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Weakly Associative Lattice Rings

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Abstract: The notion of a weakly associative lattice ring (*wal*-ring) is a generalization of that of a lattice ordered ring in which the identities of associativity of the lattice operations join and meet are replaced by the identities of weak associativity. In the paper some properties of *wal*-rings are shown. *Wal*-ideals are described and straightening, irreducible and semimaximal ideals are introduced and studied.

Key Words: Weakly associative lattice ring, *wal*-ideal, straightening ideal, irreducible ideal, semimaximal ideal

Mathematics Subject Classification: 06F25

1. Basic Notions

1.1. Basic Properties

A *semi-order* of a non-void set A is any reflexive and antisymmetric binary relation on A. If $\leq$ is a semi-order of A, then the pair $(A, \leq)$ is called a *semi-ordered set (so-set)*.

A *weakly associative lattice (wa-lattice)* is an algebra $A = (A, \wedge, \vee)$ with two binary operations satisfying the identities

(I)	$a \vee a = a\,;$	$a \wedge a = a.$
(C)	$a \vee b = b \vee a\,;$	$a \wedge b = b \wedge a.$
(Abs)	$a \vee (a \wedge b) = a\,;$	$a \wedge (a \vee b) = a.$
(WA)	$((a \wedge c) \vee (b \wedge c)) \vee c = c\,;$	$((a \vee c) \wedge (b \vee c)) \wedge c = c.$

This notion has been introduced by E. Fried in [Fr70] and by H. L. Skala in [Sk71] and [Sk72]. It is obvious that the notion of a *wa*-lattice is a generalization of that of a lattice because the identities of associativity of the operations $\vee$ and $\wedge$ are replaced by weaker conditions of weak associativity (WA). Similarly as for lattices we can define also for *wa*-lattices a binary relation $\leq$ on A as follows:

$$a, b \in A;\ a \leq b \iff_{def} a \wedge b = a \quad \text{(or equivalently } a \leq b \iff_{def} a \vee b = b).$$

This relation is reflexive and antisymmetric (i.e. $\leq$ is a semi-order) and every two-element subset $\{a, b\} \subseteq A$ has the join $\sup\{a, b\} = a \vee b$ and the meet $\inf\{a, b\} = a \wedge b$ in A. Moreover (also as for lattices), each such binary relation defines on A a structure of a *wa*-lattice. So, from the point of view of the relation theory, the notion of a weakly associative lattice is based on semi-order relations.

A semi-ordered set $(A, \leq)$ is said to be a *totally semi-ordered set (tournament)* if any elements $a, b \in A$ are comparable, that means

$$\forall a, b \in A;\ a \leq b \ \text{ or } \ b \leq a.$$

A tournament is a special case of a *wa*-lattice.

Definition. A system $G = (G, +, \leq)$ is called a *semi-ordered group (so-group)* if

(G1) $G = (G, +)$ is a group;

(G2) $(G, \leq)$ is a semi-ordered set;

(G3) $\forall a, b, c, d \in G;\ a \leq b \implies c + a + d \leq c + b + d.$

If $(G, \leq)$ is a *wa*-lattice, then we say that $G = (G, +, \leq)$ is a *weakly associative lattice group (wal-group).*

Definition. A system $R = (R, +, \cdot, \leq)$ is called a *semi-ordered ring (so-ring)* if

(R1) $(R, +, \cdot)$ is a (associative) ring;

(R2) $(R, \leq)$ is a semi-ordered set;

(R3) $\forall a, b, c \in R;\ a \leq b \implies a + c \leq b + c;$

(R4) $\forall a, b, c \in R;\ 0 \leq c,\ a \leq b \implies ac \leq bc$ and $ca \leq cb$.

If $(R, \leq)$ is a *wa*-lattice, then we say that $R = (R, +, \cdot, \leq)$ is a *weakly associative lattice ring (wal-ring).* If $(R, \leq)$ is a lattice, then $R = (R, +, \cdot, \leq)$ is said to be a *lattice ordered ring (l-ring).* If for *wal*-ring R the corresponding *wa*-lattice $(R, \leq)$ is a tournament, then R is called a *totally semi-ordered ring (to-ring).*

The axiom (R3) expresses that the additive group $R = (R, +, \leq)$ of a *so*-ring $R = (R, +, \cdot, \leq)$ is a semi-ordered group. Each commutative *so*-group can be studied as a *so*-ring; it is sufficient to define multiplication on R by $ab = 0$ for any $a, b \in R$.

(For some properties concerning of *so*-groups and *wal*-groups see [Ra79] and [Ra92], for these of *l*-rings see [BiKeWo77].)

All what is known about *so*-groups and *wal*-groups, respectively in [Ra79] and [Ra92] holds in additive groups of *so*-rings and *wal*-rings, respectively. In particular, knowledge of the following propositions will be useful for our examples and further explanation. (See [Ra79], Th. 7 and Th. 4, and [Ra92] Prop .1.5.)

Proposition 1.1.1. *If $(G, +, \leq)$ is a so-group, then the following conditions are equivalent:*

(1) G is a wal-group.

(2) For each $g \in G$ there exists $g \vee 0$.

