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ABSTRACT. The paper contains a simple construction of some types of commutative hyperstructures as hypergroupoids, weakly associative semi-hypergroupoids, hypergroups, quasi-hypergroups and weakly associative hyperrings from non-commutative hyperstructures. It is proved that the used construction induces reflectors on suitable categories of the mentioned hyperstructures.

There are some important reasons for introducing and investigation of so called H_ν structures, that is H_ν-group [20], H_ν-ring [29], and so on, which are defined from the well known classes of hyperstructures in a certain simple way. The idea consists in replacing some axioms, such as the associative law, the distributive law and others by the corresponding weak ones.

In particular, a H_ν*-semigroup* is a set H ($H \neq \emptyset$) equipped with a weak associative (we write WASS) hyperoperation $\star\colon H \times H \to \mathcal{P}^*(H)$, where for all $a, b, c \in H$, the following axiom is valid:

$$a \star (b \star c) \cap (a \star b) \star c \neq \emptyset.$$

A H_ν-semigroup is called a H_ν*-group* if moreover the reproduction axiom, i.e. $a \star H = H = H \star a$ is satisfied for any $a \in H$. It is to be noticed that H_ν-structures were introduced in [31] and investigated in the mentioned paper and in a series of others [5, 16, 17, 19-22, 25, 27-32]. In the classical group theory there is a well known construction called *abelization* of groups. From the point of view of the category theory, which allows one to make the notion of "universality", the mentioned construction yields an example of a reflector from the category of all groups and their homomorphisms into its subcategory of all commutative, i.e. abelian groups [13]. This contribution aims to present simple constructions of abelization of some types of hyperstructures, especially weak hyperstructures and quasi-hypergroups [3, 24, 25]. The bellow described constructions preserve weak associativity law, but

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Dedicated to Profesor Miroslav Novotný on the occasion of his 80$^{\text{th}}$ birthday.

not associativity law, which also shows a certain compatibility of used constructions with the concept of weakly associative hyperstructures.

Let $\mathbb{A}$ be a subcategory of $\mathbb{B}$ with embedding functor $E\colon \mathbb{A} \hookrightarrow \mathbb{B}$. If $\mathbb{E}$ is a class of $\mathbb{B}$-morphisms, then $\mathbb{A}$ is called $\mathbb{E}$-reflective in $\mathbb{B}$ provided that for each $\mathbb{B}$-object B there exists an $\mathbb{A}$-reflection (r_B, A_B) such that each $r_B \in \mathbb{E}$. By an $\mathbb{A}$-reflection (r_B, A_B) we mean — as usually — an E-universal map (r_B, A_B) for a $\mathbb{B}$-object B, i.e. $r_B\colon B \to A_B$ is a $\mathbb{B}$-morphism for $A_B \in \operatorname{Ob}\mathbb{A}$ and for each $A' \in \operatorname{Ob}\mathbb{A}$ and each morphism $f\colon B \to E(A')$ there exists a unique $\mathbb{A}$-morphism $\bar{f}\colon A_B \to A'$ such that the following triangle

$$\begin{array}{ccc} B & \xrightarrow{\;r_B\;} & A_B \\ & \searrow^{f} & \downarrow{\scriptstyle E(\bar{f})} \\ & & E(A') \end{array} \qquad \textbf{(D1)}$$

commutes. By this construction a functor $R\colon \mathbb{B} \to \mathbb{A}$ is defined, which is a left adjoint of $E\colon \mathbb{A} \to \mathbb{B}$, called a *reflector* for A. In case $\mathbb{E}$ is the class of all epimorphisms (monomorphisms) of $\mathbb{B}$ we say that $\mathbb{A}$ is epireflective (monoreflective) in $\mathbb{B}$. For the definition of a reflector a quadratic diagram can be also used (which is more convenient for our purposes) instead of the above triangle:

$$\begin{array}{ccc} B & \xrightarrow{\;f\;} & B' \\ {\scriptstyle r_B}\downarrow & & \downarrow{\scriptstyle r_{B'}} \\ A_B & \xrightarrow{\;\bar{f}\;} & A_{B'} \end{array} \qquad \textbf{(D2)}$$

Thus, for any $\mathbb{B}$-object B there exists a unique pair (r_B, A_B), $A_B \in \operatorname{Ob}\mathbb{A}$, $r_B\colon B \to A_B$ such that for any object $B' \in \operatorname{Ob}\mathbb{B}$ and any $\mathbb{B}$-morphism $f\colon B \to B'$ there exists a unique $\mathbb{A}$-morphism $\bar{f}\colon A_B \to A_{B'}$ making the diagram (D2) commutative. Then, by $R(B) = A_B$, $R(f) = \bar{f}$ a reflector $R\colon \mathbb{B} \to \mathbb{A}$ is defined.

Recall the other basic notions. *A hypergroupoid* (or a *multigroupoid)* is a pair $(M, \circ)$, where M is a nonempty set and $\circ\colon M \times M \to \mathcal{P}^*(M)$ is a binary hyperoperation called also a multioperation. ($\mathcal{P}^*(M)$ is the system of all nonempty subsets of M). *A semihypergroup* is an associative hypergroupoid, i.e. hypergroupoid satisfying the equality $(a \circ b) \circ c = a \circ (b \circ c)$ for every triad $a, b, c \in M$. *A quasi-hypergroup* is a hypergroupoid $(M, \circ)$ fulfilling the reproduction axiom, i.e. $a \circ M = M = M \circ a$ for any $a \in M$. *A hypergroup* is an associative hypergroupoid $(M, \circ)$, i.e. a semihypergroup, satisfying the reproduction axiom.

Let $(H, \circ)$ be a hypergroupoid; by Δ_H we mean the diagonal of the Cartesian product $H \times H$, i.e. $\Delta_H = \{[x, x]; x \in H\}$.

