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Global Stability of Sets for Impulsive Differential-Difference Equations by Lyapunov's Direct Method

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Abstract: In the present paper the question of global stability of sets with respect to systems of impulsive differential-difference equations with impulse effect at fixed moments is considered. It is proved that the existence of piecewise continuous functions of the type of Lyapunov's functions with certain properties is a sufficient condition for various types of global asymptotic stability.
Key Words: Global stability of sets, Impulsive differential-difference equations, Lyapunov's direct method

Mathematics Subject Classification: 34A37

1. Introduction

The impulsive differential-difference equations are adequate mathematical models of numerous real processes and phenomena studied by physics, biology, population dynamics, theory of optimal control, industrial robotics and economics. For instance, if the population of given species depends on their maturity and on the natural growth rate for the preceeding generations, then the equation which simulates the population dynamics of these species is differential-difference. If, moreover, the population is regulated by some impulsive factors acting at certain moments, then we have no reasons to expect that the process will be simulated by regular control. On the contrary, the solutions must have jumps at these moments and the jumps are given beforehand.

In spite of the great possibilities for application, the theory of the impulsive differential-difference equations is developing rather slowly due to obstacles of theoretical and technical character.[1–4]

In the present paper the problem of global stability of sets with respect to systems of impulsive differential-difference equations with impulse effect at fixed moments is considered by means of Lyapunov's direct method. The use of classical (continuous) Lyapunov's functions, however, considerably restricts the facilities provided by the method.

The fact that the solutions of the systems considered are piecewise continuous functions requires the use of analogues of Lyapunov's functions which have discontinuities of the first kind. By means of these functions the possibility of application of Lyapunov's direct method to impulsive systems of differential-difference equations considerably increases [6]. Moreover, the technique of investigation essentially depends on the choice of minimal subsets of a suitable space of piecewise continuous func tions, by the elements of which the derivatives of Lyapunov's functions are estimated. [7]

We shall note that analogous results for impulsive systems of ordinary differential equations (without delay) were obtained in [5].

2. Statement of he problem. Preliminary notes and definitions

Let $R_+ = [0, \infty)$; R^n be the n-dimensional Euclidean space with norm $|.|$, scalar product $< .,. >$ and distance $d(.,.)$. Let $h > 0$, $t_0 \in R$ and $\varphi_0 \in C[[t_0 - h, t_0], R^n]$.

Consider the initial value problem

$$\dot{x}(t) = f(t, x(t), x(t-h)), \quad t > t_0, \ t \neq \tau_k, \tag{1}$$

$$x(t) = \varphi_0(t), \quad t \in [t_0 - h, t_0], \tag{2}$$

$$\Delta x(\tau_k) = I_k(x(\tau_k)), \quad \tau_k > t_0, \ k = 1, 2, \ldots, \tag{3}$$

where $f : (t_0, \infty) \times R^n \times R^n \to R^n$; $I_k : R^n \to R^n$, $k = 1, 2, \ldots$; $\Delta x(\tau_k) = x(\tau_k + {}+ 0) - x(\tau_k - 0)$; $t_0 \equiv \tau_0 < \tau_1 < \tau_2 < \ldots$ and $\lim\limits_{k\to\infty} \tau_k = \infty$.

The solutions $x(t)$ of problems of the type (1), (2), (3) are piecewise continuous functions with points of discontinuity of the first kind $\tau_k > t_0$, $k = 1, 2, \ldots$ at which they are continuous from the left, i.e. at the moments of impulse effect τ_k the following relations are valid

$$x(\tau_k - 0) = x(\tau_k), x(\tau_k + 0) = x(\tau_k) + I_k(x(\tau_k)), \ k = 1, 2, \ldots$$

If for some positive integer j we have $\tau_k < \tau_j + h < \tau_{k+1}$, $k = 0, 1, 2, \ldots$, then in the interval $[\tau_j + h, \tau_{k+1}]$ the solution $x(t)$ of problem (1), (2), (3) coincides with the solution of the problem

$$\begin{cases} \dot{y}(t) = f(t, y(t), x(t - h + 0)), \\ y(\tau_j + h) = x(\tau_j + h), \end{cases}$$

and if $\tau_j + h \equiv \tau_k$ for $j = 0, 1, 2, \ldots, k = 1, 2, \ldots$,then in the interval $[\tau_j + h, \tau_{k+1}]$ the solution $x(t)$ coincides with the solution of the problem

$$\begin{cases} \dot{y}(t) = f(t, y(t), x(t - h + 0)), \\ y(\tau_j + h) = x(\tau_j + h) + I_k(x(\tau_j + h)). \end{cases}$$

Let $M \subset [t_0 - h, \infty) \times R^n$.