Proposition 1.1.2. *Let $G = (G, +, \leq)$ be a so-group, A a subgroup of G. Then A is convex if and only if $0 \leq x,\ x \leq a$ imply $x \in A$ for each $a \in A$, $x \in G$.*

Proposition 1.1.3. *Let for elements x, y in a so-group G $x \wedge y$ exist. Let $x = a + $ $+ (x \wedge y)$, $y = b + (x \wedge y)$, $z = x - y$. Then $a \wedge b = 0$, $a - b = z$, $a = z \vee 0$, $b = -z \vee 0$.*

Let us set out elementary properties of a *wal*-ring.

Proposition 1.1.4. *Let R be a wal-ring. Then for any a, b, $c \in R$ it holds:*

(1) $c + (a \wedge b) = (c + a) \wedge (c + b)$;
(2) $c + (a \vee b) = (c + a) \vee (c + b)$;
(3) $a \wedge b = -(-a \vee -b)$;
(4) $a + b = (a \wedge b) + (a \vee b)$;
(5) $c \geq 0 \Rightarrow \; ac \vee bc \leq (a \vee b)c,$
$ca \vee cb \leq c(a \vee b),$
$(a \wedge b)c \leq ac \wedge bc,$
$c(a \wedge b) \leq ca \wedge cb.$

Proof. The properties (1) – (3) are shown in [Ra79] Th. 6. The property (4) we obtain in the following way: $a \wedge b = (b - b + a) \wedge (b - a + a) = b + (-b \wedge -a) + $ $+ a = b - (a \vee b) + a$ by applying (1) and (3) gradually. From the commutativity of the ring addition, it follows that $a + b = (a \wedge b) + (a \vee b)$. We verify (5). Let a, $b \in R$, then $a \leq a \vee b$ and $b \leq a \vee b$. If $c \geq 0$, then $ac \leq (a \vee b)c$ and $bc \leq (a \vee b)c$. Hence $ac \vee bc \leq (a \vee b)c$. Similarly the other inequalities. □

Definition. Let R be a *so*-ring. Denote $R^+ = \{x \in R;\ 0 \leq x\}$, R^+ will be called the *positive cone of R*.

Here are some elementary properties of this concept.

Proposition 1.1.5. *a) Let $R = (R, +, \cdot, \leq)$ be a so-ring. The positive cone R^+ has the following properties*

(1) $R^+ \cap -R^+ = \{0\}$
(2) $R^+ \cdot R^+ \subseteq R^+$

b) If $(R, +, \cdot)$ is a ring, P a subset with 0 in R, $P \subseteq R$ satisfies (1) and (2), then $R = (R, +, \cdot, \leq)$, where $a \leq b$ iff $b - a \in P$ for all a, $b \in R$, is a so-ring and $R^+ = P$.

Proof. a) The property (1) is obvious. Let a, $b \in R^+$, i. e. $a \geq 0$, $b \geq 0$. Applying (R4) yields $0 \cdot b \leq a \cdot b$. That means $0 \leq ab$ and so $R^+ \cdot R^+ \subseteq R^+$.

b) Let P satisfy the above assumptions. We first prove that relation $\leq$ defined by means of P is reflexive and antisymmetric. So that $a - a = 0 \in P$, thus $a \leq a$ for any $a \in R$. Let $a \leq b$ and $b \leq a$, then $b - a \in P$ and $-(b - a) = a - b \in P$. From (1) it follows that $b - a = 0$, hence $a = b$. It remains to verify (R3) and (R4).

Let a, b, $c \in R$, $a \leq b$, hence $b - a \in P$. But $b - a = b + c - c - a = (b + c) - (a + c)$, thus $(b + c) - (a + c) \in P$, too. Therefore $a + c \leq b + c$, (R3) is satisfied.

Let a, b, $c \in R$, $a \leq b$, $0 \leq c$. Hence $0 \leq b - a \in P$, $0 \leq c \in P$. According (2) we have $0 \leq (b - a)c \in P$, thus $0 \leq bc - ac \in P$ and $ac \leq bc$. Similarly $ca \leq cb$. The proof is complete. □

1.2. Examples

In contrast to lattice ordered rings (l-rings), there are many non-trivial finite *so*-rings and *wal*-rings.

Example 1.2.1. Let us consider the ring $\mathbb{Z}_3 = \{0, 1, 2\}$ with the addition and multiplication mod3. We denote $R = (R, +, \cdot) = (\mathbb{Z}_3, +, \cdot)$, $\mathbb{Z}_3^+ = R^+ = \{0,1\}$. It is clear that $\mathbb{Z}_3^+$ is the positive cone of a total semi-order of the ring $\mathbb{Z}_3$.

Example 1.2.2. Let us consider the ring $(\mathbb{Z}_5, +, \cdot)$, $\mathbb{Z}_5^+ = \{0,1\}$. It is obvious that $\mathbb{Z}_5^+$ defines a semi-order of the ring $\mathbb{Z}_5$. But this semi-order is not weakly associative lattice, because e.g. $2 \vee 0$ does not exist in the ring $\mathbb{Z}_5$.

Remark 1.2.3. $\{0, 1\}$ is the non-trivial positive cone on every ring $\mathbb{Z}_n$, $n > 2$. So we will not mention it further.

We give the following examples briefly.

Example 1.2.4. The ring $(\mathbb{Z}_7, +, \cdot)$

a) with the positive cone $\mathbb{Z}_7^+ = \{0, 1, 2, 4\}$ is a *to*-ring.

b) with the positive cone $\mathbb{Z}_7^+ = \{0, 1, 5\}$ is a *wal*-ring, not a *to*-ring.