Let us define a mapping $D\colon H \to H \times H$ by $D(x) = [x, x]$ for all $x \in H$, i.e. $\Delta_H = D(H)$.

Remark. As a mapping, the operator D possesses usual properties, e.g. it is additive, i.e. for an arbitrary system $\{M_\gamma; \gamma \in \Gamma\} \subseteq \mathcal{P}^*(H)$, where $M_\gamma \subseteq H$ for each $\gamma \in \Gamma$, the equality $D(\bigcup_{\gamma\in\Gamma} M_\gamma) = \bigcup_{\gamma\in\Gamma} D(M_\gamma)$ holds. Also the inclusion $D(\bigcap_{\gamma\in\Gamma} M_\gamma) \subseteq \bigcap_{\gamma\in\Gamma} D(M_\gamma)$ is evident.

Let $(H,\cdot)$ be a hypergroupoid and $(\Delta_H,\star)$ be the hypergroupoid defined above. The basic properties of the used construction yield the following auxiliary assertions.

Lemma 1. *Let $(H,\cdot)$ be a hypergroupoid. Define a hyperoperation "$\star$" on the diagonal Δ_H as follows: $[x,x]\star[y,y] = D(x\cdot y \cup y\cdot x) = \{[u,u]; u \in x\cdot y\cup y\cdot x\}$ for any pair $[x,x],[y,y]\in\Delta_H$. Then the following assertions hold:*

1° *For any hypergroupoid $(H,\cdot)$ we have that $(\Delta_H,\star)$ is a commutative hypergroupoid.*

2° *If $(H,\cdot)$ is a weakly associative hypergroupoid, then the hypergroupoid $(\Delta_H,\star)$ is weakly associative, as well.*

3° *If $(H,\cdot)$ is a quasi-hypergroup, the hypergroupoid $(\Delta_H,\star)$ also satisfies the reproduction law, i.e. it is a quasi-hypergroup.*

4° *If $(H,\cdot)$ is associative, i.e., it is a semihypergroup, then the hypergroupoid $(\Delta_H,\star)$ is weakly associative (but not associative in general).*

Proof. The assertion 1° follows immediately from the above definition of the hyperoperation "$\star$".

2° Suppose, $[x,x],[y,y],[z,z]\in\Delta_H$. Then

$$\begin{aligned}
([x,x]\star[y,y])\star[z,z] &= D(x\cdot y\cup y\cdot x)\star[z,z] = \big(D(x\cdot y)\cup D(y\cdot x)\big)\star[z,z]\\
&= \big(D(x\cdot y)\star[z,z]\big)\cup\big(D(y\cdot x)\star[z,z]\big)\\
&= \Big(\bigcup_{u\in x\cdot y}[u,u]\star[z,z]\Big)\cup\Big(\bigcup_{v\in y\cdot x}[v,v]\star[z,z]\Big)\\
&= \bigcup_{u\in x\cdot y} D(u\cdot z\cup z\cdot u)\cup\bigcup_{v\in y\cdot x} D(v\cdot z\cup z\cdot v)\\
&= \bigcup_{u\in x\cdot y} D(u\cdot z)\cup\bigcup_{u\in x\cdot y} D(z\cdot u)\cup\bigcup_{v\in y\cdot x} D(v\cdot z)\cup\bigcup_{v\in y\cdot x} D(z\cdot v)\\
&= D\Big(\bigcup_{u\in x\cdot y} u\cdot z\Big)\cup D\Big(\bigcup_{u\in x\cdot y} z\cdot u\Big)\cup D\Big(\bigcup_{v\in y\cdot x} v\cdot z\Big)\cup D\Big(\bigcup_{v\in y\cdot x} z\cdot v\Big)\\
&= D\big((x\cdot y)\cdot z\big)\cup D\big(z\cdot(x\cdot y)\big)\cup D\big((y\cdot x)\cdot z\big)\cup D\big(z\cdot(y\cdot x)\big)\\
&= D\big((x\cdot y)\cdot z\big)\cup D\big(z\cdot(y\cdot x)\cup z\cdot(x\cdot y)\cup(y\cdot x)\cdot z\big).
\end{aligned}$$

On the other hand

$$\begin{aligned}
[x,x]\star([y,y]\star[z,z]) &= ([z,z]\star[y,y])\star[x,x]\\
&= D\big((z\cdot y)\cdot x\cup x\cdot(z\cdot y)\cup(y\cdot z)\cdot x\cup x\cdot(y\cdot z)\big)\\
&= D\big(x\cdot(y\cdot z)\big)\cup D\big((z\cdot y)\cdot x\cup x\cdot(z\cdot y)\cup(y\cdot z)\cdot x\big).
\end{aligned}$$

As by the assumption $(x\cdot y)\cdot z\cap x\cdot(y\cdot z)\neq\emptyset$ we get $D\big((x\cdot y)\cdot z\big)\cap D\big(x\cdot(y\cdot z)\big)\neq\emptyset$. Thus $\Big(\big([x,x]\star[y,y]\big)\star[z,z]\Big)\cap\Big([x,x]\star\big([y,y]\star[z,z]\big)\Big)\neq\emptyset$.

