Quadratic Extensions of Quintic Fields of signature (3,1)

Schehrazad Selmane

Abstract. This paper enumerates number fields of degree ten having discriminant less than $4 \cdot 10^{11}$ and containing a quintic field of signature $(3,1)$. It considers coincidences of discriminants and other interesting aspects of these fields.

In a supplementary section, we give examples of the first coincidence of discriminant of a maximum number of non-isomorphic fields with the same discriminant for each signature.

1. Introduction

We enumerate all number fields of degree ten of discriminant smaller than $4 \cdot 10^{11}$ in absolute value containing quintic fields having three real places. To establish such lists we have constructed all quadratic relative extensions of quintic fields, of signature $(3,1)$ and of discriminant smaller than 632391 in absolute value, using the number-geometric methods. For each one of the 2570 (*resp.* 12224, 14844, 4658) found number fields of signature $(0,5)$ (*resp.* $(2,4)$, $(4,3)$, $(6,2)$) of discriminant smaller than $4 \cdot 10^{11}$ in absolute value, the field discriminant, the quintic field discriminant, a polynomial defining the relative quadratic extension, the corresponding relative discriminant, the corresponding polynomial over $\mathbb{Q}$, and the Galois group of the Galois closure are given. We examined, in detail, the existence of several non-isomorphic fields, of composite fields, of unramified extensions as well as other characteristics.

Finally, in a supplementary section, we give for each signature of degree ten number fields containing a quintic field of signature $(3,1)$ having a fixed class number and narrow class number, the first coincidence of discriminant with a maximum number of non-isomorphic fields with the same discriminant, with a value of the discriminant out of the originally chosen bound.

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If L is a number field of degree n, we denote by ϑ_L its ring of integers, by d_L its discriminant and by h_L (*resp.* h_L^+) its class number (*resp.* narrow class number). For $\beta \in L$, we denote the corresponding conjugates by $\beta^{(1)}, ..., \beta^{(n)}$ and set $T_2(\beta) = \sum_{i=1}^{n} |\beta^{(i)}|^2$.

2. The method

To define a number field it is sufficient to give an irreducible polynomial a root of which is a primitive element of the field. We will show how to construct such polynomials using methods of the geometry of numbers which allowed us to find bounds for the coefficients of the searched polynomials.

Let K be a number field of degree $[K : \mathbb{Q}] = 10$, of signature (r, s) and of discriminant d_K such that $|d_K| \leq 4 \cdot 10^{11}$ containing a quintic field F having three real places. Theorem 2.8 of J. Martinet [3] asserts the existence of an integer $\theta \in K$, $\theta \notin F$ such that

- $K = F(\theta)$
- $Min(\theta, F) = P(x) = x^{2} + ax + b \in \vartheta_F[x]$
- $$\sum_{i=1}^{10} |\theta^{(i)}|^2 \leq \frac{1}{2} \sum_{i=1}^{5} |\sigma_i(a)|^2 + \left(\frac{|d_K|}{4\,|d_F|}\right)^{\frac{1}{5}} \qquad (1)$$

where σ_i $(1 \leq i \leq 5)$ denote the distinct $\mathbb{Q}$-isomorphisms of F into $\mathbb{C}$. Moreover the signature of K must be compatible with that of F, namely, $s \geq 2$.

To construct all polynomials P a root of which generates one of the searched fields K over F, we will work in the field F.

As the discriminants of K and F are related by

$$d_K = (-1)^s \cdot d_F^2 \cdot N_{F/\mathbb{Q}}(\delta_{K/F}), \qquad (2)$$

where $\delta_{K/F}$ is the relative discriminant of K over F, we must consider all quintic fields whose discriminant satisfy $|d_F| \leq 2 \cdot 10^{\frac{11}{2}}$ (all used quintic fields were taken in the tables of F. Diaz y Diaz, M. Pohst and A. Schwartz [2]). We assume that the discriminant d_F and an integral basis $W = \{w_1 = 1, w_2, ..., w_5\}$ of F are already known and let g be the minimal polynomial of ϱ, where $F = \mathbb{Q}(\varrho)$.

