Marcin Stępień

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Abstract. Given an (abstract) Witt ring W, there is only one quaternionic structure (G, Q, q) associated to it (cf. M. Marshall [2]). This paper is concerned with automorphisms of Witt rings described in the terminology of quaternionic structures. The main goal is to describe the Harrison automorphisms of products of Witt rings of local type.

1. Introduction

We consider an abstract Witt ring in the terminology of Marshall [2]. For the reader's convenience we state the definition. A *Witt ring* is a pair $W = (R, G)$, where R is a commutative ring with unity 1 and G is a subgroup of the multiplicative group of units R^* which has exponent 2 and contains -1. Let I denotes the fundamental ideal of R generated by elements of the form $x + y$, where $x, y \in G$. Moreover the following three axioms hold:

$\mathbf{W_1}$: G generates R additively, that means every element of R is of the form $a = a_1 + \cdots + a_n$ with $a_1, a_2, \ldots, a_n \in G$ and $n \geq 1$.

$\mathbf{W_2}$: The following Arason-Pfister property holds for $k = 1$ and 2. If $a = a_1 + \cdots + a_n \in I^k$ with $n < 2^k$, then $a = 0$.

$\mathbf{W_3}$: If $a_1 + \cdots + a_n = b_1 + \cdots + b_n$ and $n \geq 3$, then there exist $a, b, c_3, \ldots, c_n \in G$ such that $a_2 + \cdots + a_n = a + c_3 + \cdots + c_n$, $a_1 + a = b_1 + b$ (and, hence, $b_2 + \cdots + b_n = b + c_3 + \cdots + c_n$).

In this paper we describe the group of all automorphisms of a product of Witt rings of local type. A *Harrison automorphism* of $W = (R, G)$ is a ring automorphism σ of R, such that $\sigma(a) \in G$ for every $a \in G$. A convenient tool for investigations of Harrison automorphisms is a notion of quaternionic structure.

Let G be an elementary 2-group (i.e., $a^2 = 1$ for all $a \in G$) with distinguished element -1, with $-a := -1 \cdot a$. Let Q be a pointed set with distinguished element θ, and let $q\colon G \times G \to \mathrm{Q}$ be a surjective mapping.

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Definition 1.1. The triplet (G, Q, q) is said to be a *quaternionic structure* (Q-structure, for short), if q satisfies:
Q_1: $q(a,b) = q(b,a)$
Q_2: $q(a,-a) = \theta$
Q_3: $q(a,b) = q(a,c) \Leftrightarrow q(a,bc) = \theta$
Q_4: $q(a,b) = q(c,d) \Rightarrow \underset{x \in G}{\exists} \; q(a,b) = q(a,x),\ q(c,d) = q(c,x)$
for all $a,b,c,d \in G$.

A *(quadratic) form* of dimension $n \geq 1$ over G is an n-tuple $\phi = (a_1, \ldots, a_n)$, where $a_1, \ldots, a_n \in G$. There is defined an *isometry* relation of one and two-dimensional forms by:

$$\text{(i)} \begin{cases} (a) \cong (b) \Leftrightarrow a = b \\ (a,b) \cong (c,d) \Leftrightarrow ab = cd \text{ and } q(a,b) = q(c,d), \end{cases}$$

and for forms of dimension $n > 2$ inductively by:
(ii) $(a_1, \ldots, a_n) \cong (b_1, \ldots, b_n) \Leftrightarrow \exists a, b, c_3, \ldots, c_n \in G$ such that $(a_2, \ldots, a_n) \cong (a, c_3, \ldots, c_n)$, $(a_1, a) \cong (b_1, b)$, and $(b_2, \ldots, b_n) \cong (b, c_3, \ldots, c_n)$.
Isometry is an equivalence relation. We say that a form ϕ of dimension n *represents* an element $x \in G$ if there exist $x_2, \ldots, x_n \in G$ such that $\phi \cong (x, x_2, \ldots, x_n)$. We shall write $D(\phi)$ for the set of all elements $x \in G$ represented by the form ϕ in this sense (the value set of the form ϕ).