Introduce the following notations:

$M(t) = \{x \in R^n : (t, x) \in M, t \in (t_0, \infty)\}$;

$M_0(t) = \{x \in R^n : (t,x) \in M, \ t \in [t_0 - h, t_0]\}$;
$d(x, M(t)) = \inf_{y \in M(t)} |x - y|$ is the distance between $x \in R^n$ and $M(t)$;
$M(t, \varepsilon) = \{x \in R^n : d(x, M(t)) < \varepsilon\}$ $(\varepsilon > 0)$ is an ε-neighbourhood of $M(t)$;
$C_0 = C[[t_0 - h, t_0], R^n]$;
$d_0(\varphi, M_0(t)) = \max_{t \in [t_0 - h, t_0]} d(\varphi(t), M_0(t)), \ \varphi \in C_0$;
$M_0(t, \varepsilon) = \{\varphi \in C_0 : d_0(\varphi, M_0(t)) < \varepsilon\}$;
$S_\alpha = \{x \in R^n : |x| < \alpha\}, \ \alpha > 0$;
$\overline{S_\alpha} = \{x \in R^n : |x| \le \alpha\}$;
$\overline{S_\alpha}(C_0) = \{\varphi \in C_0 : ||\varphi|| \le \alpha\}$, where $||\varphi|| = \max_{t \in [t_0 - h, t_0]} |\varphi(t)|$ is the norm of the function $\varphi \in C_0$;
$G_k = \{(t,x) \in [t_0, \infty) \times R^n : \tau_{k-1} < t < \tau_k\}, k = 1, 2, \ldots$;
$G = \bigcup_{k=1}^{\infty} G_k$;
$K = \{a \in C[R_+, R_+] : a(r) \ \textit{is strictly increasing and} \ a(0) = 0\}$;
$CK = \{a \in C[(t_0, \infty) \times R_+, R_+] : a(t, .) \in K \ \textit{for any fixed} \ t \in (t_0, \infty)\}$;
$K^* = \{a \in C[R_+ \times R_+, R_+] : a(., s) \in K \ \textit{for any fixed} \ s \in R_+\}$;
$x(t; t_0, \varphi_0)$is the solution of problem (1),(2),(3);
$J^+(t_0, \varphi_0)$ is the maximal interval of type (t_0, β) in which the solution $x(t; t_0, \varphi_0)$ is defined. P

Introduce the following conditions:
H1. $f \in C[(t_0, \infty) \times R^n \times R^n, R^n]$.
H2. The function f is Lipschitz continuous with respect to its second and third arguments in $(t_0, \infty) \times R^n \times R^n$ uniformly on $t \in (t_0, \infty)$.
H3. $|f(t, x, \tilde{x})| \le L < \infty$ for$(t, x, \tilde{x}) \in (t_0, \infty) \times R^n \times R^n, \ L > 0$.
H4. For any $k = 1, 2, \ldots$ the following inequality is valid

$$|I_k(x_1) - I_k(x_2)| \le c|x_1 - x_2|, \ x_1, x_2 \in R^n, \ c > 0.$$

H5. $t_0 \equiv \tau_0 < \tau_1 < \tau_2 < \ldots$.
H6. $\lim_{k \to \infty} \tau_k = \infty$.
H7. $M(t) \ne \emptyset$ for $t \in (t_0, \infty)$.
H8. $M_0(t) \ne \emptyset$ for $t \in [t_0 - h, t_0]$.
H9. For any compact subset F of $(t_0, \infty) \times R^n$ there exists a constant $K > 0$ depending on F such that if $(t, x), \ (t', x) \in F$, then the following inequality is valid

$$|d(x, M(t)) - d(x, M(t'))| \le K|t - t'|.$$

We shall give definitions of stability of the set M with respect to problem (1),(2),(3).

Definition 1. The set M is said to be:
(a) *stable* with respect to problem (1),(2),(3) if

$$(\forall t_0 \in R)(\forall \alpha > 0)(\forall \varepsilon > 0)(\exists \delta = \delta(t_0, \alpha, \varepsilon) > 0)$$
$$(\forall \varphi_0 \in \overline{S_\alpha}(C_0) \cap M_0(t, \delta))(\forall t > t_0) :$$
$$x(t; t_0, \varphi_0) \in M(t, \varepsilon);$$

(*b*) $t(\alpha)$-*uniformly stable* with respect to problem (1), (2), (3) if the number δ from point (*a*) does not depend on t_0 (on α);

(*c*) *uniformly stable* with respect to problem (1), (2), (3) if the number δ from point (*a*) depends only on ε.

We shall give definitions of boundedness of the solutions of problem (1), (2), (3) with respect to the set M.

Definition 2. The solutions of problem (1), (2), (3) are said to be:

(*a*) *equi-M-bounded (equi-bounded with respect to the set M)* if

$$(\forall\eta > 0)(\forall\alpha > 0)(\forall t_0 \in R)(\exists\beta = \beta(t_0, \eta, \alpha) > 0)$$
$$(\forall\varphi_0 \in \overline{S_\alpha}(C_0) \cap \overline{M_0(t, \eta)})(\forall t > t_0) : \ x(t; t_0, \varphi_0) \in M(t, \beta);$$

(*b*) $t(\alpha)$-*uniformly-M-bounded* if the number β from (*a*) does not depend on t_0 (*on* α);

(*c*) *uniformly-M-bounded* if the number β from (*a*) depends only on η.

Finally we shall give definitions of global asymptotic stability of the set M with respect to problem (1), (2), (3).