Choice of a

As inequality (1) remains valid by translation by an element of ϑ_F, we only have to make a run through a system of representatives of ϑ_F modulo $2\vartheta_F$, and therefore only 32 values should be considered for a :

$$a = \sum_{i=1}^{5} x_i w_i \quad \text{with } x_i \in \{0, 1\} \text{ for } i = 1, ..., 5.$$

The running time for the computation of the possible b's, for a fixed value of a, strongly depends upon the size of the real constant $\kappa = \frac{1}{2} \sum_{i=1}^{5} |\sigma_i(a)|^2 + \left(10^{11}\,|d_F|^{-1}\right)^{\frac{1}{5}}$. Let us show that the coefficient a can be chosen such that κ is minimal.

Indeed, as inequality (1) remains valid if we exchange θ by $\theta+\gamma$ for an arbitrary $\gamma \in \vartheta_F$, and as $\theta + \gamma$ is also a generator of the extension K/F, then if we set $\beta = -Tr_{K/F}(\theta + \gamma) = a - 2\gamma$ and represent it by means of the basis W of F as

$$\beta = \sum_{i=1}^{5} \beta_i w_i,$$

then $T_2(\beta)$ becomes a positive definite quadratic form

$$q(\beta) = \beta \cdot A \cdot \beta^t$$

in the coefficients $\beta_1, ..., \beta_5 \quad (c = (\beta_1, ..., \beta_5))$, where $A = B \cdot \overline{B^t}$ and B is the matrix whose (i, j) entry is $w_i^{(j)}$. Moreover there exists at least one choice of γ which makes $T_2(\beta)$ minimal.

To obtain such an integer γ, we start by computing the coefficients of the real symmetric matrix $A = (m_{i\,j}) \left(m_{i\,j} = \sum_{k=1}^{5} w_i^{(k)} \overline{w_j}^{(k)} \qquad 1 \le i, j \le 5\right)$ of the quadratic form q, then we decompose the matrix A into a sum of squares by Cholesky's method [4]

$$q(\beta) = \sum_{i=1}^{5} m_{ii} \left(\beta_i + \sum_{j=i+1}^{5} m_{ij}\beta_j\right)^2$$

finally, we make $\beta_1, ..., \beta_5$ run through the integer values for which $q(\beta) \le \kappa$ and for which the relationship $\beta \equiv a \pmod{2\vartheta_F}$ is satisfied. We shall attach to a the value of β for which $q(\beta)$ is minimal and we shall set $C = \frac{1}{2}\sum_{i=1}^{5} |\sigma_i(a)|^2 + \left(10^{11}\, |d_F|^{-1}\right)^{\frac{1}{5}}$.

Remark 2.1. *The choice of a is independent of the signature of K, it is also independent of the chosen bound, it depends only on the field F.*

Choice of b

Once a convenient value of a is determined, we evaluate the possible values of b from the inequality

$$\sum_{i=1}^{5} |\sigma_i(s_2)|^2 \le C^2$$

which comes from inequality (1) and the inequality

$$\sum_{i=1}^{5} |\sigma_i(s_2)| \le T_2(\theta)$$

where s_2 denotes the second relative symmetric function. As $\sum_{i=1}^{5} |\sigma_i(s_2)|^2$ is just the quadratic form $q(y)$ in the variables $y_1, ..., y_5$ $(y = (y_1, ..., y_5))$ where we represented s_2 by means of the basis $\mathcal{W}$ in the form $s_2 = \sum_{i=1}^{5} y_i w_i$. Then, as we have already computed the coefficients matrix A and decomposed it into a sum of squares, we compute all $y \in \mathbb{Z}^5$ subject to $q(y) \leq C^2$ and such that $s_2 \equiv a^2 \mathrm{mod} 2\vartheta_F$. Thus we obtain the set of suitable values for $b = (a^2 - s_2)/2$.

For each of the polynomials constructed, we computed the conjugates of the polynomial discriminant $\Delta = a^2 - 4b$, on the one hand to determine the signature of the field K and on the other hand to eliminate polynomials having too large value of $T_2(\theta)$ by checking whether the inequality

$$\sum_{i=1}^{5} |\Delta^{(i)}| \leq 2\left(10^{11} \cdot |d_F|^{-1}\right)^{\frac{1}{5}}$$

is fulfilled since $T_2(\theta) = \sum_{i=1}^{10} |\theta^{(i)}|^2 = \frac{1}{2} \sum_{i=1}^{5} \left(|a^{(i)}|^2 + |\Delta^{(i)}|\right)$.

