Acta Mathematica et Informatica Universitatis Ostraviensis

Takao Yamazaki

Tate duality and ramification of division algebras

Acta Mathematica et Informatica Universitatis Ostraviensis, Vol. 10 (2002), No. 1, 153--160

Persistent URL: `http://dml.cz/dmlcz/120581`

Acta Mathematica et Informatica Universitatis Ostraviensis 10 (2002) 153–160

Tate duality and ramification of division algebras

Takao Yamazaki

Abstract. We survey a ramification theory for division algebras of certain type, and apply it to a study on the Tate duality for abelian varieties over local fields. We give a proof of a formula which express the different of a division algebra in terms of the Swan conductors.

1. Introduction

Let k be a complete discrete valuation field with perfect residue field. For a finite extension k'/k, there is a beautiful ramification theory. The wildness of ramification in the extension k'/k is measured by the *conductor*. The different of k'/k is determinated by the conductor, and hence so is the discriminant. Furthermore, when k is an usual local field (that is, the residue field is a finite field), the conductor has a local class field theoretic meaning. We shall review this theory in Section 2, in a form suitable for our purpose.

Now let K be a complete discrete valuation field with imperfect residue field. Then it becomes highly difficult to construct a ramification theory for a finite extension of K. Although there are a lot of excellent works about it, the whole theory is far from completion. However, in [8] a ramification theory was constructed for a finite dimensional division algebra of certain type. To be precise, we assume that the residue field F of K satisfies $[F : F^p] = p$ where p is the characteristic of F (hence F is imperfect), and let D be a finite dimensional central division algebra over K. There is a definition of the *conductor* of D. The different of D is determinated by the conductor. Furthermore, when K is a two-dimensional local field in the sense of Kato (that is, F is an usual local field), then the conductor is related to the two-dimensional local class field theory. We shall review this theory in Section 3.

Received: January 17, 2002.
1991 *Mathematics Subject Classification:* Primary: 11S25, Secondary: 14H40, 14G20, 11G10.
Key words and phrases: p-adic field, Galois cohomology, abelian variety, wild ramification, Brauer group.

As an application of our ramification theory, we study the Tate duality for abelian varieties over (usual) local fields in Section 4. The main purpose of this note is to survey those theories. No proof is given, except a proof of a different formula which is given in Section 5.

This note is written while the author is a visitor at Utrecht University, whose hospitality is gratefully appreciated. He is supported by JSPS Postdoctoral Fellowships for Research Abroad.

2. Ramification theory for field extensions

In this section, we briefly review the classical ramification theory for field extensions. Our main reference is [6]. Let k be a complete discrete valuation field whose residue characteristic is $p > 0$. The general theory is established for an arbitrary algebraic extension (at least when the residue field is perfect), but for our purpose we restrict to the following case: We assume k'/k is a totally ramified cyclic extension of degree p^n. (So we do not have to assume the residue field to be perfect.) Let O_k and $O_{k'}$ be the ring of integers in k and k' respectively.

2.1. Conductor

Usually, the conductor of an extension is defined by using the *ramification subgroups* (cf. Remark 20). However, we avoid to use them and give a direct definition in our special case. For $j = 0, 1, \cdots, n-1$, we define

$$t_j = \inf\{\mathrm{ord}_{k'}(\sigma(a)/a - 1) \mid a \in (k')^*,\ \sigma \in \mathrm{Gal}(k'/k_j)\},$$

where $\mathrm{ord}_{k'}$ is the normalized valuation on k' and k_j is the (unique) intermediate field of k'/k satisfying $[k_j : k] = p^{n-j-1}$. Then, we define for $j = 0, 1, \cdots, n-1$

$$s_j = \frac{t_j}{p^{n-j-1}} + (p-1)(\frac{t_{n-1}}{p} + \frac{t_{n-2}}{p^2} \cdots + \frac{t_{j+1}}{p^{n-j+1}}).$$

The number s_0 is called the conductor of the extension k'/k and is denoted by $f(k'/k)$. (If $j \geq 1$, s_j is the conductor of the extension k_{j-1}/k.) The numbers s_j are integers by the Hasse-Arf theorem.