Example 1.2.5. The ring $(\mathbb{Z}_9, +, \cdot)$

a) with the positive cone $\mathbb{Z}_9^+ = \{0, 1, 3, 4, 7\}$ is a *to*-ring.

b) with the positive cone $\mathbb{Z}_9^+ = \{0, 1, 4, 7\}$ is a *so*-ring, not a *wal*-ring.

c) with the positive cone $\mathbb{Z}_9^+ = \{0, 1, 3\}$ is a *so*-ring, not a *wal*-ring.

d) with the positive cone $\mathbb{Z}_9^+ = \{0, 1, 6\}$ is a *so*-ring, not a *wal*-ring.

e) with the positive cone $\mathbb{Z}_9^+ = \{0, 3\}$ is a *so*-ring, not a *wal*-ring.

f) with the positive cone $\mathbb{Z}_9^+ = \{0, 6\}$ is a *so*-ring, not a *wal*-ring.

Example 1.2.6. The Galois field $\mathbb{F}_8$ does not admit non-trivial semi-orders because its characteristic is 2 and so each element is opposite to itself.

Example 1.2.7. The Galois field $\mathbb{F}_9$ has the only non-trivial positive cone of a semi-order $\mathbb{F}_9{}^+ = \{0, 1\}$.

Example 1.2.8. The ring $R = (\mathbb{Z}, +, \cdot)$

a) with the positive cone $R^+ = \{0, 1, 2, 4, 6, \dots\}$ is a *wal*-ring, not a *to*-ring.
If $x \in R$ then it holds:
1) $x \in R^+ \Rightarrow x \vee 0 = x$;
2) $-x \in R^+ \Rightarrow x \vee 0 = 0$;
3) $x \notin R^+,\ -x \notin R^+ \Rightarrow x \vee 0 = max\{x, 0\} + 1$,
where $max\{x, 0\}$ is meant in the natural ordering of $\mathbb{Z}$.

b) with the positive cone $R^+ = \{0, 1\}$ is a *so*-ring, not a *wal*-ring.

The following example is an illustration of an infinite *to*-ring which is not an *o*-ring.

Example 1.2.9. Let us consider the ring $R = (\mathbb{Z}, +, \cdot)$ and define its positive cone R^+ as follows:

1) $0, 1 \in R^+$.
   Let $1 \neq n \in \mathbb{N}$.
2) If n is the product of an odd number of prime factors (for example $12 = 2 \cdot 2 \cdot 3$), then $-n \in R^+$.
3) If n is the product of an even number of prime factors, then $n \in R^+$. That means

$$R^+ = \{0,\ 1, -2, -3,\ 4, -5,\ 6, -7, -8,\ 9,\ 10, -11, -12, -13,\ 14,\ 15,\ 16, -17, \dots\}.$$

Then R^+ defines a total semi-order of the ring R. However, it is not a linear order because e.g. $4 \leq 1,\ 1 \leq -2$ but $4 \geq -2$.

Example 1.2.10. The ring of diagonal matrices of degree n over a division *to*-ring is a *so*-ring with the positive cone as follows:

$$\mathbf{M} = (a_{ij}) \geq 0 \text{ iff } a_{ij} \geq 0 \text{ for every } i,\ j.$$

1.3. Direct Products

Let us consider a family $\{R_i\,;\ i \in I\}$ of semi-ordered rings. The *direct product*, denoted by $R = \prod_{i \in I} R_i$, is the ring whose elements are all $(a_i)_{i \in I}$ in the cartesian product of the R_i and whose operations are

$$(a_i)_{i \in I} + (b_i)_{i \in I} = (a_i + b_i)_{i \in I};$$
$$(a_i)_{i \in I} \cdot (b_i)_{i \in I} = (a_i \cdot b_i)_{i \in I}.$$

We define a relation $\leq$ in R:

If $a = (a_i)_{i \in I}$ and $b = (b_i)_{i \in I}$, $a \leq b \iff_{def} a_i \leq_i b_i$ for every $i \in I$.

This relation is a semi-order.

If we suppose every R_i to be a *wal*-ring then R is the *wal*-ring and

$$a \vee b = (a_i \vee_i b_i)_{i \in I}\,,\ a \wedge b = (a_i \wedge_i b_i)_{i \in I}.$$

1.4. Homomorphisms

Let $R = (R, +, \cdot, \leq)$ be a *so*-ring, $\emptyset \neq A \subseteq R$. Then we say that A is a *convex subset* of R if $a \leq x,\ x \leq b$ imply $x \in A$ for all $a,\ b \in A,\ x \in R$. An ideal I of the ring R is called a *convex ideal* of R if I is a convex subset of R.

Let $R = (R, +, \cdot, \leq)$ be a *wal*-ring, S a subring of R. Then we say that S is a *wal-subring* of R, if S is a *wa*-sublattice of $(R, \leq)$.

Let $(R, +, \cdot, \leq)$ and $(R', +, \cdot, \leq)$ be *so*-rings. A mapping $h : R \longrightarrow R'$ will be called a *so-homomorphism* $(R, +, \cdot, \leq) \longrightarrow (R', +, \cdot, \leq)$ if h is a ring homomorphism $(R, +, \cdot) \longrightarrow (R', +, \cdot)$ and simultaneously h is a homomorphism $(R, \leq) \longrightarrow (R', \leq)$ (i.e. $a \leq b$ implies $h(a) \leq h(b)$ for all $a, b \in R$).