Example 1.2. Let F be a field of characteristic $\neq 2$, $G(F) := F^*/F^{*2}$. A *form over* G of dimension n is a sequence $\phi = (a_1, \ldots, a_n)$, where $a_1, \ldots, a_n \in G(F)$. Let $\mathrm{Q}(F)$ be the pointed set of all isometry classes of quadratic forms of the type $\langle 1, -a, -b, ab \rangle$ with the point $\theta = \langle 1, -1, 1, -1 \rangle$. Finally, let $q_F \colon G(F) \times G(F) \to \mathrm{Q}(F)$ be the map sending (a,b) to the isometry class of $\langle 1, -a, -b, ab \rangle$. The triplet $(G(F), \mathrm{Q}(F), q_F)$ is a Q-structure called *the quaternionic structure associated to* F (for the proof see [2]).

The category of Witt rings and the category of Q-structures are naturally equivalent ([2], Th. 4.5). This means that for given an abstract Witt ring one can construct a Q-structure (G, Q, q) associated to it and conversely, for given a Q-structure (G, Q, q) there exists a Witt ring R with the Q-structure (G', Q', q') isomorphic to (G, Q, q).

For $W = (R, G)$ an arbitrary Witt ring we define Q to be the subset of R consisting of all elements $(1-a)(1-b)$, with $a, b \in G$ and the mapping $q \colon G \times G \to \mathrm{Q}$ by $q(a,b) = (1-a)(1-b)$. The triplet (G, Q, q) is a Q-structure associated to W (cf. [2], Prop. 4.2).

Conversely, let (G, Q, q) be a Q-structure. Define R to be the quotient of the integral group ring $\mathbf{Z}[G]$ obtained by factoring by the ideal J generated by $[1] + [-1]$ and all elements $([1] - [a])([1] - [b])$, where $a, b \in G$ satisfy $1 \in D(a,b)$ and by $[a]$ we denote an element of $\mathbf{Z}[G]$. Then $W = (R, G)$ is a Witt ring associated to the Q-structure (G, Q, q).

Let $\mathcal{G} = (G, \mathrm{Q}, q)$ and $\mathcal{G}' = (G', \mathrm{Q}', q')$ be Q-structures and let $\varphi \colon G \to G'$ be a group isomorphism with $\varphi(-1) = -1'$. If for every $a, b \in G$, $q(a,b) = \theta \Leftrightarrow q'(\varphi(a), \varphi(b)) = \theta'$ then φ is said to be a Q-*isomorphism*. The structures will be called *equivalent* and we shall write $(G, \mathrm{Q}, q) \sim (G', \mathrm{Q}', q')$. If for the Q-structure

(G, Q, q) there is a field F such that $(G(F), Q(F), q_F) \sim (G, Q, q)$, then we say that the structure (G, Q, q) is *realized by the field* F. If for a quaternionic structure (G, Q, q) there is a mapping $\sigma: G \to G$, which is a Q-isomorphism of the structure (G, Q, q) onto itself, then σ is said to be a Q-*automorphism* or an *automorphism of the quaternionic structure* (G, Q, q). The group of all Q-automorphisms of (G, Q, q) will be denoted by $\mathrm{Aut}(G, Q, q)$.

If (G, Q, q) is the Q-structure of a Witt ring $W = (R, G)$ and $\sigma \in \mathrm{Aut}(G, Q, q)$, then the mapping $a_1 + \cdots + a_n \to \sigma(a_1) + \cdots + \sigma(a_n)$ is well defined Harrison automorphism of W. This establishes a canonical isomorphism of the group of Q-automorphisms of (G, Q, q) with the group $\mathrm{Aut}_H(W)$ of Harrison automorphisms of W.

Example 1.3. Let F be a field and $f \in \mathrm{Aut}\, F$. It is easy to see that the mapping $\sigma: G(F) \to G(F)$, $\sigma(aF^2) := f(a)F^2$ is a Q-automorphism of the structure $(G(F), Q(F), q_F)$.