Remark 20. Using the notion of the upper numbering ramification group, the conductor of an (arbitrary algebraic) extension k'/k is defined to be the number

$$\inf\{w \geq 0 \mid k' \subset \bar{k}^{\mathrm{Gal}(\bar{k}/k)^{(w-1)}}\},$$

where $\bar{k}$ is a separable closure of k.

2.2. Different

The different of k'/k is defined to be the inverse of the fractional ideal

$$\{\ x \in k' \mid \mathrm{Tr}_{k'/k}(xy) \in O_k \text{ for any } y \in O_{k'}\ \}$$

of $O_{k'}$ and we denote it by $\mathcal{D}(k'/k)$. This ideal is determinated in terms of the conductors, by the following equation (see [6] IV Proposition 4)

$$\begin{aligned}\mathrm{ord}_{k'}(\mathcal{D}(k'/k)) &= (p-1)(p^{n-1}s_0 + p^{n-2}s_1 + \cdots + s_{n-1}) + p^n - 1\\ &= (p-1)(t_0 + pt_1 + \cdots + p^{n-1}t_{n-1}) + p^n - 1.\end{aligned}$$

2.3. Relation with the local class field theory

Assume the residue field of k is finite. Then, by the local class field theory, giving an cyclic extension k'/k of degree p^n (together with an isomorphism $\mathrm{Gal}(k'/k) \cong \mathbb{Z}/p^n\mathbb{Z}$) is equivalent to giving a surjective homomorphism $\xi_{k'/k} : k^* \to \mathbb{Z}/p^n\mathbb{Z}$. The conductor of k'/k can be written in terms of $\xi_{k'/k}$:

$$f(k'/k) = \inf\{i \mid \xi_{k'/k}(U_k^{(i+1)}) = 0\},$$

where $U_k^{(i)} = \{x \in k^* \mid \mathrm{ord}_k(x-1) \geq i\}$ is the i-th unit group of k and ord_k is the normalized valuation on k. The proof is given in [6] Chapter XV Section 2. A similar interpretation can be given in a more general situation (see [4] Remark 6.5).

3. Ramification theory for Division algebras

In this section, we review the ramification theory for division algebras, following [8]. This theory is an analogue of the one for field extensions, so the reader is recommended to keep comparing with the preceding section. Let K be a complete discrete valuation field and F its residue field. We assume that the characteristic of F is $p > 0$ and $[F : F^p] = p$. Let D be a finite dimensional central division algebra over K and C its residue algebra. We assume that C is commutative and purely inseparable over F. This is equivalent to the condition that $D \otimes_K K_{\mathrm{ur}}$ is a division algebra, where K_{ur} is the maximal unramified extension of K. We have $[D : K]^{1/2} = [C : F] =$ the order of w, and we write this number by p^n. (See [8] Proposition 2.1.) Let O_K and O_D be the ring of integers in K and D respectively.

3.1. Conductor

For any subset S of D^*, we write

$$t(S) = \inf\{\mathrm{ord}_D(aba^{-1}b^{-1} - 1) \mid a, b \in S\},$$

where ord_D is the normalized valuation on D. For $j = 0, 1, \cdots, n-1$, we define

$$t_j = \sup\left\{ t(D_1{}^*) \;\middle|\; \begin{array}{c} D_1 \text{ is a division algebra such that} \\ K \subset D_1 \subset D, \\ (D_1 : \text{center of } D_1)^{1/2} = p^{j+1}, \\ (\text{center of } D_1 : K) = p^{n-j-1}. \end{array} \right\}.$$

Then, we define for $j = 0, 1, \cdots, n-1$

$$s_j = \frac{t_j}{p^{n-j-1}} + (p-1)\left(\frac{t_{n-1}}{p} + \frac{t_{n-2}}{p^2} \cdots + \frac{t_{j+1}}{p^{n-j+1}}\right).$$

The number s_0 is called the Swan conductor of D and is denoted by $\mathrm{sw}(D)$.