3° Let $x \in H$ be an arbitrary element. Then $x \cdot H = H = H \cdot x$ and we have

$$\begin{aligned}[x,x] \star \Delta_H &= \bigcup_{y\in H} \big([x,x] \star [y,y]\big) = \bigcup_{y\in H} D(x\cdot y \cup y \cdot x) \\ &= \bigcup_{y\in H} D(x\cdot y) \cup \bigcup_{y\in H} D(y\cdot x) = D\Big(\bigcup_{y\in H} x\cdot y\Big) \cup D\Big(\bigcup_{y\in H} y\cdot x\Big) \\ &= D(x\cdot H) \cup D(H\cdot x) = D(H) \cup D(H) = D(H) = \Delta_H.\end{aligned}$$

4° Since a semihypergroup is also weakly associative, the assertion 4° follows from 2°. □

Example 1. Let $(\mathbb{R}, \leq)$ be the naturally ordered set of all real numbers and $H = \mathrm{End}(\mathbb{R}, \leq)$ the monoid (with a binary operation of composition "$\circ$" of functions) of all endomorphisms preserving ordering of the chain $(\mathbb{R}, \leq)$, i.e., the monoid of all non-decreasing functions of one real variable. If we define $f \preceq g$ for any pair $f, g \in H$ such that $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, it is easy to show that $(H, \circ, \preceq)$ is an ordered monoid. Let "$\bullet$" be a hyperoperation defined in the following way:

$$f \bullet g := \{h \in H;\ g \circ f \preceq h\}, \text{ i.e., } h \in f \bullet g \Leftrightarrow g[f(x)] \leq h(x)$$

for all $x \in \mathbb{R}$. Then $(H, \bullet)$ is a semihypergroup.

As above $D(H) = \{[f,f];\ f \in H\}$. Let "$\star$" be a hyperoperation on $D(H)$ given by the rule:

$$[f,f] \star [g,g] := D(f \bullet g \cup g \bullet f) = \{[h,h];\ h \in f \bullet g \cup g \bullet f\}.$$

Notice that $f \bullet g \bullet h = \{k(x);\ h(g[f(x)]) \leq k(x)\}$ (see [8]). Using a concrete triad of functions we will show that the hyperstructure $(D(H), \star)$ satisfying the weak associativity law is not associative.

From the proof of Lemma $1, 2^o$ we have:

$$\big([f,f] \star [g,g]\big) \star [h,h] = D(f \bullet g \bullet h \cup h \bullet f \bullet g \cup g \bullet f \bullet h \cup h \bullet g \bullet f) = D(M_1),$$

$$[f,f] \star \big([g,g] \star [h,h]\big) = D(f \bullet g \bullet h \cup f \bullet h \bullet g \cup g \bullet h \bullet f \cup h \bullet g \bullet f) = D(M_2).$$

We will show that $D(M_1) \neq D(M_2)$, in general. Choosing, e.g. $f(x) = x + 1$, $g(x) = x^3$, $h(x) = 2^x$, we obtain

$$\begin{aligned} f \bullet g \bullet h &= h(g[f(x)]) = 2^{(x+1)^3}, & f \bullet g \bullet h &= h(g[f(x)]) = 2^{(x+1)^3}, \\ h \bullet f \bullet g &= g(f[h(x)]) = (2^x + 1)^3, & f \bullet h \bullet g &= g(h[f(x)]) = 2^{3x+3}, \\ g \bullet f \bullet h &= h(f[g(x)]) = 2^{x^3+1}, & g \bullet h \bullet f &= f(h[g(x)]) = 2^{x^3} + 1, \\ h \bullet g \bullet f &= f(g[h(x)]) = 2^{3x} + 1, & h \bullet g \bullet f &= f(g[h(x)]) = 2^{3x} + 1. \end{aligned}$$

Then M_1 and M_2 have the following form:

$$M_1 = \bigcup_{k=1}^{4} \{\varphi\colon R \to R; \varphi_k(x) \leq \varphi(x) \text{ for any } x \in R\}, \text{ where}$$
$$\varphi_1(x) = 2^{(x+1)^3},\ \varphi_2(x) = (2^x+1)^3,\ \varphi_3(x) = 2^{x^3+1},\ \varphi_4(x) = 2^{3x}+1,$$
$$M_2 = \bigcup_{k=1}^{4} \{\psi\colon R \to R; \psi_k(x) \leq \psi(x) \text{ for any } x \in R\}, \text{ where}$$
$$\psi_1(x) = 2^{(x+1)^3},\ \psi_2(x) = 2^{3x+3},\ \psi_3(x) = 2^{x^3}+1,\ \psi_4(x) = 2^{3x}+1.$$

It is easy to see that e.g. $\varphi_3 \in M_1$, $\psi_2 \in M_2$, $\varphi_3 \notin M_2$, $\psi_2 \notin M_1$, hence $M_1 \neq M_2$ and consequently $D(M_1) \neq D(M_2)$.

On the other hand, e.g. $\varphi_1 \in M_1 \cap M_2$, $\psi_1 \in M_1 \cap M_2$ (since $\varphi_1 = \psi_1$), thus $\emptyset \neq \{\varphi\colon R \to R;\ \varphi_1 \leq \varphi\} \subset M_1 \cap M_2$, $\{\psi\colon R \to R;\ \psi_1 \leq \psi\} \subset M_1 \cap M_2$, which implies $D(M_1) \cap D(M_2) \neq \emptyset$. This follows, of course, from Lemma 1, 4°.

Let $(H, \cdot)$ be a quasi-hypergroup. In connection with the concept of a reflector in the category theory it will be useful to write r_H instead of $D\colon H \to H \times H$ because r_H will be considered as a morphism in a suitable category. That means $r_H\colon (H, \cdot) \to (D(H), \star)$ is a homomorphism of quasi-hypergroups (Lemma 1, 2°) because for all pairs $x, y \in H$ we have $r_H(x.y) = D(x.y) \subset D(x.y) \cup D(y.x) = D(x.y \cup y.x) = [x, x] \star [y, y] = r_H(x) \star r_H(y)$.

Let quasi-hypergroups $(H_1, \cdot_1)$, $(H_2, \cdot_2)$ be given. Suppose $f\colon (H_1, \cdot_1) \to (H_2, \cdot_2)$ is a homomorphism. For an arbitrary $[x, x] \in D(H_1)$ we define

$$\bar{f}([x, x]) = [f(x), f(x)] \in D(H_2).$$

Consider the following diagram:

$$\begin{array}{ccc} (H_1, \cdot_1) & \xrightarrow{\ f\ } & (H_2, \cdot_2) \\ {\scriptstyle r_{H_1}}\downarrow & & \downarrow{\scriptstyle r_{H_2}} \\ (D(H_1), \star_1) & \xrightarrow{\ \bar{f}\ } & (D(H_2), \star_2) \end{array} \qquad \textbf{(D3)}$$

Lemma 2. *The following assertions hold:*

1° *The mapping* $\bar{f}\colon (D(H_1), \star_1) \to (D(H_2), \star_2)$ *is a homomorphism.*

2° *The diagram* (D3) *is commutative for any homomorphism* $f\colon (H_1, \cdot_1) \to (H_2, \cdot_2)$.

