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Abstract. For an indefinite quadratic form $f(x_1, \ldots, x_n)$ of discriminant d let $P(f)$ denote the greatest lower bound of the positive values assumed by f for integers $x_1, \ldots, x_n$. This paper uses recent isolation results about ternary forms of signature -1 to reduce the previously known upper bound of P^4/d for non-zero quaternary forms of signature -2. This gives a new bound for the lattice constant of the body $0 \leqslant x^2 - y^2 - z^2 - t^2 < 1$.

1. INTRODUCTION

This paper is concerned with the lattice constants of bodies in $\mathbb{R}^n$ associated with quadratic forms. So we begin with some relevant geometrical definitions.

A *lattice* Λ in $\mathbb{R}^n$ is the set of all integral linear combinations of linearly independent vectors $u_1, \ldots, u_n$ and its *determinant* $d(\Lambda)$ is $|\det\{u_1, \ldots, u_n\}|$. If K is a body in $\mathbb{R}^n$ that is centred at the origin then the lattice Λ is *admissible* for K if the origin is the only point of Λ in K (see eg [1]). The lattice constant of K is defined as

$$\Delta(K) = \inf\{d(\Lambda) : \Lambda \text{ is admissible for } K\} \tag{1.1}$$

So every lattice with determinant less than $\Delta(K)$ contains a point of K other than the origin. If $d(\Lambda) = \Delta(K)$ then Λ is a *critical lattice* for K.

Beginning with the work of Hurwitz and Markoff in the 19th century a great deal of effort has been put into finding the lattice constants of bodies associated with quadratic forms. For a given $n \geqslant 2$ and $r < n$ write

$$F(X_1, \ldots, X_n) = X_1^2 + \cdots + X_r^2 - \cdots - X_n^2. \tag{1.2}$$

The signature s of F is $2r - n$ and satisfies $|s| < n$ and $s \equiv n \pmod 2$. The lattice constants of the bodies $|F| < 1$ are now known for every n and s (both for

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$0 \leqslant |F| < 1$ and for $0 < |F| < 1$). The last results here are due to Margulis (see [2]). In the 1940s Segre, Mahler and Davenport introduced the asymmetric problems of finding the lattice constants of the bodies $0 < F < 1$ and $0 \leqslant F < 1$. The lattice constants of the bodies $0 < F < 1$ are also now known for every n and s (and for most n are the same as for the bodies $0 < |F| < 1$. However for $n = 4$ the bodies given by $0 \leqslant F < 1$ have proved difficult and $n = 4, s = -2$ is the one case still outstanding. So we concentrate on this body which is

$$0 \leqslant X_1^2 - X_2^2 - X_3^2 - X_4^2 < 1. \tag{1.3}$$

An admissible lattice is known for this body because in 1931 Oppenheim [8] showed in effect that the lattice with basis

$$(1,0,0,0), (\frac{1}{2}, \frac{\sqrt{5}}{2}, 0, 0), (\frac{1}{2}, \frac{1}{\sqrt{20}}, \frac{\sqrt{6}}{\sqrt{5}}, 0), (\frac{1}{2}, \frac{1}{\sqrt{20}}, \frac{1}{\sqrt{30}}, \frac{\sqrt{7}}{\sqrt{6}}) \tag{1.4}$$

is admissible for the body in (1.3). This lattice has determinant $\frac{\sqrt{7}}{2} = 1.32\ldots$ but it is not known whether it is a critical lattice. The most that is known up to now is the 1971 result of Worley (see [12]) that the lattice constant must be at least $\frac{9}{8\sqrt{2}} = 0.795\ldots$. In this paper we prove the following result.

Theorem 1. *The lattice constant of the body* $0 \leqslant X_1^2 - X_2^2 - X_3^2 - X_4^2 < 1$ *is at least* $\sqrt{\frac{27}{32}} = 0.918\ldots$.

