinal λ, the λ-topology on V is defined by the base of the neighborhoods of 0 of the form $\{V(\alpha) : \alpha < \lambda\}$, where $V(\alpha) = \{x \in V : v(x) \geq \alpha\}$.

Theorem 4 *Let κ be an (uncountable) regular cardinal and $\aleph \geq \kappa$ another cardinal. Then, there is a κ-tree with (at least) $\aleph$ κ-branches, if and only if, for every field F of cardinality $< \kappa$, there exists a valuated vector space V with the following properties:*

(1) $|V| = \kappa$,
(2) $V(\kappa) = 0$ *(the κ-topology is Hausdorff),*
(3) *for every* $i < \kappa$, $|V/V(i)| < \kappa$,
(4) *the completion $\hat{V}$ of V in the κ-topology is of cardinality (at least) $\aleph$.*

Proof. This has been proved already in the context of Kurepa trees in [Cutler and Dimitric, 1993] and we only sketch the proof with the necessary changes. Thus, given a κ-tree T with levels T_α, $\alpha < \kappa$ and the set of κ-branches B of cardinality $\aleph$ we can take its well-pruned subtree with same characteristics and use (FC2) to get a surjective inverse system of free R-modules $F_\alpha = \oplus_{i \in T_\alpha} R x_i^\alpha$ (the ring R may be taken to be any field of cardinality $\leq \kappa$); denote their product by $P = \prod_{\alpha<\kappa} F_\alpha$ and define a valuation $v : P \longrightarrow Ord \cup \{\infty\}$ ($b \in B$ is viewed as an element of P), by $v(b) = \min\{\alpha < \kappa : b(\alpha) \neq 0\}$. In this way (P, v) is a valuated vector space over R. In the second fundamental correspondence, every branch $b \in B$ of T corresponds to an element of P. For $b \in B$ and $\alpha < \kappa$, $b_\alpha \in P$ is defined via $b_\alpha(\beta) = b(\beta)$, if $\beta < \alpha$ and $b_\alpha(\beta) = 0$ if $\alpha \leq \beta < \kappa$. Now $V = \{b_\alpha : b \in B, \alpha < \kappa\}$ is a valuated vector space that has the specified properties. The κ-topology is Hausdorff because the height of the tree is κ. Cardinality of V is κ, because T is a κ-tree. Given a $b \in B$, $\{b_i\}_{i<\kappa}$ is a Cauchy net (in the κ-topology of V) that converges to b, hence the κ-completion $\hat{V}$ of V is of cardinality $|B|$. Conversely, given a valuated vector space V with the specified properties, after identifying the completion with the corresponding inverse limit of the appropriate inverse system, the desired tree is defined to be $T = \cup_{i<\kappa} V/V(i)$ and the order is defined as in the (FC1) above. If the cardinality of the completion is exactly $\aleph$, T will have exactly $\aleph$ many κ-branches. $\diamondsuit$

References

[1] Doyle Cutler and Radoslav Dimitrić, Valuated vector spaces, Kurepa's hypothesis and Abelian p-groups, *Rocky Mountain J. Math.*, **23**(1993), No.4, 1253-1266.
[2] Radoslav Dimitrić, A note on surjective inverse systems, *International J. Pure Appl. Math.*, **10**(2004), No.3, 349–356
[3] Kenneth Kunen, Set Theory, North-Holland, 1980

Author(s) Address(es):
DIBR, PO Box 87, Uniontown, PA 15401, USA,
E-mail address: dimitricr@member.ams.org