3° *The homomorphism* $\bar{f}$ *completes the diagram* (D3) *for any homomorphism* $f\colon (H_1, \cdot_1) \to (H_2, \cdot_2)$ *uniquely.*

4° *The homomorphism* $r_H\colon (H, \cdot) \to (\Delta_H, \star)$ *is a bimorphism, i.e., both a mono- and an epimorphism, for any quasi-hypergroup* $(H, \cdot)$.

Proof.

1° Suppose $[x,x],[y,y] \in D(H_1)$ are arbitrary elements. Then we have

$$\bar{f}\big([x,x] \star_1 [y,y]\big) = \bar{f}\big(D(x._1y \cup y._1x)\big)$$
$$= \bar{f}(\{[u,u]; u \in x._1y \cup y._1x\}) = \{\bar{f}([u,u]; u \in x._1y \cup y._1x)\}$$
$$= \{[f(u),f(u)]; u \in x._1y \cup y._1x\}.$$

Since for any $u \in x._1y \cup y._1x$ we have

$$f(u) \in f(x._1y \cup y._1x) = f(x._1y) \cup f(y._1x) \subset (f(x) \cdot_2 f(y)) \cup (f(y) \cdot_2 f(x)),$$

consequently

$$\{[f(u),f(u)]; u \in x._1y \cup y._1x\} \subset \{[v,v]; v \in (f(x) \cdot_2 f(y)) \cup (f(y) \cdot_2 f(x))\}$$
$$= [f(x),f(x)] \star_2 [f(y),f(y)] = \bar{f}([x,x]) \star_2 \bar{f}([y,y]),$$

therefore $\bar{f}\colon (D(H_1),\star_1) \to (D(H_2),\star_2)$ is the homomorphism.

2° Suppose $x \in H_1$ is an arbitrary element. Then

$$(r_{H_2} \circ f)(x) = r_{H_2}(f(x)) = [f(x),f(x)] = \bar{f}([x,x]) = \bar{f}(r_{H_1}(x)) = \big(\bar{f} \circ r_{H_1}\big)(x).$$

Thus $r_{H_2} \circ f = \bar{f} \circ r_{H_1}$. Consequently the diagram (D3) commutes.

3° Suppose $g\colon (D(H_1),\star_1) \to (D(H_2),\star_2)$ is a homomorphism which creates the diagram (D3) with $f\colon (H_1,\cdot_1) \to (H_2,\cdot_2)$ commutative. Then for arbitrary $[x_0,x_0] \in D(H_1)$ we have

$$g([x_0,x_0]) = \big(g \circ \operatorname*{id}_{D(H_1)}\big)([x_0,x_0]) = \big(g \circ r_{H_1} \circ r_{H_1}^{-1}\big)([x_0,x_0])$$
$$= \big(g \circ r_{H_1}\big)(r_{H_1}^{-1}[x_0,x_0]) = \big(r_{H_2} \circ f\big)(x_0) = r_{H_2}(f(x_0)) = \bar{f}([x_0,x_0]).$$

Hence, for any homomorphism $f\colon (H_1,\cdot_1) \to (H_2,\cdot_2)$ there exists a unique homomorphism $\bar{f}\colon (D(H_1),\star_1) \to (D(H_2),\star_2)$ making the diagram (D3) commutative.

4° Let $(A,\cdot)$ be an arbitrary quasi-hypergroup from $\mathrm{Ob}(\mathbb{QHG})$ and $\varphi,\psi\colon (A,\cdot) \to (H,\cdot)$ be homomorphisms such that $r_H \circ \varphi = r_H \circ \psi$. Let $a \in A$ be an arbitrary element. Suppose $\varphi(a) \neq \psi(a)$. Then

$$(r_H \circ \varphi)(a) = r_H(\varphi(a)) = [\varphi(a),\varphi(a)] \neq [\psi(a),\psi(a))] = r_H(\psi(a)] = (r_H \circ \psi)(a).$$

Thus the morphism r_H is a monomorphism.

Now we will show that it is an epimorphism. Let $x \in H$, $[x,x] \in \Delta_H$ be an arbitrary element and φ,ψ be a homomorphisms such that $\varphi \circ r_H = \psi \circ r_H$. Suppose, that $\varphi([x,x]) \neq \psi([x,x])$. Then

$$(\varphi \circ r_H)(x) = \varphi(r_H(x)) \neq \psi(r_H(x)) = (\psi \circ r_H)(x),$$

hence $\varphi \circ r_H \neq \psi \circ r_H$. Thus $\varphi \circ r_H = \psi \circ r_H$ implies $\varphi = \psi$. Therefore r_H is an epimorphism and simultaneously a monomorphism, thus it is a bimorphism. □

Let $\mathbb{QHG}$ be the category of all quasi-hypergroups and their homomorphisms, $\mathbb{AQHG}$ be its full subcategory of all commutative (i.e. abelian) quasi-hypergroups. Define a functor $F\colon \mathbb{QHG} \to \mathbb{AQHG}$ by $F((H,\cdot)) = (D(H),\star) = (\Delta_H,\star)$ for any quasi-hypergroup $(H,\cdot) \in \mathrm{Ob}(\mathbb{QHG})$, $F(f) = \bar{f}\colon F((H_1,\cdot_1)) \to F((H_2,\cdot_2))$ for any pair of quasi-hypergroups and any homomorphism $f\colon (H_1,\cdot_1) \to (H_2,\cdot_2)$. Similarly, let us denote by $\mathbb{H}_\nu\mathbb{G}$ the category of all $\mathbb{H}_\nu$-groups and their homomorphisms, by $\mathbb{AH}_\nu\mathbb{G}$ its subcategory of all commutative $\mathbb{H}_\nu$-groups. In fact, $\mathbb{H}_\nu\mathbb{G}$ is a full subcategory of the category $\mathbb{QHG}$. Thus, define a functor $G\colon \mathbb{H}_\nu\mathbb{G} \to \mathbb{AH}_\nu\mathbb{G}$ as a restriction of the functor F, i.e. $G(H,\cdot) = F(H,\cdot)$ for any $(H,\cdot) \in \mathrm{Ob}\ (\mathbb{H}_\nu\mathbb{G})$ and similarly for morphisms.