Theorem 1.4.1. *Let $R = (R, +, \cdot, \leq)$ be a so-ring. Then an ideal I of the ring R is the kernel of a so-homomorphism if and only if I is convex.*

Proof. a) Let $h : R \longrightarrow R'$ be a *so*-homomorphism, $0'$ the zero-element in R'. Let $I = \operatorname{Ker} h$. Assume $a \in I$, $x \in R$, $0 \leq x$, $x \leq a$. Then $h(0) \leq h(x)$, $h(x) \leq h(a)$, i.e. $0' \leq h(x)$, $h(x) \leq 0'$, hence $h(x) = 0'$, from this $x \in I$.

b) Let I be a convex ideal of R, $\bar{R} = R/I$. Let us consider the relation $\leq$ on $\bar{R}$ defined as: $x + I \leq y + I \iff_{def}$ there exists $a \in I$ such that $x + a \leq y$. We must show correctness of this definition. Suppose that $x, x_1, y, y_1 \in R$ and that $x_1 + I = x + I$, $y_1 + I = y + I$. Then there exist $b, c \in I$ such that $x_1 + b = x$, $y_1 + c = y$, i.e. $x_1 + b + a \leq y_1 + c$. From this $x_1 + (b + a - c) \leq y_1$ and hence $x_1 + I \leq y_1 + I$.

The reflexivity of $\leq$ is evident. We show that $\leq$ is antisymmetric. Let $x, y \in R$, $x + I \leq y + I$, $y + I \leq x + I$. Then there exist $a, b \in I$ such that $x + a \leq y$, $y + b \leq x$. From this $y + b + a \leq x + a$, $x + a \leq y$, thus $b + a \leq -y + x + a$, $-y + x + a \leq 0$. Since I is convex, $-y + x + a \in I$. Therefore $-y + x \in I$, and so $x + I = y + I$.

We now suppose $x, y, z \in R$, $x + I \leq y + I$. Then there exists $a \in I$ such that $x + a \leq y$. Thus $x + a + z \leq y + z$ and since the addition in R is commutative, $x + z + a \leq y + z$. Therefore $(x + I) + (z + I) \leq (y + I) + (z + I)$.

It remains to prove the monotony rule of the multiplication by a positive element. Let $x, y, z \in R$, $x + I \leq y + I$, $0 + I \leq z + I$. Then there exist $a, b \in I$ such that $a \leq z$, $x + b \leq y$. By this $x + b \leq y$, $0 \leq z - a$, thus $(x + b)(z - a) \leq y(z - a)$. Hence $xz + bz - xa - ba \leq yz - ya$, $xz + bz - xa - ba + ya \leq yz$. Let $c = bz - xa - ba + ya$. Then $c \in I$ because I is an ideal, thus $(x + I)(z + I) \leq (y + I)(z + I)$. Similarly $(z + I)(x + I) \leq (z + I)(y + I)$. Thus R/I is a *so*-ring.

Finally, it is obvious that the natural mapping $\nu : R \longrightarrow R/I$ is a *so*-homomorphism. □

The semi-order $\leq$ of the quotient ring R/I defined in the proof of the previous theorem is called the *induced semi-order*.

Definition. Let $R = (R, +, \cdot, \leq)$ be a *wal*-ring and I an ideal of R. If a convex ideal I is a *wa*-sublattice of $(R, \leq)$ and satisfies the condition:

(I_{wal}) For any $a, b \in I$, $x, y \in R$ such that $x \leq a$, $y \leq b$ there exists $c \in I$ such that $x \vee y \leq c$,

then I is called a *wal-ideal* of R.

Let $(R, +, \cdot, \leq)$ and $(R', +, \cdot, \leq)$ be *wal*-rings. A mapping $h : R \longrightarrow R'$ will be called a *wal-homomorphism* $(R, +, \cdot, \leq) \longrightarrow (R', +, \cdot, \leq)$ if simultaneously h is a ring homomorphism $(R, +, \cdot) \longrightarrow (R', +, \cdot)$ and a *wa*-lattice homomorphism $(R, \leq) \longrightarrow (R', \leq)$.

It is evident that each *wal*-homomorphism is a *so*-homomorphism.

Theorem 1.4.2. *Let $R = (R, +, \cdot, \leq)$ be a wal-ring. A subset $L \subseteq R$ is a wal-ideal if and only if L is the kernel of a wal-homomorphism.*

Proof. Let R, R' be *wal*-rings and $h : R \longrightarrow R'$ be a *wal*-homomorphism. Let $0'$ be the zero-element in R'. Let $L = \operatorname{Ker} h$. By Theorem 1.4.1, L is convex. Let $a, b \in L$, then $h(a \vee b) = h(a) \vee h(b) = 0' \vee 0' = 0'$, in this way $a \vee b \in L$. Let $a, b \in L$, $x, y \in R$; $x \leq a$, $y \leq b$. Then $h(x) \leq h(a) = 0'$, $h(y) \leq h(b) = 0'$, from this $h(x \vee y) = h(x) \vee h(y) \leq 0'$, hence $h(x \vee y) \vee 0' = 0'$. Let $d \in L$. Then $h((x \vee y) \vee d) = h(x \vee y) \vee h(d) = h(x \vee y) \vee 0' = 0'$ and so $(x \vee y) \vee d \in L$. From this the existence of $c \in L$ such that $(x \vee y) \vee d = c$ follows. Consequently, $x \vee y \leq c$.