Now we recall well-known facts concerning quaternionic structures, which will give us a convenient tool for future investigations.

Lemma 1.4. *Let* (G, Q, q) *and* (G', Q', q') *be* Q-*structures and let* $\varphi: G \to G'$ *be a group isomorphism with* $\varphi(-1) = -1'$. *Then*
(a) $b \in D(1, -a)$ *iff* $q(a, b) = \theta$ *for all* $a, b \in G$.
(b) φ *is a* Q-*isomorphism iff* $\varphi(D(1, a)) = D'(1', \varphi(a))$ *for every* $a \in G$.

Proof. see [2], Chapter 2, section 1. ■

A Q-structure (G, Q, q) is said to be of *local type* if G is finite and $|D(1, a)| = \frac{1}{2}|G|$ for all $-1 \neq a \in G$. Note that in this case $|Q| = 2$. A Witt ring is said to be of *local type* if the associated Q-structure is of local type.

Construction of the product of quaternionic structures

Let (G_k, Q_k, q_k), $1 \leq k \leq n$ be quaternionic structures, such that $-1_k \in G_k$, $\theta_k \in Q_k$. Put $G := G_1 \times \cdots \times G_n$, $Q := Q_1 \times \cdots \times Q_n$, $-1 := [-1_1, \ldots, -1_n]$, $\theta := [\theta_1, \ldots, \theta_n]$, $q : G \times G \to Q$, $q := q_1 \times \cdots \times q_n$, $q([a_1, \ldots a_n], [b_1, \ldots, b_n]) := [q_1(a_1, b_1), \ldots, q_n(a_n, b_n)]$. The triplet (G, Q, q) is a quaternionic structure called the *product of the quaternionic structures* and is denoted by $\prod_{k=1}^{n}(G_k, Q_k, q_k)$ or $(G_1, Q_1, q_1) \sqcap \cdots \sqcap (G_n, Q_n, q_n)$. Applying the lemma 1.4 one can conclude the expression of the value set of the binary forms (1-fold Pfister forms) in the product of Q-structures. It's $D(1, a) = D([1_1, \ldots, 1_n], [a_1, \ldots, a_n]) = D_1(1_1, q_1) \times \cdots \times D_n(1_n, a_n)$ for all $a \in G$.

Product of Witt rings

Let $(R_1, G_1), \ldots, (R_n, G_n)$ be Witt rings. Let R denote the subring of $R_1 \times \cdots \times R_n$ generated (additively) by $G_1 \times \cdots \times G_n$. The pair $W = (R, G)$ is called a *product of Witt rings*. Obviously the Q-structure associated to a product of Witt rings is isomorphic to the product of the Q-structures associated to the Witt rings which are the factors of the product.

2. Automorphisms of Q-structures and Witt rings

First we shall notice an easy

Proposition 2.1. *Let* $(G, \mathrm{Q}, q) := \prod_{k=1}^{n}(G', \mathrm{Q}', q')$ *be the product of* n *copies of a quaternionic structure* (G', Q', q'). *For every automorphisms* $\sigma_1, \sigma_2, \ldots, \sigma_n \in \mathrm{Aut}(G', \mathrm{Q}', q')$ *and every permutation* $\alpha \in S(n)$ *the mapping* $\sigma\colon G \to G$ *defined by*

$$\sigma([a_1, \ldots, a_n]) := [\sigma_1(a_{\alpha(1)}), \ldots, \sigma_n(a_{\alpha(n)})]$$

is a Q*-automorphism of the structure* (G, Q, q).