Definition 3. The set M is said to be:

(*a*) *globally equi-attractive* with respect to problem (1), (2), (3) if

$$(\forall\eta > 0)(\forall\alpha > 0)(\forall t_0 \in R)(\forall\varepsilon > 0)$$
$$(\exists\sigma = \sigma(t_0, \eta, \alpha, \varepsilon) > 0)(\forall\varphi_0 \in \overline{S_\alpha}(C_0) \cap M_0(t, \eta))$$
$$(\forall t \geq t_0 + \sigma) : \ x(t; t_0, \varphi_0) \in M(t, \varepsilon);$$

(*b*) $t(\alpha)$-*uniformly globally attractive* with respect to problem (1), (2), (3) if the number σ from (*a*) does not depend on t_0 (on α);

(*c*) *uniformly globally attractive* with respect to problem (1), (2), (3) if the number σ from (*a*) depends only on η and ε.

Definition 4. The set M is said to be:

(*a*) *globally equi-asymptotically stable* with respect to problem (1), (2), (3) if M is a stable set and a globally equi-attractive set of problem (1), (2), (3) and the solutions of problem (1), (2), (3) are equi-M- bounded;

(*b*) $t(\alpha)$-*uniformly globally asymptotically stable* with respect to problem (1), (2), (3) if M is $t(\alpha)$-uniformly stable and $t(\alpha)$-uniformly globally attractive set of problem (1), (2), (3) and if the solutions of problem (1), (2), (3) are $t(\alpha)$-uniformly M-bounded;

(*c*) *uniformly globally asymptotically stable* with respect to problem (1), (2), (3) if M is a uniformly stable and uniformly globally attractive set of problem (1), (2), (3) and if the solutions of problem (1), (2), (3) are uniformly M-bounded;

(*d*) *exponentially globally asymptotically stable* with respect to problem (1), (2), (3) if

$$(\exists c > 0)(\forall \alpha > 0)(\forall \eta > 0)(\exists k = k(\alpha, \eta) > 0)$$
$$(\forall t_0 \in R)(\forall \varphi_0 \in \overline{S_\alpha}(C_0) \cap \overline{M_0(t,\eta)})(\forall t > t_0):$$
$$d(x(t; t_0, \varphi_0), M(t)) \leq k(\alpha, \eta) d_0(\varphi_0, M_0(t)) \exp[-c(t - t_0)].$$

In the further considerations we shall use the class V_0 of piecewise continuous auxiliary functions $V : [t_0, \infty) \times R^n \to R$ which are an analogue of Lyapunov's functions [6].

Definition 5. We shall say that the function $V : [t_0, \infty) \times R^n \to R$ belongs to the class V_0 if:

1. The function V is continuous in G and locally Lipschitz continuous with respect to its second argument in each of the sets G_k, $k = 1, 2, \ldots$.
2. $V(t, x) = 0$ for $(t, x) \in M$, $t \geq t_0$ and $V(t, x) > 0$ for $(t, x) \in \{[t_0, \infty) \times R^n\} \setminus M$.
3. For each $k = 1, 2, \ldots$ and $x \in R^n$ there exist the finite limits
$$V(\tau_k - 0, x) = \lim_{\substack{t \to \tau_k \\ t < \tau_k}} V(t, x), \quad V(\tau_k + 0, x) = \lim_{\substack{t \to \tau_k \\ t > \tau_k}} V(t, x).$$
4. The equality $V(\tau_k - 0, x) = V(\tau_k, x)$ is valid.
5. The following inequality is valid
$$V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) \leq V(\tau_k, x(\tau_k)), \; k = 1, 2, \ldots \tag{4}$$

We also introduce the following classes of functions:

$PC[[t_0, \infty), R^n] = \{x : [t_0, \infty) \to R^n$: x is piecewise continuous with points of discontinuity of the first kind $\tau_1, \tau_2, \ldots$ at which it is continuous from the left$\}$;

$\Omega = \{x \in PC[[t_0, \infty), R^n] : \; V(s, x(s)) \leq V(t, x(t)), \; t - h \leq s \leq t, \; t \geq t_0, V \in V_0\}$.

Let $V \in V_0$, $t \geq t_0$, $t \neq \tau_k$, $k = 1, 2, \ldots$ and $x \in PC[[t_0, \infty), R^n]$.

Introduce the function

$$D_- V(t, x(t)) = \lim_{\sigma \to 0^-} \inf \sigma^{-1}[V(t + \sigma, x(t) + \sigma f(t, x(t), x(t - h))) - V(t, x(t))].$$

Definition 6. Let $\lambda : (t_0, \infty) \to R_+$ be measurable. Then we say that $\lambda(t)$ is *integrally positive* if $\int_J \lambda(t) dt = \infty$ whenever $J = \bigcup_{k=1}^{\infty} [\alpha_k, \beta_k]$, $\alpha_k < \beta_k < \alpha_{k+1}$ and $\beta_k - \alpha_k \geq \theta > 0$, $k = 1, 2, \ldots$.