By the above considerations (concentrated in Lemma 1, Lemma 2) we have proved the next

Theorem 1. *The following assertions hold:*

1° *Let $\mathbb{QHG}$ be the category of all quasi-hypergroups and their homomorphisms, $\mathbb{AQHG}$ be its full subcategory of all commutative (i.e. abelian) quasi-hypergroups. Then the functor $F\colon \mathbb{QHG} \to \mathbb{AQHG}$ is a reflector; more precisely the pair $(r_H, (\Delta_H, \star))$ is $\mathbb{AQHG}$-reflection for $(H,\cdot) \in \mathrm{Ob}(\mathbb{QHG})$, hence the category $\mathbb{AQHG}$ is a bireflective (i.e. mono- and epireflective) full subcategory of the category $\mathbb{QHG}$.*

2° *Let $\mathbb{H}_\nu\mathbb{G}$ be the full subcategory of all $\mathbb{H}_\nu$-groups of the category $\mathbb{QHG}$, $\mathbb{AH}_\nu\mathbb{G}$ its full subcategory of all commutative $\mathbb{H}_\nu$-groups. Then the functor $G\colon \mathbb{H}_\nu\mathbb{G} \to \mathbb{AH}_\nu\mathbb{G}$ is a reflector and $\mathbb{AH}_\nu\mathbb{G}$ is a bireflective full subcategory of the category $\mathbb{H}_\nu\mathbb{G}$.*

Example 2. Let S be a nonempty set, $\mathcal{P}^*(S)$ be the system of all its nonempty subsets, i.e., $\mathcal{P}^*(S)\cup\{\emptyset\}$ is the power set of the set S. For any nonempty subsystem $\mathcal{C} \subset \mathcal{P}^*(S)$ (possibly a covering of S, which means $X \in \mathcal{C}$ implies $\emptyset \neq X \subset S$ and $\bigcup \mathcal{C} = S$) we denote by $\mathrm{Cst}(M,\mathcal{C})$ the combinatorial star of a nonempty set $M \subset S$, i.e., $\mathrm{Cst}(M,\mathcal{C}) = \{X \in \mathcal{C}; X \cap M \neq \emptyset\}$, cf. [8]. If we define

$$A \cdot B = \mathrm{Cst}(A \setminus B, \mathcal{C}) \cup \{A, B\} \quad \text{for any pair of sets } A, B \in \mathcal{P}^*(S),$$

then it is easy to verify that $(\mathcal{P}^*(S),\cdot)$ is a non-commutative H_ν-group. Indeed, for an arbitrary triad of nonempty subsets $X, Y, Z \subset S$ we have

$$\begin{aligned}(X\cdot Y)\cdot Z &= \big(\mathrm{Cst}(X\setminus Y,\mathcal{C})\cup\{X,Y\}\big)\cdot Z \cup \{X,Y\}\cdot Z\\ &= \{V\in\mathcal{C}; V\cap(X\setminus Y)\neq\emptyset\}\cdot Z\cup\mathrm{Cst}(X\setminus Z,\mathcal{C})\cup\mathrm{Cst}(Y\setminus Z,\mathcal{C})\cup\{X,Y,Z\}\\ &= \bigcup_{\substack{U\in\mathcal{C}\\ U\cap(X\setminus Y)\neq\emptyset}}\big(\mathrm{Cst}(U\setminus Z,\mathcal{C})\cup\{U,Z\}\big)\cup\mathrm{Cst}(X\setminus Z,\mathcal{C})\cup\mathrm{Cst}(Y\setminus Z,\mathcal{C})\cup\{X,Y,Z\}.\end{aligned}$$

On the other hand

$$\begin{aligned}X\cdot(Y\cdot Z) &= X\cdot\mathrm{Cst}(Y\setminus Z,\mathcal{C})\cup X\cdot\{Y,Z\}\\ &= X\cdot\{V\in\mathcal{C}; V\cap(Y\setminus Z)\neq\emptyset\}\cup\mathrm{Cst}(X\setminus Y,\mathcal{C})\cup\mathrm{Cst}(X\setminus Z,\mathcal{C})\cup\{X,Y,Z\}\\ &= \bigcup_{\substack{V\in\mathcal{C}\\ V\cap(Y\setminus Z)\neq\emptyset}}\big(\mathrm{Cst}(X\setminus V,\mathcal{C})\cup\{X,V\}\big)\cup\mathrm{Cst}(X\setminus Y,\mathcal{C})\cup\\ &\qquad\cup\,\mathrm{Cst}(X\setminus Z,\mathcal{C})\cup\{X,Y,Z\}.\end{aligned}$$

Now, it is evident that $\emptyset \neq \mathrm{Cst}(X\setminus Z,\mathcal{C})\cup\{X,Y,Z\} \subset \big((X\cdot Y)\cdot Z\big)\cap\big(X\cdot(Y\cdot Z)\big)$ and it is easy to see that the reproduction axiom is satisfied.