Proof. It is clear, that σ is well defined automorphism of the group G and $\sigma(-1) = -1$. Now let $a = [a_1, \ldots, a_n], b = [b_1, \ldots, b_n] \in G$ and let $q(a, b) = \theta$ (in the product). By definition of q it is equivalent to $q'(a_k, b_k) = \theta' \in G'$ for all $1 \leq k \leq n$. Since σ_k are Q-automorphisms, then $q'(a_k, b_k) = \theta'$ if and only if $q'(\sigma_k(a_l), \sigma_k(b_l)) = \theta'$ for all $1 \leq k \leq n$ and $1 \leq l \leq n$. In particular, applying the permutation $\alpha \in S(n)$ to this items we get that the above is equivalent to $[q'(\sigma_1(a_{\alpha(1)}), \sigma_1(b_{\alpha(1)})), \ldots, q'(\sigma_n(a_{\alpha(n)}), \sigma_n(b_{\alpha(n)}))] = \theta$ and by definition of q that means that $q([\sigma_1(a_{\alpha(1)}), \ldots, \sigma_n(a_{\alpha(n)})], [\sigma_1(b_{\alpha(1)}), \ldots, \sigma_n(b_{\alpha(n)})]) = \theta$. Finally by the definition of σ the last statement is equivalent to
$q(\sigma[a_1, \ldots, a_n], \sigma[b_1, \ldots, b_n]) = q(\sigma(a), \sigma(b)) = \theta$, as required. ■

Let (G, Q, q) be a Q-structure of local type. Since $|\mathrm{Q}| = 2$, one can regard the group G as a bilinear space (G, q) over the field $\mathbf{F_2}$ of order two with the non-degenerate bilinear mapping $q\colon G \times G \to \mathbf{F_2}$ (cf. [2]). Hence the group of automorphisms of the Q-structure (G, Q, q) is a group of automorphisms of the orthogonal space (G, q) over $\mathbf{F_2}$.

Let us consider the (finite) product of structures of local type $\prod_{k=1}^{n}(G_k, \mathrm{Q}_k, q_k)$. In the sequel the subgroup $\{1\} \times \cdots \times \{1\} \times G_k, \times \{1\} \times \cdots \times \{1\}$ of $G_1 \times \cdots \times G_n$ will be denoted by G'_k.

Lemma 2.2. *Let* (G_k, Q_k, q_k) *for* $k = 1, \ldots, n$ *be quaternionic structures of local type and* $\sigma \in \mathrm{Aut}(\prod_{k=1}^{n}(G_k, \mathrm{Q}_k, q_k))$. *For every* $k \in \{1, \ldots, n\}$ *there exists* $j \in \{1, \ldots, n\}$ *such that* $\sigma(G'_k) = G'_j$. *In this case* $(G_k, \mathrm{Q}_k, q_k) \cong (G_j, \mathrm{Q}_j, q_j)$.

Proof. Let $a = [a_1, \ldots, a_n] \in G$, $a \neq -1$. Since all G_k are of local type, $|D(1, a)| = |D_1(1_1, a_1) \times \cdots \times D_n(1_n, a_n)| = \frac{1}{2^{n-k}}|G_1 \times \cdots \times G_n|$ if k of $a_1, \ldots, a_n$ equals to -1. Since σ is an automorphism of a quaternionic structure, hence by Lemma 1.4(b) we get $\sigma([-1, \ldots, -1]) = [-1, \ldots, -1]$ and if $a_k \neq -1$, $a_k \in G_k$ for some $k \in \{1, \ldots, n\}$, then $\sigma([-1, \ldots, -1, a_k, -1, \ldots, -1]) = [-1, \ldots, -1, b_j, -1, \ldots, -1]$, where $-1 \neq b_j \in G_j$. It follows that for a_k as above $\sigma([1, \ldots, 1, -a_k, 1, \ldots, 1]) = [1, \ldots, 1, -b_j, 1, \ldots, 1]$ and $-b_j \neq 1$. This shows that for every $a_k \in G_k$ there exists $l \in \{1, \ldots, n\}$ such that

$$\sigma([1, \ldots, 1, a_k, 1, \ldots, 1]) \in G'_l. \tag{2.1}$$

Moreover, if $a_k \neq 1$ then $\sigma([1, \ldots, 1, a_k, 1, \ldots, 1]) \neq 1$. Assume, that for $a'_k, a''_k \in G_k \setminus \{1\}$, $k \in \{1, \ldots, n\}$
$\sigma([1, \ldots, 1, a'_k, 1, \ldots, 1]) = [1, \ldots, 1, d'_i, 1, \ldots, 1] \in G'_i$

