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Projective Objects in the Category of Chain Complexes

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Abstract: Projective objects in the category of chain complexes are characterized as mapping cones of projective graded modules. Also, injective objects are mapping cones of chain complexes with injective modules and zero differentials.

Key Words: projective, injective, chain complex, mapping cone

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1. Introduction

The present paper shows that projective objects in the category of chain complexes possess a mapping cone structure. More precisely, a projective object is isomorphic to a mapping cone $C(1_K)$ of a chain complex K whose modules K_n are projective and whose differentials ∂^K are zero. In the same fashion injective objects correspond to mapping cones of chain complexes with injective modules and zero differentials. Along the way, projective and injective objects prove to be contractible; hence, all these objects are homotopy equivalent.

In the literature a chain complex is labeled projective if all its constituent modules are projective. For example, [3] investigates projective chain complexes. However, a projective chain complex need not be a projective object in the category of chain complexes – a counterexample follows below. To avoid confusion with established definitions we adopt the following terminology:

Definition. A chain complex P is a *projective object* if for each diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle l} & & \\ A & \xrightarrow[g]{} & B & \longrightarrow & 0 \end{array}$$

with exact row there is a chain map $k\colon P \to A$ that lifts l along g, i.e., $g\,k = l$. A chain complex Q is an *injective object* if for each diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \\ & & {\scriptstyle l}\downarrow & & \\ & & Q & & \end{array}$$

with exact row there is a chain map $k: B \to Q$ that extends l with respect to f, i.e., $k f = l$.

Definition. A *weak chain complex* K is a chain complex whose differentials ∂^K are zero.

2. Mapping cones

Our characterization of projective objects uses the notion *mapping cone*; this section lists its pertinent properties (cf. [2]).

The *mapping cone* of a chain map $f: K \to L$ is the complex Cf with modules $(Cf)_n := L_n \oplus K_{n-1}$ and differentials

$$\partial^{Cf}\begin{pmatrix} l \\ k \end{pmatrix} := \begin{pmatrix} \partial^L & f \\ 0 & -\partial^K \end{pmatrix}\begin{pmatrix} l \\ k \end{pmatrix} .$$

Matrix representations of maps from or into mapping cones will be used constantly.

There are two important special cases: The mapping cone of the zero map $0: K \to L$ is the direct sum $L \oplus K^+$ where ${K^+}_n := K_{n-1}$ and $\partial_n^{K^+} := -\partial_{n-1}^K$. Secondly, the mapping cone of the identity map $1_K: K \to K$ which is called the mapping cone of the chain complex K. We write $CK := C(1_K)$.

The *cone sequence* of $f: K \to L$ is the short exact sequence

$$0 \longrightarrow L \xrightarrow{\binom{1}{0}} Cf \xrightarrow{(0,1)} K^+ \longrightarrow 0 .$$

So, the cone sequences of the above special cases are

$$0 \longrightarrow L \longrightarrow L \oplus K^+ \longrightarrow K^+ \longrightarrow 0$$

and

$$0 \longrightarrow K \longrightarrow CK \longrightarrow K^+ \longrightarrow 0 .$$

Lemma 2.1. *(cf. [4]) The cone sequences*

$$0 \longrightarrow L \xrightarrow{\binom{1}{0}} Cf \xrightarrow{(0,1)} K^+ \longrightarrow 0$$

and

$$0 \longrightarrow L \xrightarrow{\binom{1}{0}} Cg \xrightarrow{(0,1)} K^+ \longrightarrow 0$$

are congruent (cf. [5], congruence of extensions) if and only if the defining chain maps f and g are homotopic.

Proof. Congruence of the cone sequences corresponds to a commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\binom{1}{0}} & Cf & \xrightarrow{(0,1)} & K^+ \\ \| & & \big\downarrow \phi & & \| \\ L & \xrightarrow[\binom{1}{0}]{} & Cg & \xrightarrow[(0,1)]{} & K^+ \end{array}$$

The chain map $\phi\colon Cf \to Cg$ has a 2 by 2 matrix representation. Due to the diagram's commutativity this representation has the form

$$\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = \begin{pmatrix} 1 & \phi_{12} \\ 0 & 1 \end{pmatrix} .$$