Statements about the form F in (1.2) and different lattices are equivalent to statements about different quadratic forms of dimension n and signature s and the one lattice of integer vectors. This is because when $f(x_1, \ldots, x_n)$ is a quadratic form of signature s there are linear forms $X_i = \sum l_{ij}x_j$ such that $f(x_1, \ldots, x_n) = F(X_1, \ldots, X_n) = X_1^2 + \cdots + X_r^2 - \cdots - X_n^2$. So the values of f at integral points $(x_1, \ldots, x_n)$ are the values of F at points $(X_1, \ldots, X_n)$ of the lattice Λ with basis $u_j = (l_{1j}, l_{2j}, \ldots, l_{nj})$. Here the discriminants of f and F are related by $d(f) = d(\Lambda)^2 d(F)$. We use the non-Gaussian discriminant defined as in [11] and in particular the discriminant of $X_1^2 - X_2^2 - X_3^2 - X_4^2$ is -16. So the lattice Λ of determinant δ will have a non-zero point in the body given by (1.3) if and only if for the corresponding form f there is an integer vector $x \neq 0$ giving a non-negative value of $f(x)$ with

$$0 \leqslant f^4(x) < \frac{1}{16\delta^2}|d(f)|. \tag{1.5}$$

This enables us to use the machinery of quadratic forms. If for a fixed δ we have (1.5) for all forms with $(n, s) = (4, -2)$ that will show that any admissible lattice for the body (1.3) has to have determinant greater than δ. In trying to establish (1.5) we need only consider those forms for which there is no non-zero x with $f(x) = 0$. For these *non-zero forms* we define

$$P(f) = \inf\{\text{positive values of } f\}, \quad N(f) = -\sup\{\text{negative values of } f\}. \tag{1.6}$$

Theorem 1 will then follow immediately from the following result about quadratic forms.

Theorem 2. *If f is a real non-zero quaternary form of signature -2 and discriminant d then*

$$\varphi(f) = P^4/|d| < \frac{2}{27} \tag{1.7}$$

2. PRELIMINARY RESULTS

Lemma 1. *If g is a nonzero ternary form of signature -1 with $\varphi(g) = P^3(g)/|d(g)| \geqslant 1/2$ then it must be equivalent to a multiple of one of the following forms:*

i	g_i	$P(g_i)$	$\varphi(g_i)$
1	$-(x+\frac{1}{2}z)^2 + 15(y+\frac{1}{2}z)^2 - \frac{9}{2}z^2$	6	$\frac{4}{5}$
2	$-(x+\frac{1}{2}y)^2 + \frac{117}{4}(y+\frac{6}{13}z)^2 - \frac{120}{13}z^2$	9	$\frac{27}{40}$
3	$-x^2 + 8(y+\frac{1}{2}z)^2 - 3z^2$	4	$\frac{2}{3}$
4	$-x^2 + 24(y+\frac{1}{2}z)^2 - \frac{26}{3}$	8	$\frac{8}{13}$
5	$-(x+\frac{1}{2}y)^2 + \frac{45}{4}(y+\frac{2}{5}z)^2 - \frac{24}{5}z^2$	5	$\frac{125}{216}$
6	$-(x+\frac{1}{2}z)^2 + 15(y+\frac{1}{2}z)^2 - \frac{25}{4}z^2$	6	$\frac{72}{125}$
7	$-x^2 + 15(y+\frac{2}{5}z)^2 - \frac{32}{5}z^2$	6	$\frac{9}{16}$
8	$-x^2 + \frac{39}{5}(y+\frac{6}{13}z)^2 - \frac{45}{13}z^2$	$\frac{19}{5}$	$\frac{6859}{13500}$
9	$-(x+\frac{1}{2}y)^2 + \frac{45}{4}(y+\frac{4}{9}z)^2 - \frac{50}{9}z^2$	5	$\frac{1}{2}$

Proof. This is Theorem 2 of [6] (with the values of $P(g_i)$ coming also from [4] and [5]). □

Lemma 2. *Suppose $F = F(x,y)$ is an indefinite binary form that does not represent 0 non-trivially and, as in (1.6), let $P(F)$ be the infimum of the positive values of F with $N(F) = P(-F)$. Then*

$$P^2(F)N(F) \leqslant \frac{1}{\sqrt{108}} d^{3/2}(F) \tag{2.1}$$

Proof. This is the case $n = 2$ of the first part of Theorem 2 in [3]. □

Lemma 3. *Let $f = f(x,y,z,t)$ be a nonzero quaternary form of signature -2 with $\varphi(f) = P^4/|d| \geqslant \frac{2}{27}$. Then we may assume that*

$$f(x,y,z,t) = -(x+\alpha y+\beta z+\gamma t)^2 + g(y,z,t) \tag{2.2}$$

where

$$0 \leqslant \alpha \leqslant \frac{1}{2}, \quad -\frac{1}{2} < \beta \leqslant \frac{1}{2}, \quad -\frac{1}{2} < \gamma \leqslant \frac{1}{2} \tag{2.3}$$

and

$$N(f) = 1 \tag{2.4}$$

We may also assume that the non-zero ternary form g is equivalent to a positive multiple of one of the forms g_i in Lemma 1.