Remark 21. If $[D]$ is the class of D in the Brauer group of K, the number s_j is the Swan conductor of the division algebra whose class is $p^j[D]$. The numbers s_j are integers by the comparison with Kato's definition of the Swan conductor; see Subsection 3.3.

3.2. Different

The different of D is defined to be the inverse of the fractional ideal

$$\{ x \in D \mid \mathrm{Trd}_{D/K}(xy) \in O_K \text{ for any } y \in O_D \}$$

of O_D, and we denote it by $\mathcal{D}(D/K)$. (Here, $\mathrm{Trd}_{D/K}$ is the reduced trace.) This ideal is determinated in terms of the conductors:

$$\begin{aligned}\mathrm{ord}_D(\mathcal{D}(D/K)) =&(p-1)(p^{n-1}s_0 + p^{n-2}s_1 + \cdots + s_{n-1}) + p^n - 1\\ =&(p-1)(t_0 + pt_1 + \cdots + p^{n-1}t_{n-1}) + p^n - 1.\end{aligned}$$

We shall prove this formula in Section 5.

3.3. Relation with the two-dimensional local class field theory

Assume K is a two-dimensional local field in the sense of Kato, that is, the residue field F is a complete discrete valuation field with finite residue field (so F is isomorphic to a field of Laurent power series over a finite field). Thanks to the two-dimensional local class field theory of Kato [3], giving an division algebra D over K with $[D : K]^{1/2} = p^n$ is equivalent to giving a surjective homomorphism $\eta_{D/K} : K^* \to \mathbb{Z}/p^n\mathbb{Z}$. The conductor of D can be written in terms of $\eta_{D/K}$:

$$\mathrm{sw}(D) = \inf\{i \mid \eta_{D/K}(U_K^{(i+1)}) = 0\}. \tag{3.1}$$

Here $U_K^{(i)} = \{x \in K^* \mid \mathrm{ord}_K(x-1) \geq i\}$ and ord_K is the normalized valuation on K. This is proven in [8] Theorem 5.1, in a more general situation.

Remark 22. Since K is a two-dimensional local field, there are rank 2 valuations on K and D, namely, normalized valuations $\mathrm{ord}_K^{(2)} : K^* \to \mathbb{Z} \oplus \mathbb{Z}$ and $\mathrm{ord}_D^{(2)} : D^* \to \mathbb{Z} \oplus \mathbb{Z}$. (If we compose them with the first projection, we get the usual valuations ord_K and ord_D.) We regard $\mathbb{Z} \oplus \mathbb{Z}$ as an ordered abelian group with lexicographic order.

If we replace ord_D with $\mathrm{ord}_D^{(2)}$ in the definition of $\mathrm{sw}(D)$ in Subsection 3.1, we can define the rank two Swan conductor $\mathrm{sw}^{(2)}(w) \in \mathbb{Z} \oplus \mathbb{Z}$. On the other hand, we can define for each $(i,j) \in \mathbb{Z} \oplus \mathbb{Z}$ the (i,j)-th unit group $U_K^{(i,j)}$ of K to be $\{x \in K^* \mid \mathrm{ord}_K^{(2)}(x-1) \geq (i,j)\}$. In this generalized context, the equation (3.1) is still valid. In fact, by the same proof as in [8] Theorem 5.1 one can show

$$\mathrm{sw}^{(2)}(D) = \inf\{(i,j) \in \mathbb{Z} \oplus \mathbb{Z} \mid \eta_{D/K}(U_K^{(i,j+1)}) = 0\}.$$

4. Tate duality for abelian varieties

As an application of our theory, we state a result concerning the Tate duality for abelian varieties over a local field, which is proved in [10]. Let k a finite extension of $\mathbb{Q}_p$, O_k the ring of integers in k, and $\mathcal{P}_k$ its maximal ideal.