We obtain the products

$$\partial^{Cg}\phi = \begin{pmatrix} \partial^L & \partial^L\phi_{12} + g \\ 0 & -\partial^K \end{pmatrix} \quad \text{and} \quad \phi\,\partial^{Cf} = \begin{pmatrix} \partial^L & f - \phi_{12}\partial^K \\ 0 & -\partial^K \end{pmatrix} ,$$

which are equal by ϕ's being a chain map. Equality of the upper right components yields the homotopy condition and vice versa. □

Corollary 2.2. *The cone sequence of $f\colon K \to L$ splits if and only if f is nullhomotopic.*

3. Cones of projective weak chain complexes are projective objects

Forming the mapping cone $CK := C(1_K)$ of a chain complex K is key to the characterization of projective and injective objects. Here we apply this cone construction to projective weak chain complexes.

Theorem 3.1. *Let K be a chain complex with projective modules K_n and zero differentials ∂^K. Then the mapping cone CK is a projective object.*

Proof. Let

$$\begin{array}{ccccc} & & CK & & \\ & & \big\downarrow{\scriptstyle (l',l)} & & \\ A & \underset{g}{\longrightarrow} & B & \longrightarrow & 0 \end{array} \tag{3.2}$$

be a diagram with exact row; the matrix notation (l', l) refers to the direct sum structure $(CK)_n = K_n \oplus K_{n-1}$. We will construct a lifting $(k'\ k)\colon CK \to A$.

First, note that the chain map condition $\partial^B(l', l) = (l', l)\,\partial^{CK}$, in full matrix representation

$$\partial^B\ (\,l' \quad l\,) = (\,l' \quad l\,)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,$$

renders the relation $l' = \partial^B l$.

Now, diagram (3.2) induces for each integer n the module diagram

$$\begin{array}{ccccc} & & K_{n-1} & & \\ & & \big\downarrow{\scriptstyle l_n} & & \\ A_n & \underset{g_n}{\longrightarrow} & B_n & \longrightarrow & 0 \end{array} .$$

By assumption K_{n-1} is projective; so there is a homomorphism $k_n\colon K_{n-1} \to A_n$ with $g_n k_n = l_n$. We set $k' := \partial^A k$ because $g\,k' = g\,\partial^A k = \partial^B g\,k = \partial^B l = l'$. The map $(k'\ k)\colon CK \to A$ is indeed a chain map and a lifting of (3.2). □

Theorem 3.3. *Let K be a chain complex with injective modules K_n and zero differentials ∂^K. Then the mapping cone CK is an injective object.*

The proof is analogous to the one of Theorem (3.1).

4. Projective objects are cones of projective weak chain complexes

The preceding section's cone construction actually generates all projective and all injective objects. We prove this statement in several steps.

Lemma 4.1. *Let P be a projective object. Any chain map $f\colon P \to K$ is nullhomotopic.*

Proof. Since P is a projective object, the cone sequence

$$0 \longrightarrow K \longrightarrow Cf \longrightarrow P \longrightarrow 0$$

splits; by Lemma (2.1), f is nullhomotopic. □

Corollary 4.2. *Every projective object is contractible.*

Proof. The identity map of a projective object is nullhomotopic. □

Corollary 4.3. *Two projective objects are homotopy equivalent.*

Lemma 4.4. *(for another presentation cf. [1] IV.2.3.) A contractible chain complex is isomorphic to the mapping cone of its boundary subcomplex. In symbols, $K \cong C(B(K))$ for all $K \simeq 0$.*

Proof. Let K be contractible, $1_K = \partial^K s + s\,\partial^K$, and let $B(K)$ denote the boundary subcomplex of K. Consider the image-restricted boundary operator $\partial'\colon K \to B(K)$ and the inclusion map $i\colon B(K) \subseteq K$. We obtain the calculation rules $\partial'\partial^K = 0$, $i\,\partial' = \partial^K$, $\partial^K i = 0$, and, from the contraction relation, $i = \partial^K s\,i$. Further $1_{B(K)} = \partial' s\, i$.