In the proof of the main results we shall use the following lemmas:

Lemma 1. *Let the conditions H1, H2, H4-H6 hold.*
Then $J^+(t_0, \varphi_0) = (t_0, \infty)$.

Proof. By conditions H1, H2 and H4 the solution $x(t) = x(t; t_0, \varphi_0)$ of problem (1), (2), (3) is defined in each of the intervals $(\tau_{k-1}, \tau_k]$, $k = 1, 2, \dots$. From conditions H5 and H6 we conclude that it is continuable for each $t > t_0$. □

Lemma 2. *Let the following conditions hold:*

1. *Conditions H1, H2, H4-H6 are met.*
2. $g \in PC[[t_0, \infty) \times R_+, R]$ *and* $g(t, 0) = 0$ *for* $t \in [t_0, \infty)$.
3. $B_k \in C[R_+, R_+]$, $B_k(0) = 0$ *and* $\psi_k(u) = u + B_k(u)$ *are nondecreasing with respect to* u, $k = 1, 2, \dots$.
4. *The maximal solution* $r(t; t_0, u_0)$ *of the problem*

$$\begin{cases} \dot{u}(t) = g(t, u(t)), \quad t > t_0, \ t \neq \tau_k, \ k = 1, 2, \dots, \\ u(t_0 + 0) = u_0 \geq 0, \\ \Delta u(\tau_k) = B_k(u(\tau_k)), \ k = 1, 2, \dots \end{cases} \tag{5}$$

   is defined in the interval $[t_0, \infty)$.

5. *The function* $V \in V_0$ *is such that*

$$V(t_0, \varphi_0(t_0)) \leq u_0$$

   and the inequalities

$$\begin{aligned} &D_- V(t, x(t)) \leq g(t, V(t, x(t))), \ t \neq \tau_k, \ k = 1, 2, \dots, \\ &V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) \leq \psi_k(V(\tau_k, x(\tau_k))), \ k = 1, 2, \dots \end{aligned}$$

   are valid for each $t > t_0$ *and* $x \in \Omega$.

Then

$$V(t, x(t; t_0, \varphi_0)) \leq r(t; t_0, u_0) \quad \text{for } t \in [t_0, \infty). \tag{6}$$

Proof. The maximal solution $r(t; t_0, u_0)$ of problem (5) is defined by the equality

$$r(t; t_0, u_0) = \begin{cases} r_0(t; t_0, u_0 + 0), \ t_0 < t \leq \tau_1, \\ r_1(t; \tau_1, u_1 + 0), \ \tau_1 < t \leq \tau_2, \\ \dots\dots\dots\dots \\ r_k(t; \tau_k, u_k + 0), \ \tau_k < t \leq \tau_{k+1}, \\ \dots\dots\dots\dots, \end{cases}$$

where $r_k(t; \tau_k, u_k + 0)$ is the maximal solution of the equation without impulses $\dot{u} = g(t, u)$ in the interval $(\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \dots$, for which $u_k + 0 = \psi_k(r_{k-1}(\tau_k; \tau_{k-1}, u_{k-1} + 0))$, $k = 1, 2, \dots$ and $u_0 + 0 = u_0$.

Let $t \in (t_0, \tau_1]$. Then from the corresponding comparison lemma for the continuous case [7] it follows that

$$V(t, x(t; t_0, \varphi_0)) \leq r(t; t_0, u_0),$$

i.e. inequality (6) is valid for $t \in (t_0, \tau_1]$.

Suppose that (6) is satisfied for $t \in (\tau_{k-1}, \tau_k]$, $k > 1$. Then, using condition 5 of Lemma 2 and the fact that the function ψ_k are nondecreasing, we obtain

$$V(\tau_k + 0, x(\tau_k + 0; t_0, \varphi_0)) \le \psi_k(V(\tau_k, x(\tau_k; t_0, \varphi_0))) \le$$
$$\le \psi_k(r(\tau_k; t_0, u_0)) = \psi_k(r_{k-1}(\tau_k; \tau_{k-1}, u_{k-1} + 0)) = u_k + 0.$$

We apply again the comparison lemma for the continuous case in the interval $(\tau_k, \tau_{k+1}]$ and obtain

$$V(t, x(t; t_0, \varphi_0)) \le r_k(t; \tau_k, u_k + 0) = r(t; t_0, u_0),$$

i.e. the inequality (6) is valid for $t \in (\tau_k, \tau_{k+1}]$.

The proof is completed by induction. □

Corollary 1. *Let the following conditions hold:*

1. Conditions H1, H2, H4-H6 are met.

2. The function $V \in V_0$ is such that the inequality

$$D_- V(t, x(t)) \le 0, \ t \ne \tau_k, \ k = 1, 2, \ldots$$

is valid for each $t > t_0$ and $x \in \Omega$.