$\sigma([1, \ldots, 1, a''_k, 1, \ldots, 1]) = [1, \ldots\ldots, 1, d''_j, 1, \ldots, 1] \in G'_j$
where $i \neq j$, $1 \leq i, j \leq n$.
Suppose, that $i < j$. Then
$\sigma([1, \ldots, 1, a'_k a''_k, 1, \ldots, 1]) =$
$= \sigma([1, \ldots, 1, a'_k, 1, \ldots, 1] \cdot [1, \ldots, 1, a''_k, 1, \ldots, 1]) =$
$= \sigma([1, \ldots, 1, a'_k, 1, \ldots, 1]) \cdot \sigma([1, \ldots, 1, a''_k, 1, \ldots, 1]) =$
$= [1, \ldots, 1, d'_i, 1, \ldots, 1] \cdot [1, \ldots, 1, d''_j, 1, \ldots, 1] =$
$= [1, \ldots, 1, d'_i, 1, \ldots, 1, d''_j, 1, \ldots, 1]$
which contradicts to (2.1), hence $i = j$ and consequently $\sigma(G'_k) \subseteq G'_j$.
Therefore we have $G'_k \subseteq \sigma^{-1}(G'_j)$. Analogously, one can show that there exists $l \in \{1, ..., n\}$ such that $\sigma^{-1}(G'_j) \subseteq G'_l$. This provides the inclusion $G'_k \subseteq \sigma^{-1}(G'_j) \subseteq G'_l$ and consequently $k = l$.

Now we show that the map $\varphi\colon G_k \to G_j$, $\varphi(a) = b$, where $\sigma([1, \ldots, a, \ldots, 1]) = [1, \ldots, b, \ldots, 1]$ is a Q-isomorphism of the Q-structures (G_k, Q_k, q_k) and (G_j, Q_j, q_j). Indeed, let $a_k, b_k \in G_k$, $a_j, b_j \in G_j$ be such that $\sigma([1, \ldots, a_k, \ldots, 1]) = [1, \ldots, a_j, \ldots, 1]$ and $\sigma([1, \ldots, b_k, \ldots, 1]) = [1, \ldots, b_j, \ldots, 1]$ and let $q_k(a_k, b_k) = \theta_k$. Then $[q(1,1), \ldots, q(a_k, b_k), \ldots, q(1,1)] = \theta$ in the product. Next we obtain $q([1, \ldots, a_k, \ldots, 1], [1, \ldots, b_k, \ldots, 1]) = \theta$ and by the property of σ we get $q(\sigma([1, \ldots, a_k, \ldots, 1]), \sigma([1, \ldots, b_k, \ldots, 1])) = \theta$, which means that $q([1, \ldots, a_j, \ldots, 1], [1, \ldots, b_j, \ldots, 1]) = \theta$. By the definition of q we have now $[q(1,1), \ldots, q(a_j, b_j), \ldots, q(1,1)] = \theta$ hence, in particular $q_j(a_j, b_j) = \theta_j$. Similarly one can prove the converse implication $q_j(a_j, b_j) = \theta_j \Rightarrow q_k(a_k, b_k) = \theta_k$.

Finally the fact that φ is a group isomorphism with $\varphi(-1) = -1$ is obvious. ■

Corollary 2.3. *Under the hypothesis of Lemma 2.2 there exists a permutation $\alpha \in S_n$ and Q-isomorphism $\sigma_{\alpha(i)}\colon G_i \to G_{\alpha(i)}$ for $i = 1, \ldots, n$ such that*

$$\sigma([a_1, \ldots, a_n]) = [\sigma_1(a_{\alpha^{-1}(1)}), \ldots, \sigma_n(a_{\alpha^{-1}(n)})]$$

for all $[a_1, \ldots, a_n] \in G_1 \times \cdots \times G_n$.

Now we can express the main result.