Conversely, let L be a *wal*-ideal of R. By the proof of Theorem 1.4.1, R/L is a *so*-ring with respect to the induced semi-order. Suppose that $x, y \in R$. Then $x + L \leq (x \vee y) + L$ and $y + L \leq (x \vee y) + L$. Let $z \in R$ be such that $x + L \leq z + L$ and $y + L \leq z + L$. Then there exist $a, b \in L$ satisfying $x + a \leq z$, $y + b \leq z$. By this $-z + x \leq -a$, $-z + y \leq -b$. Since L is a *wal*-ideal, there exists $c \in L$ such that $(-z + x) \vee (-z + y) \leq -c$. From this $-z + (x \vee y) \leq -c$, hence $(x \vee y) + c \leq z$ and so $(x \vee y) + L \leq z + L$. This means $(x + L) \vee (y + L) = (x \vee y) + L$. Hence R/L is a *wal*-ring and the natural homomorphism $\nu : R \longrightarrow R/L$ is a *wal*-homomorphism.

□

Lemma 1.4.3. *Let $R = (R, +, \cdot, \leq)$ be a wal-ring and I its convex ideal which is its wa-sublattice simultaneously. Then I is a wal-ideal of R if and only if*

$$(\mathrm{I'}_{\mathrm{wal}}) \qquad \forall a, b, c \in I,\ x, y \in R;\ x \leq a,\ y \leq b \implies (x \vee y) \vee c \in I.$$

Proof. Let I be a *wal*-ideal, $a, b, c \in I$, $x, y \in R$; $x \leq a$, $y \leq b$. Then I is the kernel of a *wal*-homomorphism $h : R \longrightarrow R'$ for a *wal*-ring R'. It holds $h((x \vee y) \vee c) = h(x \vee y) \vee h(c) = h(x \vee y) \vee 0'$, where $0'$ is the zero-element in R'. Since $h(x) \leq h(a) = 0'$, $h(y) \leq h(b) = 0'$, we have $h(x \vee y) = h(x) \vee h(y) \leq 0'$, thus $h((x \vee y) \vee c) = 0'$. That is why $(x \vee y) \vee c \in I$.

Conversely, let I be a convex ideal of R which is a *wa*-sublattice of R simultaneously and let I satisfy the condition $(\mathrm{I'}_{\mathrm{wal}})$. Let $a, b, c \in I$, $x, y \in R$; $x \leq a$, $y \leq b$. Then there exists $d \in I$ such that $(x \vee y) \vee c = d$, and so $x \vee y \leq d$. Therefore I is a *wal*-ideal of R. □

Notation. If there exists some *wal*-isomorphism $R \longrightarrow R'$, i.e. if R and R' are isomorphic, we will write $R \cong R'$.

Theorem 1.4.4. (First Isomorphism Theorem) *Let $h : R \longrightarrow R'$ be a surjective wal-homomorphism of wal-rings with the kernel I. Then it holds $R' \cong R/I$.*

Proof. Define $\varphi : R/I \longrightarrow R'$ by $\varphi(a + I) = h(a)$. The fact that φ is the ring isomorphism is known. We only need to show that it is the *wal*-isomorphism. According to the proof of Theorem 1.4.2, we have $(x + I) \vee (y + I) = (x \vee y) + I$. Thus, $\varphi((a+I) \vee (b+I)) = \varphi((a \vee b)+I) = h(a \vee b) = h(a) \vee h(b) = \varphi(a+I) \vee \varphi(b+I)$. We have shown that φ is a *wal*-isomorphism. □

Notation. We denote the set of all *wal*-ideals of the ring $(R, +, \cdot, \leq)$ by $\mathcal{I}(R)$.

Theorem 1.4.5. (Second Isomorphism Theorem) *Let R be a wal-ring, $I, J \in \mathcal{I}(R)$, $I \subseteq J$. Then $J/I \in \mathcal{I}(R/I)$ and $(R/I)/(J/I) \cong R/J$.*

Proof. The proof is based on the first isomorphism theorem. Define $f : R/I \longrightarrow R/J$ by $f(a+I) = a+J$. It is plain that f is a surjective *wal*-homomorphism with the kernel J/I and hence the theorem above holds. □

Some properties of the set of *wal*-ideals of a *wal*-ring come in handy for the proof of Third Isomorphism Theorem. That is why we will give it subsequently.

2. The Set of wal-ideals

2.1. The Lattice of wal-ideals

Let $(R, +, \cdot, \leq)$ be a *wal*-ring. We have denoted the set of all *wal*-ideals of the ring $(R, +, \cdot, \leq)$ by $\mathcal{I}(R)$. Further we denote the set of all *wal*-ideals of the additive *wal*-group $(R, +, \leq)$ by $\mathcal{L}(R)$. $\mathcal{L}(R)$ ordered by set inclusion forms a complete lattice with the least element $\{0\}$ and the greatest element R. The infima are formed by set intersections and the supremum of any system of *wal*-ideals of a *wal*-group $(R, +, \leq)$ coincides with the subgroup of the additive group $(R, +)$ generated by these ideals as subgroups. (See [Ra92] and [Ra96].)

We will denote the subgroup of the additive group R generated by a system $\{A_i,\ i \in J\}$ of subgroups of R by $[\bigcup_{i\in J} A_i]$.