Proof. We may assume $P(f) > 0$ since otherwise (1.7) certainly holds; and then Theorem 1 of [9] gives $N(f) \neq 0$. We can then scale f to have $N(f) = 1$. Since $N(f)$ is non-zero and is rational, the form f must have rational coefficients by [2]. So f takes the value $-N(f)$ and a unimodular transformation puts it in the shape (2.2). Then simple parallel transformations give (2.3) and g is a non-zero form as otherwise we would have $f(x) \in [-\frac{1}{4}, 0]$ for some $x \neq 0$. We may also suppose that $P(g) = g(l,m,n) = v$ say. Now consider the binary section F of f given by $F(x,y) = f(x, yl, ym, yn)$. Since f represents all the values of F we have F a non-zero form, $P(f) \leqslant P(F)$ and $N(f) = 1 = N(F)$. So, using the asymmetric inequalities about binary forms in Lemma 2,

$$P^4(f) \leqslant P^4(F)N^2(F) \leqslant \frac{2}{6^3}d^3(F) = \frac{16}{27}v^3.$$

When g is not equivalent to a multiple of one of the forms $g_1, \ldots, g_9$ we have $v^3 < \frac{1}{2}|d(g)|$ and therefore

$$P^4(f) < \frac{8}{27}|d(g)| = \frac{2}{27}|d(f)|.$$

□

We now use the results of Lemma 3 to begin the proof of Theorem 2. In particular we will always take f to be of the shape (2.2) and can assume that for some $i = 1, \ldots 9$ we have $g = \frac{k}{P(g_i)} g_i$ for a positive parameter k. So $\varphi(g) = \varphi(g_i)$ and

$$P(g) = k \text{ and } N(g) = -g(1,0,0) = \frac{k}{P(g_i)} \tag{2.5}$$

because each g_i has $N(g_i) = 1$. Choosing x such that $|x + \alpha| \leqslant \frac{1}{2}$ then gives $-\frac{1}{4} - \frac{k}{P(g_i)} \leqslant f(x,1,0,0) = -(x+\alpha)^2 - \frac{k}{P(g_i)} < 0$ so $N(f) = 1$ forces

$$k \geqslant \frac{3}{4}P(g_i). \tag{2.6}$$

We can also suppose that $\frac{1}{4}a^2 - 1 < k \leqslant \frac{1}{4}(a+1)^2 - 1$ for some $a \geqslant 0$, so that choosing x with $\frac{a-1}{2} \leqslant |x + \alpha y + \beta z + \gamma t| \leqslant \frac{a}{2}$ gives

$$P(f) \leqslant k - \frac{(a-1)^2}{4}. \tag{2.7}$$

Since $|d(f)| = 4|d(g)| = 4k^3/\varphi(g_i)$ we therefore have

$$\frac{P^4(f)}{|d(f)|} \leqslant \frac{\varphi(g_i)\left[k - \frac{(a-1)^2}{4}\right]^4}{4k^3} \tag{2.8}$$

and so

$$\varphi(f) \leqslant \frac{16\varphi(g_i)(a-1)}{(a+3)^3}. \tag{2.9}$$

This makes $\varphi(f) < \frac{2}{27}$ for sufficiently large a. Indeed when $i = 9$ the inequality (2.6) gives $k \geqslant \frac{15}{4}$, with consequently $a \geqslant 4$, and then (2.9) gives $\varphi(f) < \frac{2}{27}$. For

$i < 9$ we try to improve (2.7) and thus (2.8). We do this by looking for small positive values of g other than the value k.

3. COMPLETION OF THE PROOF

When $g = \frac{k}{9}g_2$ the bounds (2.6) and (2.9) give (1.7) except for $\frac{27}{4} \leqslant k \leqslant \frac{45}{4}$ with $a = 5$ or 6. In this interval (2.8) gives (1.7) for $k \leqslant \frac{31}{4}$ when $a = 5$ and for $8 < k < 11.25$ when $a = 6$. But $g_2(6,1,1) = 11$ so that g represents $\frac{11k}{9} > \frac{(a+1)^2}{4}$ and then, with x so that $\frac{a}{2} \leqslant |x + \alpha y + \beta z + \gamma t| \leqslant \frac{a+1}{2}$, we have $P(f) \leqslant \frac{11k}{9} - \frac{a^2}{4}$. This inequality gives (1.7) for the remaining possibilities for k.