As the polynomial P is irreducible, it remains us to compute the discriminant d_K to see whether K lies within the limits of the search. The determination of the relative discriminant allows us to obtain the value of d_K directly. For each value of d_K such that $|d_K| \leq 4 \cdot 10^{11}$ we found several polynomials. To determine sets of polynomials generating isomorphic fields we used the function OrderIsSubfield implemented in KANT-Kash [1].

3. Description of results

In this section, we describe our experiences provided by the computation and we give several tables illustrating some of the obtained results.

In Table 1, we present the number of fields that were constructed, the smallest discriminant, and the number of discriminants for which there are exactly k non-isomorphic fields having the same discriminant.

The number nb of fields K containing, in addition to the quintic field F, the quadratic field L and the number nb_1 of composite fields K ($K = F \cdot L$ and $F \cap L = \mathbb{Q}$) are given in Table 2.

In Table 3 we present the number nb of fields having discriminant of the form m^i ($i \geq 2$); for $i \geq 3$ the values of discriminants are given.

Proposition 18. *For each quintic field F of signature $(3,1)$, of discriminant less than $2 \cdot 10^{\frac{11}{2}}$ in absolute value, having class number 1 and narrow class number 2 (resp. class number 2) there exists a relative quadratic extension K/F of absolute discriminant $d_K = d_F^2$ and signature $(2,4)$ (resp. $(6,2)$).*

(h_F, h_F^+)	*Number of quintic fields*	*Number of unramified extensions*	
		$(6,2)$	$(2,4)$
$(1,2)$	1140	0	1140
$(2,2)$	28	28	0
$(2,4)$	8	8	0

Proposition 19. *For each quintic field F of signature $(3,1)$ and of discriminant d_F such that $|d_F| \leq 19015$ in which 2 remains inert there exists* 6 *(resp.* 6, 2,

1) *non-isomorphic fields* K *of signature* $(2,4)$ (*resp.* $(4,3)$, $(0,5)$, $(6,2)$) *and of discriminant* $d_K = (-1)^s \cdot 2^{10} \cdot d_F^2$.

Remark 3.1. *The polynomials* f *defining the fields* K, *in proposition 3.2, are even and the corresponding* $f(x^{\frac{1}{2}})$ *are just the polynomials defining the quintic subfields.*

TABLE 1

signature	(6, 2)	(4, 3)	(2, 4)	(0, 5)
number of fields	2570	12224	14844	4658
d_K minimal	15417264029	−3120654523	836077225	−379908823
k	Number of k non-isomorphic fields			
2	98	1484	1763	204
3	3	249	249	6
4	3	30	50	2
5	–	8	5	–
6	–	41	42	–
7	–	2	1	–
8	–	1	1	–
12	–	3	3	–

TABLE 2

(6, 2)					
d_L	5	8	13	17	29
nb	38	5	1	2	1
nb_1	14	–	–	–	–
(0, 5)					
d_L	−3	−4	−7	−11	−19
nb	376	196	28	6	2
nb_1	89	31	1	–	–

TABLE 3

(r,s)	(0, 5)	(2, 4)	(4, 3)	(6, 2)
m^2				
nb	409	2773	818	175
m^3				
nb	2	2	1	1
d_K	−117865222327 −399826899863	71421719949 71421719949	−399826899863	218275947261
m^4				
nb	1	0	1	1
d_K	−256992219136	–	−256992219136	256992219136

4. Supplement

Although Hermite showed that the set of isomorphism classes of number fields of given discriminant is finite, however, for a fixed degree and signature, the maximum number of non-isomorphic fields with the same discriminant is not known generally. We especially are interested in the determination of a maximum number of number fields of degree ten having a given discriminant containing a quintic field of signature $(3,1)$ having a fixed class number and narrow class number. With the chosen bound $\Im = 4 \cdot 10^{11}$ we have obtained up to 12 non-isomorphic fields for the signatures $(4,3)$ and $(2,4)$ as a maximum number of non-isomorphic fields with the same discriminant. The obtained 12 non-isomorphic fields have a value of discriminant of the form

$$|d_K| = 2^{10} \cdot p^{f_i} \cdot d_F^2$$

where p is a prime number such that $p\vartheta_F = \prod_{i=1}^{l} \wp_i^{e_i}$ with $N(\wp_i) = p^{f_i}$ and where 2 remains inert in the quintic field F.