4.1. Tate duality and McCallum's theorem

Let A be an abelian variety over k and A^t its dual. We consider Tate's perfect pairing [7]

$$<,>_A : A(k) \times H^1(k, A^t) \to \mathbb{Q}/\mathbb{Z}. \tag{4.1}$$

We define a descending filtration $A(k) \supset U^1A(k) \supset U^2A(k) \supset \cdots$ as follows: $U^nA(k)$ is the subgroup of elements in $A(k)$ whose formal coordinates around the origin are in the n-th power of $\mathcal{P}_k$. We consider the annihilator of $U^nA(k)$ with respect to the Tate pairing (4.1). Before we state our result, we recall McCallum's important result in this direction:

Theorem 4.1. *(McCallum [5] Theorem 1.) Let A be an abelian variety over k, and n a positive integer. If $x \in U^nA(k)$ and $\xi \in \ker(H^1(k, A^t) \to H^1(k', A^t))$ where k'/k is a finite extension of conductor $\leq n$ (in the sense of Subsection 2.1), then we have $< x, \xi >_A = 0$.*

Remark 23. (1) In general, we cannot get every element of the annihilator of $U^nA(k)$ only by this theorem. See [5] Section 4.

(2) Coates and Greenberg obtained a description of the annihilator of $U^1A(k)$ when A has good ordinary reduction (see [1] p.170).

4.2. The case of Jacobian varieties

Let X be a projective smooth geometrically connected curve over k such that $X(k) \neq \phi$. Let J be the Jacobian variety of X. We remark that the dual J^t of J is canonically isomorphic to J. We shall give a description of the annihilator of $U^nA(k)$ when $A = J$. The key step is to use the ramification theory for division algebras developed in Section 3 in order to define the Swan conductor $\mathrm{sw}(w)$ of w for each element w in $\mathrm{Br}(X)$.

We take a regular model $\mathcal{X}$ of X, which is proper and flat over O_k. (The existence is due to Hironaka and Abhyankar. See [9], [10] for a more precise condition for $\mathcal{X}$.) We set $Y := (\mathcal{X} \times_{\mathrm{Spec}\, O_k} \mathrm{Spec}(O_k/\mathcal{P}_k))_{\mathrm{red}}$. If η is a generic point of Y, let Λ_η be the fraction field of the completion of the strict henselization of $\mathcal{O}_{\mathcal{X},\eta}$. Let w be an element of $\mathrm{Br}(X)$. We write by w_{Λ_η} the image of w in $\mathrm{Br}(\Lambda_\eta)$. Let $D(w_{\Lambda_\eta})$ be the division algebra over Λ_η, whose class is w_{Λ_η}. We define $\mathrm{sw}(D(w_{\Lambda_\eta}))$ by applying Subsection 3.1 to $D(w_{\Lambda_\eta})$. Now we define the Swan conductor of w to be

$$\mathrm{sw}(w) = \sup\{\ [\mathrm{sw}(D(w_{\Lambda_\eta}))/e_\eta] \mid \eta \text{ runs over the generic points of } Y\ \},$$

where $[\ ]$ denotes the least integer function and e_η denotes the multiplicity of $\bar{\{\eta\}}$ in the divisor $\mathcal{X} \times_{\mathrm{Spec}\, O_k} \mathrm{Spec}(O_k/\mathcal{P}_k)$ on $\mathcal{X}$.

On the other hand, by the Hochschild-Serre spectral sequence for the etale sheaf $\mathbb{G}_m$, we can deduce the following exact sequence, which allows us to relate $H^1(k, J)$ with $\mathrm{Br}(X)$:

$$0 \to \mathrm{Br}(k) \to \mathrm{Br}(X) \xrightarrow{\theta} H^1(k, J) \to 0. \tag{4.2}$$

Here we used the assumption $X(k) \neq \phi$.