Proposition 2.1.1. *If R is a wal-ring, then $\mathcal{I}(R)$ is a complete sublattice of the lattice $\mathcal{L}(R)$ of wal-ideals of the additive wal-group $(R, +)$.*

Proof. It is evident that the intersection of any system of *wal*-ideals of a *wal*-ring R is also a *wal*-ideal of R. It remains to verify that a join of ring *wal*-ideals in the lattice of *wal*-ideals of the additive group is simultaneously a ring ideal. Let I_i, $i \in J$ be *wal*-ideals of a *wal*-ring R, $[\bigcup_{i\in J} I_i]$ be the subgroup of the additive group $(R, +)$ generated by $\bigcup_{i\in J} I_i$. If $x \in [\bigcup_{i\in J} I_i]$, then $x = a_1 + \cdots + a_n$, $a_j \in I_{i_j}$, $j = 1, \ldots, n$. Let $r \in R$, then $rx = ra_1 + \cdots + ra_n$ and $ra_j \in I_{i_j}$, $j = 1, \ldots, n$, because I_{i_j} are ring ideals. Hence $rx \in [\bigcup_{i\in J} I_i]$ and $[\bigcup_{i\in J} I_i]$ is also a ring ideal. □

Theorem 2.1.2. (Third Isomorphism Theorem) *Let R be a wal-ring, $I, J \in \mathcal{I}(R)$. Then $I \cap J$ is a wal-ideal in J and $J/(I\cap J) \cong (I+J)/I$.*

Proof. It is obvious that if $I, J \in \mathcal{I}(R)$, then $I \in \mathcal{I}(I+J)$ and $(I+J) \in \mathcal{I}(R)$. Further $(J+I)/I$ is the *wal*-subring of R/I consisting of all those cosets $(j+i)+I$, where $j+i \in J+I$. Since $j+i+I = j+I$, it follows that $(J+I)/I$ consists precisely of all those cosets by I having a representative in J.

Let $\nu : R \longrightarrow R/I$ be the natural mapping and let $\nu' = \nu|J$ be the restriction of ν to J. Since ν' is a homomorphism whose kernel is $I \cap J$, by Theorem 1.4.2 and Theorem 1.4.4, we have $I \cap J \in \mathcal{I}(J)$ and $J/(I \cap J) \cong \mathrm{Im}\nu'$. But $\mathrm{Im}\nu'$ is just the family of all those cosets by I having a representative in J. That is, $\mathrm{Im}\nu'$ consists of $(I + J)/I$. □

Theorem 2.1.3. *The class of all wal-rings is a variety of algebras of type* $\langle +, 0, -, \cdot, \vee, \wedge \rangle$ *of signature* $\langle 2, 0, 1, 2, 2, 2 \rangle$.

Proof. It is sufficient to show, that the condition (R4) in the definition of a *wal*-ring can be replaced by some identities. Indeed the condition $0 \le c,\ a \le b \Rightarrow ac \le bc$ and $ca \le cb$ is equivalent to two following identities:

$$(a \vee b)(c \vee 0) \ge a(c \vee 0) \vee b(c \vee 0),$$
$$(c \vee 0)(a \vee b) \ge (c \vee 0)a \vee (c \vee 0)b.$$

Let the condition (R4) hold. Since $a \vee b \ge a,\ a \vee b \ge b,\ 0 \le c \vee 0 = c'$, according to (R4), we get $(a \vee b)c' \ge ac'$ and $(a \vee b)c' \ge bc'$. Hence $(a \vee b)c' \ge ac' \vee bc'$ and so $(a \vee b)(c \vee 0) \ge a(c \vee 0) \vee b(c \vee 0)$. Similarly the other identity.

Conversely, let the identities be fulfilled and $0 \le c,\ a \le b$. then $c \vee 0 = c,\ a \vee \vee b = b$. We have $bc \ge ac \vee bc$, in this way $bc \ge ac$. The proof for $ca \le cb$ is similar. □

wal-rings are Ω-groups in the sense of Kurosch (see [Ku77]), in view of satisfying the following equalities:

$$0 \cdot 0 = 0;$$
$$0 \vee 0 = 0;$$
$$0 \wedge 0 = 0.$$

The kernels of homomorphisms of an Ω-group are precisely all its ideals. Hence a *wal*-ideal of a *wal*-ring is also an ideal in the sense of an ideal of an Ω-group. Hence by [Ku77] III.2.5, a partition to blocks of any *wal*-ring R defines a congruence on R if and only if it is the partition by some *wal*-ideal in R.

Now we can show that the lattice $\mathcal{I}(R)$ is distributive. For this we will use the known properties of varieties of algebras. Let us recall that a variety of algebras is called *arithmetical* if it is both congruence-distributive and congruence-permutable.

Theorem 2.1.4. *The variety of all wal-rings is arithmetical.*

Proof. By [BuSa81] Th. II.12.5, the variety $\mathcal{V}$ is arithmetical if and only if there is a ternary term $m(x,\ y,\ z)$ such that

$$m(x,\ y,\ x) = m(x,\ y,\ y) = m(y,\ y,\ x) = x.$$

For the variety of *wal-t'mgs* we can use the term

ra(x, y, *z) ~ x — (((x* V *y) A* (x V *z)) A (y* V *z))* 4- *z.*

•

It gives, as an immediate corollary, the following theorem.

Theorem 2.1.5. *The lattice of wal-ideals of any wal-ring is distributive.*

2.2. SrreducibSe ideals and straightening ídeals

Let *R* he a ti;a/-ring and *I* G *X(R).* Consider the following conditions for 7.