Then

$$V(t, x(t; t_0, \varphi_0)) \le V(t_0, \varphi_0(t_0)), \ t \in [t_0, \infty).$$

Lemma 3. *Let conditions H5 and H6 hold and let the function $V : [t_0, \infty) \times R^n \to R$ belong to class V_0.*

Then for any choice of the numbers $t_0 \in R$, $\alpha > 0$, $\eta > 0$ there exists a number $K(t_0, \alpha, \eta) > 0$ such that for $\varphi_0 \in \overline{S_\alpha}(C_0) \bigcap \overline{M_0(t, \eta)}$ the following inequality holds:

$$V(t_0, \varphi_0(t_0)) \le K(t_0, \alpha, \eta). \tag{7}$$

Proof. Suppose that the assertion is not true. Then there exist $t_0 \in R$, constants $\alpha > 0, \eta > 0$ and a sequence $\{x_r\}_{r=1}^{\infty} \subset R^n$ such that $x_r \in \overline{S_\alpha} \bigcap \overline{M_0(t_0, \eta)}$ for $r = 1, 2, \ldots$ and the following inequalities are valid

$$V(t_0, \varphi_0(t_0)) > r, \ r = 1, 2, \ldots \tag{8}$$

Since the sequence $\{x_r\}_{r=1}^{\infty}$ is bounded, then out of it we can choose a convergent subsequence $\{x_{r_j}\}_{j=1}^{\infty}$. Let $\lim_{j \to \infty} x_{r_j} = \varphi_0(t_0)$. From the continuity of the function φ_0 at $t_0 \in R$ and of $V(t_0, x)$ at the point $(t_0, \varphi_0(t_0))$ it follows that

$$\lim_{j \to \infty} V(t_0, x_{r_j}) = V(t_0, \varphi_0(t_0)),$$

which contradicts inequality (8).

This completes the proof of Lemma 3. □

3. Main results

Theorem 1. *Let the following conditions hold:*

1. Conditions H1-H9 are met.

2. The functions $V \in V_0$ and $a \in K$ are such that

$$a(d(x, M(t))) \leq V(t, x) \ for \ (t, x) \in [t_0, \infty) \times R^n, \tag{9}$$

where $a(r) \to \infty$ as $r \to \infty$.

3. The inequality

$$D_- V(t, x(t)) \leq -cV(t, x(t)) \ for \ t \neq \tau_k, \ k = 1, 2 \ldots$$

is valid for each $t > t_0$, $x \in \Omega, V \in V_0$ and $c = const > 0$.

Then the set M is globally equi-asymptotically stable with respect to problem (1), (2), (3).

Proof. Let $\alpha > 0$, $\varepsilon > 0$. From the condition $V(t_0, x) = 0$ for $x \in M_0(t_0)$ it follows that there exists a constant $\delta = \delta(t_0, \alpha, \varepsilon) > 0$ such that if $x \in \overline{S_\alpha} \bigcap M_0(t_0, \delta)$, then $V(t_0, x) < a(\varepsilon)$.

Let $\varphi_0 \in \overline{S_\alpha}(C_0) \bigcap M_0(t, \delta)$. Then $\varphi_0(t_0) \in \overline{S_\alpha} \bigcap M_0(t_0, \delta)$, hence $V(t_0, \varphi_0(t_0)) < < a(\varepsilon)$.

Let $x(t) = x(t; t_0, \varphi_0)$ be the solution of problem (1), (2), (3).

By Lemma 1 $J^+(t_0, \varphi_0) = (t_0, \infty)$.

Since the conditions of Corollary 1 are met, then

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0, \varphi_0(t_0)), \ t \in (t_0, \infty). \tag{10}$$

Form (9) and (10) there follow the inequalities

$$a(d(x(t; t_0, \varphi_0), M(t))) \leq V(t, x(t; t_0, \varphi_0)) \leq$$
$$\leq V(t_0, \varphi_0(t_0)) < a(\varepsilon) \ for \ t \in (t_0, \infty).$$

Hence $x(t; t_0, \varphi_0) \in M(t, \varepsilon)$ for $t > t_0$, i.e. the set M is stable with respect to problem (1), (2), (3).

Let $\alpha > 0$, $\eta > 0$ and $\varepsilon > 0$. From condition 3 of Theorem 1 and (4) there follows the inequality

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0, \varphi_0(t_0)) exp[-c(t - t_0)]. \tag{11}$$

Let $N(t_0, \eta, \alpha) = sup\{V(t_0, x) : \ x \in \overline{S_\alpha} \bigcap M_0(t_0, \eta)\}$ and $\sigma = \sigma(t_0, \alpha, \eta, \varepsilon) > \frac{1}{c} ln \frac{N(t_0, \eta, \alpha)}{a(\varepsilon)}$.

Then from conditions (9) and (11) it follows that for $t \geq t_0 + \sigma$ the following inequalities are valid

$$a(d(x(t; t_0, \varphi_0), M(t))) \leq V(t, x(t; t_0, \varphi_0)) \leq$$
$$\leq V(t_0, \varphi_0(t_0)) exp[-c(t - t_0)] < a(\varepsilon),$$

which means that the set M is globally equi-attractive with respect to problem (1), (2), (3).

Finally we shall prove that the solutions of problem (1), (2), (3) are equi-M-bounded.