Applying the abelization to the hyperoperation "$\cdot$" we get a new commutative hyperoperation "$\bullet$". With respect to an evident formula

$$\mathrm{Cst}\Big(\bigcup_{\gamma\in\Gamma} M_\gamma, \mathcal{C}\Big) = \bigcup_{\gamma\in\Gamma} \mathrm{Cst}(M_\gamma, \mathcal{C})$$

for any family $\{M_\gamma; \gamma \in \Gamma\} \subset \mathcal{P}^*(S)$ we have

$$\begin{aligned} A \bullet B = A \cdot B \cup B \cdot A &= \mathrm{Cst}(A \setminus B, \mathcal{C}) \cup \{A, B\} \cup \mathrm{Cst}(B \setminus A, \mathcal{C}) \cup \{B, A\} \\ &= \mathrm{Cst}(A \triangle B, \mathcal{C}) \cup \{A, B\}, \end{aligned}$$

where $\triangle$ means the symmetrical difference of set.

Similarly as in the proof of Lemma 1 it is easy to verify that $(\mathcal{P}^*(S), \bullet)$ is the commutative H_ν-group.

Now applying Theorem 1, 2° to $(\mathcal{P}^*(S), \cdot)$ we obtain the commutative H_ν-group $G(\mathcal{P}^*(S), \cdot) = \big(D(\mathcal{P}^*(S), \star)\big)$, where G is a functor from the mentioned theorem. As in the preceeding we have

$$\begin{aligned} [A, A] \star [B, B] &= \big\{[Z, Z]; Z \in \mathrm{Cst}(A \triangle B, \mathcal{C}) \cup \{A, B\}\big\} = \\ &= \big\{[Z, Z]; Z \in \mathrm{Cst}\big((A \setminus B) \cup (B \setminus A), \mathcal{C}\big) \cup \{A, B\}\big\} \\ &= \big\{[Z, Z]; Z \in \mathrm{Cst}(A \setminus B, \mathcal{C}) \cup \mathrm{Cst}(B \setminus A, \mathcal{C}) \cup \{A, B\}\big\} \\ &= \big\{[Z, Z]; Z \in A \cdot B \cup B \cdot A\big\}. \end{aligned}$$

Evidently $G(\mathcal{P}^*(S), \cdot) = \big(D(\mathcal{P}^*(S), \star)\big) \cong \big(\mathcal{P}^*(S), \bullet\big)$. Moreover, it can be easily seen that the hyperstructure $G(\mathcal{P}^*(S), \cdot)$ is not associative, in general.

From [31, 32] it follows that non-associative hyperstructures as quasi-hypergroups play an essential role in geometry. On the other hand certain quasi-hypergroups can be obtained from quite fundamental structures as transformation groups of bijective linear real functions of one variable. In [3] a certain construction of non-commutative quasi-hypergroups is described based on a certain decomposition of the structure mentioned above.

As an application of the previous results we obtain a theorem for binary hyperstructures with two binary hyperoperations — called H_ν-rings — which is analogous to the above one. In [6], [29] H_ν-rings are defined and investigated. Recall that H_ν-rings are triads $(R, +, \cdot)$, where R is a set and $+ \colon R \times R \to \mathcal{R}$, $\cdot \colon R \times R \to \mathcal{R}$ are weakly associative (WASS) hyperoperations such that "$+$" satisfies the reproduction axiom (i.e. $(R, +)$ is H_ν-group, $(R, \cdot)$ is a H_ν-semigroup) and the hyperoperation "$\cdot$" is weakly distributive with respect to the hyperoperation "$+$", which means that

$$x.(y + z) \cap (x.y + x.z) \neq \emptyset,$$
$$(x + y) \cdot z \cap (x \cdot z + y \cdot z) \neq \emptyset$$

for all elements $x, y, z \in R$, see [32].

Recall that H_ν-ring homomorphisms or weak homomorphisms of H_ν-ring $(R, +, \cdot)$ into another one $(S, +, \cdot)$ are mappings $f \colon R \to S$ such that $f(x + y) \cap (f(x) + f(y)) \neq \emptyset$, $f(x \cdot y) \cap (f(x) \cdot f(y)) \neq \emptyset$ for any pair $x, y \in R$. However, for our purposes we will consider so called inclusion homomorphisms between H_ν-rings as the basic morphisms for this objects. Let us remind that a mapping of a

H_ν-ring $(R,+,\cdot)$ into another one $(S,+,\cdot)$ is called an inclusion homomorphism if $f(x)+f(y)\subset f(x+y)$, $f(x)\cdot f(y)\subset f(x\cdot y)$ for all elements $x,y\in R$.

Remark. In the following consideration we will apply the following useful identity valid for subsets of any hypergroupoid. Let $(R,+)$ be a hypergroupoid, $R\neq\emptyset$, X,Y,U,V nonempty subsets of the set R. Then

$$(X\cup Y)+(U\cup V)=(X+U)\cup(X+V)\cup(Y+U)\cup(Y+V).$$

Further denote the category of all H_ν-rings by $\mathbb{H}_\nu\mathbb{R}$ and their inclusion homomorphisms, by $\mathbb{AH}_\nu\mathbb{R}$ its full subcategory of all commutative H_ν-rings. Thus $(R,+,\cdot)\in \operatorname{Ob}\mathbb{AH}_\nu\mathbb{R}$ whenever $(R,+,\cdot)$ is a H_ν-ring such that $x\cdot y=y\cdot x$ for any pair $x,y\in R$. Similarly as above we define for an arbitrary H_ν-ring $(R,+,\cdot)$ the hyperoperations $\oplus$, $\odot$ on the diagonal $D(R)=\Delta_R$ by

$$[x,x]\oplus[y,y]=\{[u,u];u\in(x+y)\cup(y+x)\},$$
$$[x,x]\odot[y,y]=\{[v,v];v\in(x\cdot y)\cup(y\cdot x)\}$$

for all pairs $x,y\in R$. Then we have

Lemma 3. *Let $(R,+,\cdot)$ be a H_ν-ring. Then $(D(R),\oplus,\odot)$ is a commutative H_ν-ring.*