The following theorem can be proved by using our ramification theory for division algebras, especially the result of Subsection 3.3.

Theorem 4.2. *([10] Theorem 2.2.) Let X be a projective smooth geometrically connected curve over k such that $X(k) \neq \phi$, and J its Jacobian variety. Let $n \geq 1$*

be an integer. Then, the annihilator of $U^n J(k)$ with respect to the Tate pairing (4.1) *is*

$$\theta(\{w \in \mathrm{Br}(X) \mid \mathrm{sw}(w) \le n\}),$$

where θ is the map in (4.2).

5. Proof of a different formula

In this section, we prove the formula in Subsection 3.2. We use the notations in Section 3. Following Hyodo [2], we define the depth of D/K to be the number

$$d(D/K) = \inf\{\mathrm{ord}_D(\mathrm{Trd}_{D/K}(y)/y) \mid y \in D^*\}.$$

Then we have $d(D/K) = \mathrm{ord}_D(\mathcal{D}(D/K)) - (p^n - 1)$. We shall prove the following formula by induction on n:

$$d(D/K) = (p-1)(p^{n-1}s_0 + p^{n-2}s_1 + \cdots + s_{n-1}).$$

The case $n = 0$ is trivial. Suppose $n \ge 1$. We consider an extension K_1/K of degree p, which is contained in D. Let D_1 be the centralizer of K_1 in D. We define

$$d(D_1/K) = \inf\{\mathrm{ord}_D(\mathrm{Tr}_{K_1/K} \circ \mathrm{Trd}_{D_1/K_1}(y)/y) \mid y \in D_1^*\},$$
$$d(D_1/K_1) = \inf\{\mathrm{ord}_D(\mathrm{Trd}_{D_1/K_1}(y)/y) \mid y \in D_1^*\},$$
$$d(D/K) = \inf\{\mathrm{ord}_D(\mathrm{Tr}_{K_1/K}(y)/y) \mid y \in K_1^*\}.$$

(Note that we used ord_D instead of ord_{D_1} or ord_{K_1}, the normalized valuations on D_1 or K_1.) The following is clear from the definition:

$$d(D/K) = \inf\{d(D_1/K) \mid K_1/K \text{ runs all extensions as above}\}.$$

We remark that the similar formula does not hold for the different. We also have the following formula, which can be seen by the totally same proof as in [2] Lemma 2.4:

$$d(D_1/K) = d(D_1/K_1) + d(K_1/K).$$

So we have to show that

$$(p-1)(p^{n-1}s_0 + p^{n-2}s_1 + \cdots + s_{n-1}) \le d(D_1/K_1) + d(K_1/K)$$

for any K_1/K and that the equality holds for some K_1/K.

We apply the definition in Subsection 3.1 to the division algebra D_1 and define s'_j ($j = 0, 1, \cdots, n-2$) to be the Swan conductor of the division algebra whose class in $\mathrm{Br}(K_1)$ is $p^j[D_1]$. (Here $[D_1]$ is the class of D_1 in $\mathrm{Br}(K_1)$; see Remark 21.) By [8] p.142, we may assume either (A) or (B) below. Furthermore, there exists at least one K_1 which satisfies the equality in the inequalities in (A) or (B) (with $t = s_{n-1}$).

(A) K_1/K is a totally ramified Galois extension. Put $t = \mathrm{ord}_{K_1}(1 - \sigma(\pi)/\pi)$ where π is a prime element of K_1 and σ is a generator of $\mathrm{Gal}(K_1/K)$. Let m be an integer satisfying $s_m \le t < s_{m-1}$. Then

$$\begin{aligned} &m \le n-1, \\ &s'_j \ge ps_j + (1-p)t \qquad j = 0, 1, \cdots m-1, \\ &s'_j \ge s_j \qquad j = m+1, m+2, \cdots, n-2. \end{aligned}$$

See [8] Lemma 4.2, especially the equations (4) and (7) thereof. Note that we have

$$d(D_1/K_1) = (p-1)(p^{n-2}s_0' + p^{n-3}s_1' + \cdots + s_{n-2}'),$$
$$d(K_1/K) = p^{n-1}((p-1)(t+1) - (p-1)),$$

by the inductive hypothesis and [6] V Lemma 3 respectively.