In the table below, we give, for each possible value of (h_F, h_F^+), the number of fields of the first coincidence of discriminant.

$\mathcal{N}$umber of $\mathcal{F}$ields with the same $\mathfrak{D}$iscriminant $d_K = 2^{10} \cdot p \cdot d_F^2$

(h_F, h_F^+)	d_F	p	$(0,5)$	$(2,4)$	$(4,3)$	$(6,2)$
$(1,1)$	-7367	277	10	29	30	10
$(1,2)$	-39231	173	12	28	28	12
$(2,2)$	-550151	709	12	**36**	**36**	12
$(2,4)$	-936823	137	**16**	32	32	**16**

To obtain such number of non-isomorphic fields, we looked for the first prime number which splits completely in a quintic field of discriminant $|d_F| \leq 10^6$ where 2 remains inert and such that $p \cdot d_F^2$ is as small as possible.

Notice that the maximum number η of non-isomorphic fields with the same discriminant $2^{10} \cdot p \cdot d_F^2$ is obtained with the signature (r,s) such that $|r-3| = 1$. For the corresponding signature, namely $(6-r, \frac{1}{2}r+2)$ we obtain β ($\beta \leq \eta$) non-isomorphic fields. Moreover, it is not necessary to do the computation for the determination of polynomials defining the β non-isomorphic fields with the same discriminant of signature $(6-r, \frac{1}{2}r+2)$. Indeed, if we denote by

$$f_i(x) = x^{10} + a_1x^8 + a_2x^6 + a_3x^4 + a_4x^2 + a_5 \quad (i = 1, ..., \eta)$$

the η non-isomorphic fields with the same discriminant of discriminant $|d_K| = 2^{10} \cdot p \cdot d_F^2$, then among the η polynomials g_i $(1 \leq i \leq \eta)$

$$g_i(x) = x^{10} - a_1x^8 + a_2x^6 - a_3x^4 + a_4x^2 - a_5$$

β polynomials define non-isomorphic number fields with the same discriminant of signature $(6-r, \frac{1}{2}r+2)$ and discriminant $|d_K| = 2^{10} \cdot p \cdot d_F^2$ and $(\eta - \beta)$ polynomials define non-isomorphic fields with the same discriminant of signature $(6-r, \frac{1}{2}r+2)$ and discriminant $|d_K| = p \cdot d_F^2$. Moreover $f_i(x^{\frac{1}{2}})$ and the corresponding $g_i(x^{\frac{1}{2}})$ are just the polynomials defining the quintic subfield F.

We summarize below results given in papers [6], [7] and the present one ($L.B.O$: denotes the Odlyzko lower bound for $|d_K|^{1/10}$ [8]).

$\mathcal{Q}$uadratic $\mathcal{E}$xtensions of $\mathcal{Q}$uintic $\mathfrak{F}$ields

(r,s)	$(10,0)$	$(8,1)$	$(6,2)$	
$(r\prime,s\prime)$	$(5,0)$	$(5,0)$	$(5,0)$	$(3,1)$
chosen bound	10^{13}	10^{13}	10^{13}	4.10^{11}
number of fields	313	2845	7579	2570
d_K minimal	443952558373	-70952789611	15417264029	
$\lvert d_K\rvert^{1/10}$	14.613	12.165	10.442	
$L.B.O$	13.744	11.805	10.182	

(r,s)	$(4,3)$		$(2,4)$			$(0,5)$		
$(r\prime,s\prime)$	$(5,0)$	$(3,1)$	$(5,0)$	$(3,1)$	$(1,2)$	$(5,0)$	$(3,1)$	$(1,2)$
chosen bound	10^{13}	4.10^{11}	10^{13}	4.10^{11}	10^{11}	10^{13}	4.10^{11}	10^{11}
number of fields	7420	12224	3950	14844	18142	770	4658	21452
d_K minimal	-3120654523		799905449			-209352647		
$\lvert d_K\rvert^{1/10}$	8.9007		7.7679			6.7934		
$L.B.O$	8.824		7.685			6.730		

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15 Rue Lahcene Mimouni, Place du 1 er Mai, Algiers, Algeria

E-mail address: sc_selmane@hotmail.com