Proof. Let $(R,+,\cdot)$ be a H_ν-ring. According to Lemma 1 we obtain that $(D(R),\oplus)$ is a commutative weakly associative hypergroupoid satisfying the reproduction axiom, thus it is a commutative H_ν-group. Similarly $(D(R),\odot)$ is a commutative H_ν-semigroup. Thus it remains to prove that

$$[x,x]\odot([y,y]\oplus[z,z])\cap([x,x]\odot[y,y])\oplus([x,x]\odot[z,z])\neq\emptyset$$

for arbitrary elements $x,y,z\in R$.
Indeed, we have $[y,y]\oplus[z,z]=\{[u,u];u\in(y+z)\cup(z+y)\}$ and

$$\begin{aligned}
&[x,x]\odot([y,y]\oplus[z,z])=\\
&=\bigcup_{u\in(y+z)\cup(z+y)}[x,x]\odot[u,u]\\
&=\Big(\bigcup_{u\in(y+z)}[x,x]\odot[u,u]\Big)\cup\Big(\bigcup_{u\in(z+y)}[x,x]\odot[u,u]\Big)\\
&=\Big(\bigcup_{u\in(y+z)}\{[v,v];v\in x\cdot u\cup u\cdot x\}\Big)\cup\Big(\bigcup_{u\in(z+y)}\{[v,v];v\in x\cdot u\cup u\cdot x\}\Big)\\
&=\bigcup_{u\in(y+z)}\{[v,v];v\in x\cdot u\}\cup\bigcup_{u\in(y+z)}\{[v,v];v\in u\cdot x\}\cup\bigcup_{u\in(z+y)}\{[v,v];v\in x\cdot u\cup u\cdot x\}\\
&=\big\{[v,v];v\in x\cdot(y+x)\big\}\cup M(x,y,z)
\end{aligned}$$

where $M(x,y,z)=\bigcup_{u\in(y+z)}\{[v,v];v\in u\cdot x\}\cup\bigcup_{u\in(z+y)}\{[v,v];v\in x\cdot u\cup u\cdot x\}$.
On the other hand

$$[x,x]\odot[y,y]=\{[v,v];v\in x\cdot y\cup y\cdot x\}=\{[v,v];v\in x\cdot y\}\cup\{[v,v];v\in y\cdot x\},$$
$$[x,x]\odot[z,z]=\{[v,v];v\in x\cdot z\}\cup\{[v,v];v\in z\cdot x\}$$

and then

$$\begin{aligned}
&([x,x]\odot[y,y])\oplus([x,x]\odot[z,z]) = \\
&= \{[v,v]; v\in x\cdot y\}\cup\{[v,v]; v\in y\cdot x\}\big)\oplus\big(\{[v,v]; v\in x\cdot z\}\cup\{[v,v]; v\in z\cdot x\}\big) \\
&= \big(\{[v,v]; v\in x\cdot y\}\oplus\{[v,v]; v\in x\cdot z\}\big)\cup\big(\{[v,v], v\in x\cdot y\}\oplus\{[v,v]; v\in z\cdot x\}\big) \\
&\quad \cup\big(\{[v,v]; v\in y\cdot x\}\oplus\{[v,v]; v\in x\cdot z\}\big)\cup\big(\{[v,v], v\in y\cdot x\}\oplus\{[v,v]; v\in z\cdot x\}\big) \\
&= \Big(\bigcup_{\substack{v\in x\cdot y\\ u\in x\cdot z}}[v,v]\oplus[u,u]\Big)\cup\Big(\bigcup_{\substack{v\in x\cdot y\\ u\in z\cdot x}}[v,v]\oplus[u,u]\Big)\cup \\
&\quad \cup\Big(\bigcup_{\substack{v\in y\cdot x\\ u\in x\cdot z}}[v,v]\oplus[u,u]\Big)\cup\Big(\bigcup_{\substack{v\in y\cdot x\\ u\in z\cdot x}}[v,v]\oplus[u,u]\Big) \\
&= \bigcup_{\substack{v\in x\cdot y\\ u\in x\cdot z}}\{[t,t]; t\in(v+u)\cup(u+v)\}\cup K(x,y,z),
\end{aligned}$$

where

$$K(x,y,z) = \Big(\bigcup_{\substack{v\in x\cdot y\\ u\in z\cdot x}}[v,v]\oplus[u,u]\Big)\cup\Big(\bigcup_{\substack{v\in y\cdot x\\ u\in x\cdot z}}[v,v]\oplus[u,u]\Big)\cup\Big(\bigcup_{\substack{v\in y\cdot x\\ u\in z\cdot x}}[v,v]\oplus[u,u]\Big).$$

Now, we have

$$\begin{aligned}
&([x,x]\odot[y,y])\oplus([x,x]\odot[z,z]) = \\
&= \bigcup_{\substack{v\in x\cdot y\\ u\in x\cdot z}}\{[t,t]; t\in u+v\}\cup\bigcup_{\substack{v\in x\cdot y\\ u\in x\cdot z}}\{[t,t]; t\in u+v\}\cup K(x,y,z) \\
&= \{[t,t]; t\in x\cdot y+x\cdot z\}\cup\{[t,t]; t\in x\cdot z+x\cdot y\}\cup K(x,y,z).
\end{aligned}$$

As by the supposition $(x\cdot y+x\cdot z)\cap x\cdot(y+z)\neq\emptyset$, we have $[t_o,t_o]\in\{[v,v]; v\in x\cdot(y+z)\}$ for some $t_o\in x\cdot y+x\cdot z$, thus

$$\{[v,v]; v\in x\cdot(y+z)\}\cap\{[t,t]; t\in x\cdot y+x\cdot z\}\neq\emptyset,$$

consequently the sets $[x,x]\odot([y,y]\oplus[z,z])$, $([x,x]\odot[y,y])\oplus([x,x]\odot[z,z])$ have a nonempty intersection. □

Remark. The above proof implies that either of the laws of the weak distributivity for $(R,+,\cdot)$ (right or left) ensures the weak distributivity of $(D(R),\oplus,\odot)$.