(B) K_1/K is a Galois extension whose ramification index is 1 and whose residue extension is purely inseparable. Put $t = p\,\mathrm{ord}_{K_1}(1-\sigma(h)/h)$ where σ is a generator of $\mathrm{Gal}(K_1/K)$ and h is an unit element in K_1 whose residue class generates the residue field of K_1 over F. Let m be an integer satisfying $s_m \leq t < s_{m-1}$. Then

$$m \leq n-1,$$
$$s_j' \geq s_j + (1/p-1)t \qquad j = 0, 1, \cdots m-1,$$
$$s_j' \geq (1/p)s_j \qquad j = m+1, m+2, \cdots, n-2.$$

See [8] Lemma 4.4. Note that we have

$$d(D_1/K_1) = p(p-1)(p^{n-2}s_0' + p^{n-3}s_1' + \cdots + s_{n-2}'),$$
$$d(K_1/K) = p^{n-1}(p-1)t,$$

by the inductive hypothesis and [2] (2-10) respectively.

In the case (A), we calculate

$$\begin{aligned}(p-1)^{-1}&(d(D_1/K_1) + d(K_1/K))\\ &= p^{n-1}t + p^{n-2}s_0' + \cdots + p^{n-m-1}s_{m-1}' + p^{n-m-2}s_m' + \cdots + x_{n-2}'\\ &\geq p^{n-1}t + p^{n-2}(ps_0 + (1-p)t) + \cdots + p^{n-m-1}(ps_{m-1} + (1-p)t)\\ &\qquad + p^{n-m-2}s_{m+1} + \cdots + s_{n-1}\\ &= p^{n-1}s_0 + \cdots + p^{n-m}s_{m-1} + p^{n-m-1}t + p^{n-m-2}s_{m+1} + \cdots + s_{n-1}\\ &\geq p^{n-1}s_0 + \cdots + s_{n-1},\end{aligned}$$

and the equalities hold when $t = s_{n-1}$. A similar calculation in the case (B) completes the proof.

References

[1] Coates, J. and Greenberg, R., *Kummer theory for abelian varieties over local fields*, Invent. Math. **124**, 129-174 (1996).

[2] Hyodo, O., *Wild ramification in the imperfect residue field case*, Adv. Stud. Pure Math., 12, 287–314 (1987).

[3] Kato,K., *A generalization of local class field theory by using K-groups. I.*, J. Fac. Sci. U. of Tokyo, Sec. IA **26** ,303-376 (1989).

[4] Kato, K., *Swan conductors for characters of degree one in the imperfect residue field case*, Contemporary Math. **83**, 101-131 (1989).

[5] McCallum, W., *Tate duality and wild ramification*, Math. Ann. **288**, 553-558 (1990).

[6] Serre, J.P., Corps locaux, Hermann (1962).

[7] Tate, J., *WC-groups over p-adic fields*, Seminaire Bourbaki, **156** 13p (1957).

[8] Yamazaki, T., *Reduced norm map of division algebras over complete discrete valuation fields of certain type*, Comp. Math. **112**, 127-145 (1998).

[9] Yamazaki, T., *On Swan conductors for Brauer groups of curves over local fields*, Proc. Amer. Math. Soc. **127**, 1269-1274 (1999).

[10] Yamazaki, T., *On Tate duality for Jacobian varieties*, preprint (2001).

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA 305-8571, JAPAN
E-mail address: ytakao@math.tsukuba.ac.jp