We show that $\binom{\partial' s}{\partial'}\colon K \to C(B(K))$ is a chain map isomorphism with inverse $(i, s\,i)\colon C(B(K)) \to K$. The former map is a chain map since the products

$$\begin{pmatrix} \partial' s \\ \partial' \end{pmatrix} \partial^K = \begin{pmatrix} \partial' s \partial^K \\ \partial' \partial^K \end{pmatrix} \quad \text{and } \partial^{C(B(K))} \begin{pmatrix} \partial' s \\ \partial' \end{pmatrix} = \begin{pmatrix} 0 & 1_K \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \partial' s \\ \partial' \end{pmatrix} = \begin{pmatrix} \partial' \\ 0 \end{pmatrix}$$

are equal by $\partial' s \partial^K = \partial'(1_K - \partial^K s) = \partial'$. To check the latter map's chain map condition we compare

$$(i \quad s\,i) \begin{pmatrix} 0 & 1_K \\ 0 & 0 \end{pmatrix} = (0 \quad i) \quad \text{with} \quad \partial^K (i \quad s\,i) = (\partial^K i \quad \partial^K s\,i)\,.$$

Equality follows from above calculation rules. Finally, check the maps' being inverse; the composites are

$$(i \quad s\,i) \begin{pmatrix} \partial' s \\ \partial' \end{pmatrix} = i\partial' s + s\,i\,\partial' = \partial^K s + s\partial^K = 1_K$$

and (for upper right component see below)

$$\begin{pmatrix} \partial' s \\ \partial' \end{pmatrix} (i \quad s\,i) = \begin{pmatrix} \partial' s\,i & \partial' s\,s\,i \\ \partial' i & \partial' s\,i \end{pmatrix} = \begin{pmatrix} 1_{B(K)} & 0 \\ 0 & 1_{B(K)} \end{pmatrix}.$$

For the upper right component we used

$$\begin{aligned} i\partial' ssi &= \partial^K s\,s\,i = (1_K - s\partial^K)\,s\,i = s\,i - s\partial^K s\,i = \\ &= s\,i - s\,(1_K - s\,\partial^K)\,i = s\,s\partial^K i = 0\,; \end{aligned}$$

since i is injective, $\partial' s\,s\,i = 0$ □

To complete the characterization we prove one more statement.

Lemma 4.5. *The modules of a projective object are projective.*

Proof. Let P be a projective object and let n be an arbitrary integer. A module diagram

$$\begin{array}{ccccc} & & P_n & & \\ & & \downarrow{\scriptstyle l} & & \\ A & \xrightarrow[g]{} & B & \longrightarrow & 0 \end{array} \tag{4.6}$$

extends to a chain complex diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle (l,n)} & & \\ (A,n) & \xrightarrow[(g,n)]{} & (B,n) & \longrightarrow & 0 \end{array} \tag{4.7}$$

where (A, n) denotes the chain complex whose n-th module is A and whose other modules are 0. Since P is a projective object, (4.7) has a lifting $k: P \to (A, n)$. Its constituent module homomorphism $k_n: P_n \to A$ is a lifting for (4.6). □

Theorem 4.8. *(for another presentation cf. [1] IV.2.5.) A projective object P is isomorphic to the cone CK of a chain complex K with projective modules K_n and zero differentials ∂^K. More precisely, P is isomorphic to $C(B(P))$ where $B(P)$ is the boundary subcomplex of P.*

Proof. By Corollary (4.2) the projective object P is contractible, and Lemma (4.4) yields an isomorphism $P \cong C(B(P))$. Any boundary subcomplex has zero differentials by definition. Lemma (4.5) shows each module P_n projective. Since the above isomorphism gives $P_n \cong B_n(P) \oplus B_{n-1}(P)$ the modules $B_n(P)$ are projective as direct summands of the projective modules P_n. □

The respective result for injective objects is proved analogously.

5. Projective chain complexes need not be projective objects

In a different context the following example appears in [3]. The chain complex $K\colon \cdots \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{2} \cdots$ consists of free $\mathbb{Z}_4$-modules. Also, the modules are injective over $\mathbb{Z}_4$ since $\mathbb{Z}_4 \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_4, \mathbb{Q}/\mathbb{Z})$. Thus K is projective as well as injective. Nevertheless, K is neither a projective object nor an injective object; in fact, it is not even contractible: Lemma (4.4) states that if K were contractible it would satisfy $K \cong C(B(K))$, in particular $K_n \cong B_n(K) \oplus B_{n-1}(K)$. But each group $B_n(K)$ is isomorphic to $\mathbb{Z}_2$ and, of course, there is no isomorphism from $K_n = \mathbb{Z}_4$ to $B_n(K) \oplus B_{n-1}(K) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

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