From the above considerations it follows immediately:

Lemma 4. *Let $(R,+,\cdot)$ be a H_ν-ring and $r_R(x)=[x,x]\in D(R)$ for any $x\in R$. Then the mapping $r_R\colon (R,+,\cdot)\to(D(R),\oplus,\odot)$ is an inclusion homomorphism of H_ν-rings.*

In order to prove a theorem analogous to Theorem 1 we show that the following lemma holds.

Lemma 5. *For any pair of H_ν-rings $(R,+,\cdot)$, $(S,+,\cdot)$ and for any inclusion H_ν-ring homomorphism $f\colon (R,+,\cdot)\to(S,+,\cdot)$ there exists exactly one inclusion H_ν-ring homomorphism $\bar f\colon (D(R),\oplus,\odot)\to(D(S),\oplus,\odot)$ such that the diagram*

$$\begin{array}{ccc} (R,+,\cdot) & \xrightarrow{f} & (S,+,\cdot) \\ r_R \big\downarrow & & \big\downarrow r_S \\ (D(R),\oplus,\odot) & \xrightarrow{\bar{f}} & (D(S),\oplus,\odot) \end{array} \qquad \textbf{(D4)}$$

is commutative.

Proof. Consider an arbitrary inclusion ring homomorphism $f\colon (R,+,\cdot) \to (S,+,\cdot)$ and define $\bar{f}\colon (D(R) \to (D(S))$ as the restriction of the mapping $f \times f\colon R \times R \to S \times S$ onto $D(R) \subset R \times R$, i.e. $\bar{f} = (f \times f)|D(P)$, hence $\bar{f}([x,x]) = [f(x), f(x)]$ for any $x \in R$.
Now we have

$$\begin{aligned} \bar{f}([x,x] \oplus [y,y]) &= \bar{f}(\{[u,u]; u \in (x+y) \cup (y+x)\} \\ &= \{[f(u), f(u)], u \in (x+y) \cup (y+x)\} \\ = \{[v,v], v \in f(x+y) \cup f(y+x)\} &\subset \{[v,v], v \in (f(x)+f(y)) \cup ((f(y)+f(x))\} \\ &= [f(x)+f(y)] \oplus [f(y)+f(x)] = \bar{f}([x,x]) \oplus \bar{f}([y,y]) \end{aligned}$$

for any pair of elements $x, y \in R$ and similarly $\bar{f}([x,x] \odot [y,y]) \subset f([x,x]) \odot f([y,y])$, which we obtain immediately from the above calculation changing the operation "$\oplus$" by the operation "$\odot$". Moreover, we show that the diagram (D4) commutes.

Let us suppose $f\colon (R,+,\cdot) \to (S,+,\cdot)$ is an arbitrary H_ν-ring homomorphism. Then evidently $\bar{f}\colon (D(R),\oplus,\odot) \to (D(S),\oplus,\odot)$ is a H_ν-ring homomorphism as well. For an arbitrary $x \in R$ we have

$$\begin{aligned} (r_f \circ f)(x) &= r_f(f(x)) = [f(x), f(x)] = (f \times f)(x,x) = \bar{f}([x,x]) \\ &= \bar{f}(r_R(x)) = (\bar{f} \circ r_R)(x), \end{aligned}$$

i.e.

$$r_f \circ f = \bar{f} \circ r_R. \tag{1}$$

Now let $g\colon (D(R),\oplus,\odot) \to (D(S),\oplus,\odot)$ be a H_ν-ring homomorphism such that

$$r_f \circ f = g \circ r_R. \tag{2}$$

Since $r_R\colon R \to D(R), r_S\colon S \to D(S)$ are bijections there is well defined $r_R^{-1}\colon D(R) \to R$, $r_S^{-1}\colon D(S) \to S$. We get then that the equalities (1), (2) imply

$$\bar{f} = \bar{f} \circ \operatorname*{id}_{D(R)} = \bar{f} \circ r_R \circ r_R^{-1} = r_S \circ f \circ r_R^{-1} = g \circ r_R \circ r_R^{-1} = g \circ \operatorname*{id}_{D(R)} = g.$$

The proof is complete. □

From the above results we obtain immediately the following theorem.

Theorem 2. *Let $\mathbb{H}_\nu\mathbb{R}$ be the category of all H_ν-rings and their inclusion homomorphisms, $\mathbb{AH}_\nu\mathbb{R}$ be its full subcategory of all commutative H_ν-rings. Then the functor $\Phi\colon \mathbb{H}_\nu\mathbb{R} \to \mathbb{AH}_\nu\mathbb{R}$ defined by*

$$\Phi(R,+,\cdot) = (D(R),\oplus,\odot), \quad \Phi(f) = \bar{f} \quad \textit{for any } (R,+,\cdot) \in \operatorname{Ob}\mathbb{H}_\nu\mathbb{R}$$

and any morphism $f \in \mathrm{Mor}\,\mathbb{H}_\nu\mathbb{R}$, $f\colon (R,+,\cdot) \to (S,+,\cdot)$ is a reflector; more precisely the pair $(r_R, (\Delta_R, \oplus, \odot))$ is an $\mathbb{AH}_\nu\mathbb{R}$-reflection for any $(R,+,\cdot) \in \mathrm{Ob}(\mathbb{H}_\nu\mathbb{R})$. Thus $\mathbb{AH}_\nu\mathbb{R}$ is a reflective full subcategory of the category $\mathbb{H}_\nu\mathbb{R}$.

Remark. The results presented at the Second Conference on Mathematics and Physics at Technical Universities and published in [11] are a special case of the topic studied in the first part of the presented paper.

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