Theorem 2.4. *Let W be a finite product of Witt rings $W_1, \ldots, W_n$ of local type. Then* $\mathrm{Aut}_H(W) \cong \prod_{i=1}^n \mathrm{Aut}_H(W_i) \ltimes S_n$.

Proof. First we establish two preliminary results:

Claim 1: For any quaternionic structure (G, Q, q) of local type holds:

$$\mathrm{Aut}((G, Q, q)^n) \cong (\mathrm{Aut}(G, Q, q))^n \ltimes S_n$$

Define a map $\Phi\colon (\mathrm{Aut}(G, Q, q))^n \ltimes S_n \to \mathrm{Aut}(G, Q, q)^n$ by

$$\Phi(([\sigma_1, \ldots, \sigma_n], \alpha))([a_1, \ldots, a_n]) := \sigma$$

such that

$$\sigma([a_1, \ldots, a_n]) = [\sigma_1(a_{\alpha^{-1}(1)}), \ldots, \sigma_n(a_{\alpha^{-1}(n)})] \quad \text{for all} \quad [a_1, \ldots, a_n] \in G^n.$$

By the Proposition 2.1 we have that σ is the Q-automorphism of the Q-structure $(G, Q, q)^n$.

For proving that Φ is a group homomorphism compare the values of

$\Phi(([\sigma_1,\dots,\sigma_n],\alpha)*([\tau_1,\dots,\tau_n],\beta))$ and $\Phi(([\sigma_1,\dots,\sigma_n],\alpha)\circ\Phi([\tau_1,\dots,\tau_n],\beta))$ for all $[a_1,\dots,a_n]\in G^n$.

By the definition of operation $*$ in semidirect product of groups we get

$$\Phi([\sigma_1\circ\tau_{\alpha^{-1}(1)},\dots,\sigma_n\circ\tau_{\alpha^{-1}(n)}],\alpha\circ\beta)[a_1,\dots,a_n]=$$
$$[\sigma_1\circ\tau_{\alpha^{-1}(1)}(a_{(\alpha\circ\beta)^{-1}(1)}),\dots,\sigma_n\circ\tau_{\alpha^{-1}(n)}(a_{(\alpha\circ\beta)^{-1}(n)})].$$

On the other hand

$$\Phi([\sigma_1,\dots,\sigma_n],\alpha)\circ\Phi([\tau_1,\dots,\tau_n],\beta)[a_1,\dots,a_n]=$$
$$\Phi([\sigma_1,\dots,\sigma_n],\alpha)[\tau_1(a_{\beta^{-1}(1)}),\dots,\tau_n(a_{\beta^{-1}(n)})]=$$
$$[\sigma_1(\tau_{\alpha^{-1}(1)}(a_{\beta^{-1}(\alpha^{-1}(1))})),\dots,\sigma_n(\tau_{\alpha^{-1}(n)}(a_{\beta^{-1}(\alpha^{-1}(n))}))]=$$
$$[\sigma_1\circ\tau_{\alpha^{-1}(1)}(a_{(\alpha\circ\beta)^{-1}(1)}),\dots,\sigma_n\circ\tau_{\alpha^{-1}(n)}(a_{(\alpha\circ\beta)^{-1}(n)})]$$

like in the previous computing. This shows that Φ is a group homomorphism. ¿From Corollary 2.3 it follows that Φ is a surjection.

Now suppose that $([\sigma_1,\dots,\sigma_n],\alpha)\in(\mathrm{Aut}(G,\mathrm{Q},q))^n$ and that $\Phi([\sigma_1,\dots,\sigma_n],\alpha)$ is the identity map. Then for all $[a_1,\dots,a_n]\in G^n$ we have

$$\Phi([\sigma_1,\dots,\sigma_n],\alpha)([a_1,\dots,a_n])=[a_1,\dots,a_n] \tag{2.2}$$