For $4 \leqslant i \leqslant 8$ we have $\frac{k}{8}g_4(6,1,1) = \frac{7k}{6}$, $\frac{k}{5}g_5(2,1,1) = \frac{11k}{5}$, $\frac{k}{6}g_6(4,1,1) = \frac{29k}{24}$, $\frac{k}{6}g_7(4,1,1) = \frac{7k}{6}$, $\frac{5k}{19}g_8(3,1,1) = \frac{21k}{19}$. When reduced by an appropriate $|x + \alpha y + \beta z + \gamma t|^2$ each time, we get new upper bounds for $P(f)$. In a similar manner to the argument for $i = 2$ these new bounds can then be used in conjuction with (2.7) to eliminate the possibility $g = \frac{k}{P(g_i)}g_i$ for each $i \geqslant 4$.

When $g = \frac{k}{4}g_3$ inequalities (2.6) and (2.9) show that we only need to consider the range $3 \leqslant k \leqslant 8$ with $a = 3$ when $k = 3$, $a = 4$ or 5 otherwise. Here (2.8) immediately gives (1.7) for $3 < k \leqslant 4.95$. For $4.95 < k \leqslant 5.15$ we have $8.6 < g(1,1,0) = \frac{7k}{4} < 9.1$ whence, with x such that $2.5 \leqslant |x + \alpha y + \beta z + \gamma t| \leqslant 3$, we get $P(f) \leqslant \frac{7k}{4} - 6.25$ and then

$$\varphi(f) \leqslant \frac{(\frac{7k}{4} - 6.25)^4}{6k^3} < \frac{2}{27}.$$

For $5.25 < k \leqslant 7.8$ the inequality (2.8) again gives (1.7); and for $7.8 < k \leqslant 8$ we have $11.7 < g(3,1,1) = \frac{3k}{2} \leqslant 12$, leading to $P(f) \leqslant \frac{3k}{2} - 9$ and $\varphi(f) \leqslant \frac{(\frac{3k}{2}-9)^4}{6k^3} < \frac{2}{27}$. This leaves the possibilities $k = 3$ or $5.15 < k \leqslant 5.25$.

When $5.15 < k \leqslant 5.25$ we may suppose that $P(f) > 2.75$ or else we have at once $P^4(f)/6k^3 < \frac{2}{27}$. Then using $g(1,1,0) = \frac{7k}{4}$, $g(2,1,0) = k$ and $g(3,1,1) = \frac{3k}{2}$ we see that [1] we must have $\|\alpha+\beta\| > 0.46$, $\|2\alpha+\beta\| > 0.47$ and $\|3\alpha+\beta+\gamma\| < 0.05$ or else we would have one of $-1 < f(x,1,1,0) \leqslant 2.75$, $-1 < f(x,2,1,0) \leqslant 2.75$ or $-1 < f(x,3,1,1) \leqslant 2.75$ for some x. Hence $\alpha = (2\alpha + \beta) - (\alpha + \beta) < 0.07$ and $\|\alpha + \beta + \gamma\| = \|3\alpha + \beta + \gamma - 2\alpha\| < 0.19$ which gives

$$0.4 < f(x,1,1,1) = -(x + \alpha + \beta + \gamma)^2 + \frac{7k}{2} \leqslant 2.4 \text{ for suitable } x.$$

When $k = 3$ then for suitable choice of x we would have either $-1 < -(x+2\alpha+\beta)^2+k \leqslant 0.75$, $-1 < -(x+4\alpha+\beta+2\gamma)^2+k \leqslant 0.75$ or $-1 < -(x+\alpha+\beta)^2+\frac{7}{4}k \leqslant 1.25$, giving (1.7) immediately unless $2\alpha + \beta \equiv 0 \pmod 1$, $\alpha + \beta \equiv \frac{1}{2} \pmod 1$ and $4\alpha + \beta + 2\gamma \equiv 0 \pmod 1$. These make $\alpha = \frac{1}{2}$, $\beta = 0$ and $2\gamma \equiv 0 \pmod 1$. For $\gamma = 0$ we would then have $-(2 + 2\alpha + \beta + \gamma)^2 + \frac{11}{4}k = -\frac{3}{4}$ contradicting $N(f) = 1$; and for $\gamma = \frac{1}{2}$ we would have $-(3\alpha + \beta + \gamma)^2 + \frac{3}{2}k = \frac{1}{2}$ which gives (1.7).