Let $\alpha > 0$ and $\eta > 0$. From Lemma 3 it follows that there exists a constant $K(t_0, \alpha, \eta) > 0$ such that if $\varphi_0 \in \overline{S_\alpha}(C_0) \bigcap \overline{M_0(t, \eta)}$, then $V(t_0, \varphi_0(t_0)) \leq$ $\leq K(t_0, \alpha, \eta)$. From the condition $a(r) \to \infty$ as $r \to \infty$ it follows that there exists a constant $\beta = \beta(t_0, \alpha, \eta) > 0$ such that $a(\beta) > K(t_0, \alpha, \eta)$.

Let $\varphi_0 \in \overline{S_\alpha}(C_0) \bigcap \overline{M_0(t, \eta)}$ and let $x(t) = x(t; t_0, \varphi_0)$ be the solution of problem (1), (2), (3). Then from condition (9), condition 3 of Theorem 1 and from (4) we obtain

$$a(d(x(t; t_0, \varphi_0), M(t))) \leq V(t, x(t; t_0, \varphi_0)) \leq$$
$$\leq V(t_0, \varphi_0(t_0)) \leq K(t_0, \alpha, \eta) < a(\beta)$$

for each $t > t_0$. This shows that $d(x(t), M(t)) < \beta,\ t > t_0$, hence the solutions of problem (1), (2), (3) are equi-M- bounded. □

Theorem 2. *Let the following conditions hold:*

1. Conditions H1-H9 are met.

2. The functions $V \in V_0$ and $a, b \in K$ are such that

$$a(d(x, M(t))) \leq V(t, x) \leq \gamma(t) b(d(x, M(t))),$$

for $(t, x) \in [t_0, \infty) \times R^n$ and $a(r) \to \infty$ as $r \to \infty$, where $\gamma : [t_0, \infty) \to [1, \infty)$.

3. The inequality

$$D_- V(t, x(t)) \leq -p(t) c(d(x(t), M(t))),\ t \neq \tau_k,\ k = 1, 2 \dots$$

is valid for any $t > t_0$, $x \in \Omega, V \in V_0$, $p : [t_0, \infty) \to (0, \infty)$, $c \in K$.

4. $\int_0^\infty p(s) c[b^{-1}(\frac{\eta}{\gamma(s)})] ds = \infty$ for each sufficiently small value of $\eta > 0$.

Then the set M is α-uniformly globally asymptotically stable with respect to problem (1), (2), (3).

Proof. Let $\varepsilon > 0$. Choose $\delta = \delta(t_0, \varepsilon) > 0,\ \delta < \varepsilon$ so that $b(\delta) < \frac{a(\varepsilon)}{\gamma(t_0)}$.

Let $\alpha > 0$ be arbitrary, $\varphi_0 \in \overline{S_\alpha}(C_0) \bigcap M_0(t, \delta)$ and $x(t) = x(t; t_0, \varphi_0)$.

From conditions 2 and 3 of Theorem 2 and (4) it follows that for $t \in J^+(t_0, \varphi_0)$ the following inequalities are valid

$$a(d(x(t; t_0, \varphi_0), M(t))) \leq V(t, x(t)) \leq$$
$$\leq V(t_0, \varphi_0(t_0)) \leq \gamma(t_0) b(d(\varphi_0(t_0), M_0(t_0))) \leq$$
$$\leq \gamma(t_0) b(d_0(\varphi_0, M_0(t))) < \gamma(t_0) b(\delta) < a(\varepsilon).$$

Since $J^+(t_0, \varphi_0) = (t_0, \infty)$, then $x(t) \in M(t, \varepsilon)$ for all $t > t_0$.

Thus it is proved that the set M is α- uniformly stable.

Now let $\eta > 0$ and $\varepsilon > 0$ be given and let the number $\sigma = \sigma(t_0, \eta, \varepsilon) > 0$ be chosen so that

$$\int_{t_0}^{t_0+\sigma} p(s)c[b^{-1}(\frac{a(\varepsilon)}{2\gamma(s)})]ds > \gamma(t_0)b(\eta). \tag{12}$$

(This is possible in view of condition 4 of Theorem 2).

Let $\alpha > 0$ be arbitrary, $\varphi_0 \in \overline{S_\alpha}(C_0) \bigcap M_0(t, \eta)$ and $x(t) = x(t; t_0, \varphi_0)$.

Assume that for any $t \in [t_0, t_0 + \sigma]$ the following inequality holds:

$$d(x(t), M(t)) \geq b^{-1}(\frac{a(\varepsilon)}{2\gamma(t)}).$$

Then by condition 3 of Theorem 2 and (12) it follows that

$$\int_{t_0}^{t_0+\sigma} D_-V(s, x(s))ds \leq -\int_{t_0}^{t_0+\sigma} p(s)c[b^{-1}(\frac{a(\varepsilon)}{2\gamma(s)})]ds < \gamma(t_0)b(\eta). \tag{13}$$