(1) If a, *be R* and 0 < *a A b* G 7, then a G 7 or 6 G 7.

(2) If a, *b* G *R* and oAfe-0, then *a* G 7 or *b* G 7.

(3) 72/7 is a totally semi-ordered set.

(3) *{A* G I(-R); / C A} is a linearly ordered set.

(5) If A, B G Z(Jfl) and Af1B-J, then A = 7 or *B* = 7.

(6) If A, 5 G I(JR) and AnBC/, then *ACI ov BCL*

Theorem 2.2.1. *If I is a wal-ideal of a wal-ring R, then*
(1) <=^ (2) <=> (3) = > (4) = > (5) <=» (6).

Proo/. (1) => (2): Trivial.

(2) => (3): Let x + 7, y -f 7 G 7Č/7. By Proposition 1.1.3, there exist *a, b e R* such that *x* = (x A *y)* 4- a, y = (x A y) 4- 6, *a A b* — 0. If a G 7, then x 4- 7 = ((x A A ?/) + a) 4- 7 = (x A y) + 7 < y 4- 7. If 6 G 7, then *y* 4- I < x 4- 7. Thus 7Č/7 is a totally semi-ordered set.

(3) => (1); Let 72/7 be a totally semi-ordered set, a, *b € R\ I,* 0 < a A 6. By the assumption, a 4- 7 and 6 4-7 are comparable. If, for example, *a* 4-*1* < *b* 4- 7, then (a A 6) 4- 7 = (a 4-7) *A (b* 4-1) = *a* 4- 7, and hence *a A b* £ 7.

(3) => (4): Let A, B G I(JR), / C i , / C B and i ⊄ B. Since (by [Ra79] Th. 3) every iua/-group (hence every *wal-ring)* is generated by its positive elements, there exist 0 < *a* G *A\B* and 0 < *b* G 7?. By the assumption, *a* 4- 7 and 6 4-7 are comparable. If a 4~ 7 < 6 4-7, then there exists x G 7 such that a 4- x < 6, i.e. a < 6 — x. Since 0 < a, *a* < 6 — x G 7?, we get a G 7?, a contradiction. Hence 6 4- 7 < a 4- 7, that means there exists *y* G 7 such that 6 4- y < a, i.e. 6 < a — y. Since 0 < 6, 6 < a — y G A, we háve 6 G A. As A, 7? are waMdeals, we get *B C A.*

(4) => (5): Evident.

(5) => (6): If *A* H £ C 7, then 7 = (4 *O B)* V 7 = *(A* V 7) n (J3 V 7), because the lattice of iua/-ideals of any iua/-ring is distributive (Theorem 2.1.5). According to (5) *A* V 7 = 7 or *B* V 7 = 7. It follows that .4 C 7 or *B* C 7.

(6) =» (5): Trivial. D

Definition. A tuaJ-ideal 7 of a *wal-ňng R* satisfying the conditions (1), (2) and (3) will be called a *straightening ideál of R.*

If a *wal*-ideal I of a *wal*-ring R satisfies the conditions (5) and (6), then I is said to be an *irreducible ideal of R*.

We give the following example to show that (2) $\not\Leftrightarrow$ (5).

Example 2.2.2. Let R be the direct product $\mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z} = (\mathbb{Z}, +, \cdot)$ is semi-ordered by the same semi-order as in Example 1.2.8 a). That is $\mathbb{Z}^+ = \{0, 1, 2, 4, 6, \dots\}$. As a direct product of *wal*-rings, R is a *wal*-ring. Denote $I = \{(x, 0);\ x \in \mathbb{Z}\}$. Let us show that I is a *wal*-ideal of R. By the definition of operations in the direct product R, it is easily seen that I is a ring ideal and a *wa*-sublattice. We check that it is a convex ideal. Let $a = (a_1, 0)$, $b = (b_1, 0) \in I$, $x = (x_1, x_2) \in R$ and hold $a \leq x$, $x \leq b$. Then $a_1 \leq x_1$, $0 \leq x_2$ and $x_1 \leq b_1$, $x_2 \leq 0$. $\mathbb{Z}$ is the convex set and from the above it follows $x_2 = 0$. Therefore $x \in I$.

It remains to verify that the condition ($\mathrm{I'}_{\mathrm{wal}}$) from Lemma 1.4.3 is satisfied. Let $a = (a_1, 0)$, $b = (b_1, 0)$, $c = (c_1, 0) \in I$ and $x = (x_1, x_2)$, $y = (y_1, y_2) \in R$, and let hold $x \leq a$, $y \leq b$. Then $x_1 \leq a_1$, $x_2 \leq 0$ and $y_1 \leq b$, $y_2 \leq 0$. There exists $d_1 \in \mathbb{Z}$ such that $(x_1 \vee y_1) \vee c_1 = d_1$. Hence $(x \vee y) \vee c = ((x_1 \vee y_1) \vee c_1,\ (x_2 \vee y_2) \vee 0) = (d_1, 0) \in I$. It follows that I is a *wal*-ideal of R.

I is not a straightening ideal because, for example, $(1, 4) \wedge (4, 1) = (0, 0)$ but neither $(1, 4)$ nor $(4, 1)$ belongs to I.