When $g = \frac{k}{6}g_1$ the inequalities (2.6) and (2.9) show that we only need to consider the range $4.5 \leqslant k \leqslant 15$ with $4 \leqslant a \leqslant 7$. We split this range into 10

[1]The notation $\|t\|$ denotes the distance from t to the nearest integer.

subintervals in each of which we use different estimates for $P(f)$. Firstly (2.8), with $a = 4,5,6,7$ in turn, gives (1.7) for $4.5 \leqslant k \leqslant 4.75$, $5.25 < k \leqslant 7.55$, $8 < k \leqslant 10.9$ and $11.25 < k \leqslant 14.9$. The gaps $7.55 < k \leqslant 8$ and $10.9 < k \leqslant 11.25$ correspond to $a = 5$ and $a = 6$ respectively and in each of these cases we have $g(4,1,1) = \frac{3k}{2} > \frac{(a+2)^2}{4} - 1$. Choosing x so that $\frac{a+1}{2} \leqslant |x + 4\alpha + \beta + \gamma| \leqslant \frac{a+2}{2}$ then gives $P(f) \leqslant \frac{3k}{2} - \frac{(a+1)^2}{4}$. This is enough to make $\varphi(f) < \frac{2}{27}$ each time. For $4.85 < k \leqslant 4.95$ we similarly get $P(f) \leqslant \frac{7k}{3} - 9$ which again gives $\varphi(f) < \frac{2}{27}$. This leaves the intervals $4.75 < k \leqslant 4.85$, $4.95 < k \leqslant 5.25$ and $14.9 < k \leqslant 15$.

For $k \in (4.75, 4.85]$ we may assume that $P(f) > 2.5$ and we have $g(1,1,0) = \frac{7k}{3} \in (11, 11.32)$ and $g(2,1,0) = \frac{11k}{6} \in (8.7, 8.9)$. These imply $\|\alpha + \beta\| > 0.46$, $\|2\alpha + \beta\| > 0.47$ and so $\|\beta\| > 0.39$. Since $g(0,1,0) = \frac{5k}{2} \in (11.8, 12.2)$ we then have $-0.45 < f(x,0,1,0) < 0.75$ for some x.

For $k = g(3,1,0) \in (4.95, 5.25]$ we may assume that $P(f) > 2.55$ and we have $g(1,1,0) \in (11.55, 12.25]$ and $g(2,1,0) \in (9, 9.625]$. These imply $\|\alpha + \beta\| < 0.115$, $\|2\alpha + \beta\| > 0.34$ and $\|3\alpha + \beta\| > 0.439$ which cannot all hold simultaneously.

Finally for $k \in (14.9, 15]$ we may assume that $P(f) > 5.9$ and we have $g(1,1,0) \in (34.76, 35]$, $g(2,1,0) \in (27.3, 27.5]$, $g(3,1,1) = \frac{17k}{6} \in (42.2, 42.5]$, $g(4,1,1) = \frac{3k}{2} \in (22.35, 22.5]$. These imply $\|\alpha + \beta\| < 0.025$, $\|2\alpha + \beta\| > 0.35$, $\|3\alpha + \beta + \gamma\| < 0.05$, and $\|4\alpha + \beta + \gamma\| < 0.075$. The first two make $\alpha > 0.32$ and the second two give the contradiction $\alpha < 0.2$.

References

[1] J W S Cassels, An introduction to the geometry of numbers, Springer, 1971.

[2] S G Dani and G A Margulis, Values of quadratic forms at integral points: an elementary approach, Enseign. Math. (2) **36** (1990), 143-174.

[3] T H Jackson, Small positive values of indefinite binary quadratic forms, J. London Math. Soc. **43** (1968) 730-738.

[4] T H Jackson, One-sided inequalities for ternary forms I, J. Number Theory **13** (1981), 376-397.

[5] T H Jackson, One-sided inequalities for ternary forms II, J. Number Theory **16** (1983), 333-342.

[6] T H Jackson, Small positive values of indefinite ternary forms, Mathematika *to appear*.

[7] K Mahler, A theorem of B. Segre, Duke Math. J. **12** (1945), 367-371.

[8] A Oppenheim, Minima of indefinite quaternary quadratic forms, Ann. Math. **32** (1931), 271-298.

[9] A Oppenheim, Values of quadratic forms I, Quart. J. Math. Oxford Ser. 2 **4** (1953), 54-59.

[10] B Segre, Lattice points in infinite domains and asymmetric Diophantine approximations, Duke Math. J. **12** (1945), 337-365.

[11] G L Watson, Integral Quadratic Forms, Cambridge University Press, 1960.

[12] R T Worley, Non-negative values of quadratic forms, J. Austral. Math. Soc. **12** (1971), 224-238.

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