On the other hand, if $t_0 + \sigma \in (\tau_r, \tau_{r+1}]$, then from (4) we obtain

$$\int_{t_0}^{t_0+\sigma} D_-V(s, x(s))ds = \sum_{k=1}^{r} \int_{\tau_{k-1}}^{\tau_k} D_-V(s, x(s))ds + \int_{\tau_r}^{t_0+\sigma} D_-V(s, x(s))ds =$$

$$= \sum_{k=1}^{r} [V(\tau_k, x(\tau_k)) - V(\tau_{k-1} + 0, x(\tau_{k-1} + 0))] + V(t_0 + \sigma, x(t_0 + \sigma)) -$$

$$-V(\tau_r + 0, x(\tau_r + 0)) \geq V(t_0 + \sigma, x(t_0 + \sigma)) - V(t_0, \varphi_0(t_0)),$$

whence, in view of condition (13) and condition 3 of Theorem 2, it follows that $V(t_0 + \sigma, x(t_0 + \sigma)) < 0$, which contradicts condition 2 of Theorem 2.

The contradiction obtained shows that there exists $t^* \in [t_0, t_0 + \sigma]$, such that

$$d(x(t^*), M(t^*)) < b^{-1}(\frac{a(\epsilon)}{2\gamma(t^*)}).$$

Then for $t \geq t^*$ (hence for any $t \geq t_0 + \sigma$ as well) the following inequalities are valid

$$a(d(x(t), M(t))) \leq V(t, x(t)) \leq V(t^* + 0, x(t^* + 0)) \leq$$

$$\leq \gamma(t^*)b(d(x(t^*), M(t^*))) < \frac{a(\varepsilon)}{2} < a(\epsilon).$$

Hence $x(t) \in M(t, \epsilon)$ for $t \geq t_0 + \sigma$, i.e. the set M is α-uniformly globally attractive with respect to problem (1), (2), (3).

Finally we shall prove that the solutions of problem (1), (2), (3) are α-uniformly M-bounded.

Let $\eta > 0$ and let $\beta = \beta(t_0, \eta) > 0$ be such that $a(\beta) > \gamma(t_0)b(\eta)$.

Choose arbitrary $\alpha > 0$, $\varphi_0 \in \overline{S_\alpha}(C_0) \bigcap \overline{M_0(t, \eta)}$ and let $x(t) = x(t; t_0, \varphi_0)$.

Then for $t > t_0$ the following inequalities are valid

$$a(d(x(t), M(t))) \leq V(t, x(t)) \leq V(t_0, \varphi_0(t_0)) \leq \\ \leq \gamma(t_0) b(d(\varphi_0(t_0), M_0(t_0))) \leq \gamma(t_0) b(d_0(\varphi_0, M_0(t))) \leq \\ \leq \gamma(t_0) b(\eta) < a(\beta).$$

Hence $x(t) \in M(t, \beta)$ for $t > t_0$. □

Theorem 3. *Let the following conditions hold:*

1. Conditions H1-H9 are met.

2. The functions $V \in V_0$ and $a, b \in K$ are such that

$$a(d(x, M(t))) \leq V(t, x) \leq b((d(x, M(t))),$$

for $(t, x) \in [t_0, \infty) \times R^n$ and $a(r) \to \infty$ as $r \to \infty$.

3. There exists an integrall positive function $\lambda(t)$ such that the inequality

$$D_- V(t, x(t)) \leq -\lambda(t) c(d(x(t), M(t))), \ t \neq \tau_k, \ k = 1, 2 \ldots$$

is valid for any $t > t_0$, $x \in \Omega$, $V \in V_0$ and $c \in K$.

Then the set M is uniformly globally asymptotically stable with respect to problem (1), (2), (3).

Proof. For an arbitrary $\varepsilon > 0$ choose the positive number $\delta = \delta(\varepsilon)$ so that $b(\delta) < a(\varepsilon)$.

Let $\alpha > 0$, $\varphi_0 \in \overline{S_\alpha}(C_0) \bigcap M_0(t, \delta)$ and $x(t) = x(t; t_0, \varphi_0)$. Then for any $t \in J^+(t_0, \varphi_0)$ the following inequalities are valid

$$a(d(x(t), M(t))) \leq V(t, x(t)) \leq V(t_0, \varphi_0(t_0)) \leq \\ b(d(\varphi_0(t_0), M_0(t_0))) \leq b(d_0(\varphi_0, M_0(t))) < b(\delta) < a(\epsilon).$$

Since by Lemma 1 $J^+(t_0, \varphi_0) = (t_0, \infty)$, then $x(t) \in M(t, \varepsilon)$ for $t > t_0$.

Thus it is proved that the set M is uniformly stable with respect to the problem (1), (2), (3).

Now we shall prove that the set M is uniformly globally attractive with respect to the problem (1), (2), (3).

Let again $\varepsilon > 0$ and $\eta > 0$ be given. Choose the number $\delta = \delta(\varepsilon) > 0$ so that $b(\delta) < a(\varepsilon)$.