Suppose that α is not the identity permutation, i.e. there exists i such that $\alpha^{-1}(i)\neq i$. Denote $\alpha^{-1}(i)=j$. Consider a sequence

$$a_l=\begin{cases}1 & \text{if } l\neq j\\ a_j & \text{if } l=j\end{cases}$$

where $a_j\neq 1$.
By (2.2) we get $\sigma_i(a_{\alpha^{-1}(i)})=a_i$ for all $1\le i\le n$. Hence $\sigma_i(a_j)=a_i=1$ since $i\neq j$. That contradicts to the choice of the element $[a_1,\dots,a_n]$. That means that α is identity. Therefore $\sigma_i(a_i)=a_i$ for all $1\le i\le n$, so σ is identity map and it follows that Φ is an injection. That finishes the prove that Φ is an isomorphism of groups $(\mathrm{Aut}(G,\mathrm{Q},q))^n\ltimes S_n$ and $\mathrm{Aut}(G,\mathrm{Q},q)^n$.

Claim 2: If the quaternionic structures (G,Q,q) and (G',Q',q') are equivalent, then $\mathrm{Aut}(G,\mathrm{Q},q)\cong\mathrm{Aut}(G',\mathrm{Q}',q')$.
Let $\varphi\colon G\to G'$ be the Q-isomorphism of the structures (G,Q,q) and (G',Q',q'). Let $\sigma\in\mathrm{Aut}(G,\mathrm{Q},q)$. Easy verification shows, that the mapping $\Psi\colon\mathrm{Aut}(G,\mathrm{Q},q)\to\mathrm{Aut}(G',\mathrm{Q}',q')$ defined by $\Psi=\varphi\circ\sigma\circ\varphi^{-1}$ is well defined isomorphism of the groups of automorphisms of the Q-structures.

Combining claims 1 and 2 we get that for the quaternionic structures (G_i,Q_i,q_i), $1\le i\le n$ of local type holds: $\mathrm{Aut}(\prod_{i=1}^n(G_i,\mathrm{Q}_i,q_i))\cong\prod_{i=1}^n(\mathrm{Aut}(G_i,\mathrm{Q}_i,q_i))\ltimes S_n$.

According to the mentioned one-to-one correspondence between automorphisms of Q-structures and the Harrison automorphisms of Witt rings we obtain the theorem. ∎

Now we investigate the group of automorphisms of a finite product of Witt rings of local type, in general. Consider a finite set $\mathcal{S}$ of Witt rings of local type. Divide these Witt rings into isomorphism classes $\mathcal{C}_1, \dots, \mathcal{C}_m$ of cardinality $k_1, \dots, k_m$, respectively. Choose a unique representative W_i of the class $\mathcal{C}_i$ for all $i = 1, \dots, m$.

Theorem 2.5. *Under the above notation*

$$\mathrm{Aut}_H(\prod_{W \in \mathcal{S}} W) \cong \prod_{i=1}^{m} (\mathrm{Aut}_H(W_i))^{k_i} \ltimes S_{k_i}.$$

Proof. If σ is a Harrison automorphism of $\prod_{W \in \mathcal{S}} W$ then by Lemma 2.2 $\sigma(W_i) \in \mathcal{C}_i$ for all $i = 1, \dots, m$. This implies that

$$\mathrm{Aut}_H(\prod_{W \in \mathcal{S}} W) \cong \prod_{i=1}^{m} \mathrm{Aut}_H(\prod_{W \in \mathcal{C}_i} W).$$

Now, by the previous theorem we get

$$\mathrm{Aut}_H(\prod_{W \in \mathcal{C}_i} W) \cong (\mathrm{Aut}_H(W_i))^{k_i} \ltimes S_{k_i},$$

which completes the proof. ■

The following corollary is an immediate consequence of Theorem 2.5.

Corollary 2.6. *Let W be a finite product of Witt rings $W_1, \dots, W_n$ of local type which are prairies non-isomorphic. Then* $\mathrm{Aut}_H(W) \cong \prod_{i=1}^{n} \mathrm{Aut}_H(W_i)$.

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Department of Early Education of Children, Higher Teacher Education School, Silesian University, in Częstochowa. Institute of Mathematics, in Katowice

E-mail address: m.stepien@wsp.czest.pl