Let $A \in \mathcal{I}(R)$, let I be a proper ideal of A and let $(a_1, a_2) \in A \setminus I$. Then $a_2 \neq 0$ and $(0, a_2) = (a_1, a_2) - (a_1, 0) \in A$. Since the convex ideal of $\mathbb{Z}$ generated by a_2 is equal to $\mathbb{Z}$, we get $(x_1, x_2) = (x_1, 0) + (0, x_2) \in A$ for any element $(x_1, x_2) \in R$, hence $A = R$.

That is why I is an irreducible ideal of R which is not straightening.

Definition. A *wal*-ideal I of a *wal*-ring R is called *semimaximal* if there exists an element $a \in R$ such that I is a maximal *wal*-ideal of R with respect to the property "not containing a".

Proposition 2.2.3. *A wal-ideal $I \in \mathcal{I}(R)$ is semimaximal if and only if it is infinitely irreducible, i.e. if $I = \bigcap_{\alpha\in\Gamma} J_\alpha$, $(J_\alpha \in \mathcal{I}(R))$ implies the existence of an $\alpha_0 \in \Gamma$ such that $I = J_{\alpha_0}$.*

Proof. Let I be a semimaximal *wal*-ideal of R with respect to the property "not containing a". Let $I = \bigcap_{\alpha\in\Gamma} J_\alpha$, $J_\alpha \in \mathcal{I}(R)$. Then there exists α such that $a \notin J_\alpha$. But I is maximal with this property, hence $I = J_\alpha$.

Conversely, let I be infinitely irreducible and I^* the intersection of all *wal*-ideals containing I as a proper set $I \subset I^*$. Then there exists $a \in I^* \setminus I$. If $I \subset J$ then $a \in J$, that means I is maximal with respect to the property "not containing a", i.e. I is semimaximal. □

Proposition 2.2.4. *A wal-ideal $I \in \mathcal{I}(R)$ is semimaximal if and only if R/I is subdirectly irreducible.*

Proof. Let I be semimaximal and I^* be the *wal*-ideal covering I in $\mathcal{I}(R)$. Then I^*/I is the least non-zero *wal*-ideal in R/I and therefore R/I is subdirectly irreducible.

Conversely, let R/I be subdirectly irreducible and J/I be its least non-zero *wal*-ideal. Let $a \in J \setminus I$. Consider any $K \in \mathcal{I}(R)$ such that $I \subset K$. Then $J/I \subseteq K/I$, thus $a \in K$. Therefore I is a maximal *wal*-ideal in R with respect to the property "not containing a", i.e. I is semimaximal. □

Let us denote by $\mathrm{V}(a)$ the set of all semimaximal *wal*-ideals, maximal with respect to the property "not containing a".

Proposition 2.2.5. *If $I \in \mathcal{I}(R)$ and $a \in R \setminus I$, then there exists $H \in V(a)$ such that $I \subseteq H$.*

Proof. Let $\{J_\alpha;\ \alpha \in \Gamma\}$ be a linearly ordered system of *wal*-ideals of R such that $I \subseteq J_\alpha$ and $a \notin J_\alpha$ for each $\alpha \in \Gamma$. Denote $J = \bigcup_{\alpha\in\Gamma} J_\alpha$. Let $b, c, d \in J$ and $x, y \in R$ and let hold $x \leq b$, $y \leq c$. Then there exist $\beta, \gamma, \delta \in \Gamma$ such that $b \in J_\beta$, $c \in J_\gamma$ and $d \in J_\delta$. Let e.g. $J_\gamma \subseteq J_\beta$, $J_\delta \subseteq J_\beta$. Then $(x \vee y) \vee d \in J_\beta \subseteq J$, hence $J \in \mathcal{I}(R)$. Therefore (by the Zorn lemma) the set of all $K \in \mathcal{I}(R)$ such that $I \subseteq K$, $a \notin K$ contains a maximal element belonging to $\mathrm{V}(a)$ and so being a semimaximal *wal*-ideal of R. □

Corollary 2.2.6. *Every wal-ideal of a wal-ring R is an intersection of semimaximal wal-ideals.*

In particular, the intersection of all semimaximal wal-ideals of R is equal to $\{0\}$.

References

[BiKeWo77] Bigard, A., Keimel, K. and Wolfenstein, S.: *Groupes et anneaux réticulés*. Springer-Verlag, Berlin – Heidelberg – New York, 1977.

[BuSa81] Burris, S. and Sankappanavar, H. P.: *A Course in Universal Algebra*. Springer- Verlag, New York – Heidelberg – Berlin, 1981.

[Fr70] Fried, E.: *A generalization of ordered algebraic systems*. Acta Sci. Math. (Szeged) 31 (1970), 233–244.

[Ku77] Kurosch, A. G.: *Lectures on General Algebra*. Academia, Praha, 1977. (In Czech)

[Ra79] Rachůnek, J.: *Semi-ordered groups*. Acta UPO, Fac. Rer. Nat. 61 (1979), 5–20.

[Ra92] Rachůnek, J.: *Solid subgroups of weakly associative lattice groups*. Acta UPO, Fac. Rer. Nat. 105, Math 31 (1992), 13–24.

[Ra96] Rachůnek, J.: *On some varieties of weakly associative lattice groups*. Czechoslovak Mathematical Journal, 46 (121) 1996, 231–240.

[Sk71] Skala, H.: *Trellis theory*. Alg. Univ. 1 (1971), 218–233.

[Sk72] Skala, H.: *Trellis Theory*. Memoirs AMS, Providence, 1972.

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