We shall prove that there exists $\sigma = \sigma(\varepsilon, \eta) > 0$ such that for any solution $x(t) = x(t; t_0, \varphi_0)$ of problem (1), (2), (3) for which $t_0 \in R$, $\varphi_0 \in \overline{S_\alpha}(C_0) \bigcap M_0(t, \eta)$ ($\alpha > 0$ - arbitrary) and for any $t^* \in [t_0, t_0 + \sigma]$ the following inequality is valid

$$d(x(t^*), M(t^*)) < \delta(\varepsilon). \tag{14}$$

Suppose that this is not true. Then for any $\sigma > 0$ there exists a solution $x(t) = x(t; t_0, \varphi_0)$ of problem (1), (2), (3) for which $t_0 \in R$, $\varphi_0 \in \overline{S_\alpha}(C_0) \bigcap M_0(t, \eta)$, $\alpha > 0$, such that

$$d(x(t), M(t)) \geq \delta(\varepsilon), \tag{15}$$

for $t \in [t_0, t_o + \sigma]$.

From condition 3 of Theorem 3 and (4) it follows that

$$V(t, x(t)) - V(t_0, \varphi_0(t_0)) \leq \int_{t_0}^{t} D_- V(s, x(s)) ds \leq$$
$$\leq - \int_{t_0}^{t} \lambda(s) c(d(x(s), M(s))) ds, \ t > t_0. \tag{16}$$

From the properties of the function $V(t, x(t))$ in the interval (t_0, ∞) it follows that there exists the finite limit

$$\lim_{t \to \infty} V(t, x(t)) = v_0 \geq 0. \tag{17}$$

Then from condition 2 of Theorem 3, (15)-(17) it follows that

$$\int_{t_0}^{\infty} \lambda(t) c(d(x(t), M(t))) dt \leq b(\eta) - v_0.$$

From the integral positivity of the function $\lambda(t)$ it follows that the number σ can be chosen so that

$$\int_{t_0}^{t_0+\sigma} \lambda(t) dt > \frac{b(\eta) - v_0 + 1}{c(\delta(\varepsilon))}.$$

Then

$$b(\eta) - v_0 \geq \int_{t_0}^{\infty} \lambda(t) c(d(x(t), M(t))) dt \geq$$
$$\geq \int_{t_0}^{t_0+\sigma} \lambda(t) c(d(x(t), M(t))) dt \geq c(\delta(\varepsilon)) \int_{t_0}^{t_0+\sigma} \lambda(t) dt > b(\eta) - v_0 + 1.$$

The contradiction obtained shows that there exists a positive constant $\sigma = \sigma(\epsilon, \eta)$ such that for any solution $x(t) = x(t; t_0, \varphi_0)$ of problem (1), (2), (3) for which $t_0 \in R$, $\varphi_0 \in \overline{S_\alpha}(C_0) \bigcap M_0(t, \eta)$, $\alpha > 0$, there exists $t^* \in [t_0, t_0 + \sigma]$ such that inequality (14) holds.

Then for $t \geq t^*$ (hence for any $t \geq t_0 + \sigma$ as well) the following inequalities are valid

$$a(d(x(t), M(t))) \leq V(t, x(t)) \leq V(t^* + 0, x(t^* + 0)) \leq$$
$$\leq b(d(x(t^*), M(t^*))) < b(\delta) < a(\epsilon),$$

which proves that the set M is uniformly globally attractive with respect to problem (1), (2), (3).

The uniform M-boundedness of the solutions of problem (1), (2), (3) is proved as in the proof of Theorem 2.

The proofs of the following two theorems are analogous to the proof of Theorem 3.

□

Theorem 4. *Let the conditions of Theorem 3 hold, condition 2 being replaced by the condition*

$$a(d(x, M(t))) \leq V(t, x) \leq b(d(x, M(t)), |x|),$$

for $(t, x) \in [t_0, \infty) \times R^n$, *where the function* $b \in K^*$.

Then the set M *is t-uniformly globally asymptotically stable with respect to problem* (1), (2), (3).

Theorem 5. *Let the conditions of Theorem 3 hold, condition 2 being replaced by the condition*

$$a(d(x, M(t))) \leq V(t, x) \leq b(t, d(x, M(t))),$$

for $(t, x) \in [t_0, \infty) \times R^n$, *where the function* $b \in CK$.

Then the set M *is* α*-uniformly globally asymptotically stable with respect to problem* (1), (2), (3).

Corollary 2. *Let the conditions of Theorem 3 hold, condition 3 being replaced by the condition*

$$D_{-}V(t, x(t)) \leq -c(d(x(t), M(t))), \ t \neq \tau_k, \ k = 1, 2, \ldots, \tag{18}$$

for $t > t_0, \ x \in \Omega, \ c \in K$.

Then the set M *is uniformly globally asymptotically stable with respect to problem* (1), (2), (3).

Corollary 3. *Let the conditions of Theorem 3 hold, condition 3 being replaced by the condition*

$$D_{-}V(t, x(t)) \leq -cd(x(t), M(t)), \ t \neq \tau_k, \ k = 1, 2, \ldots, \tag{19}$$

for $t > t_0, \ x \in \Omega, \ c = const > 0$.

Then the set M *is uniformly globally asymptotically stable with respect to problem* (1), (2), (3).

Corollary 4. *Let the conditions of Theorem 4 hold, condition 3 being replaced by the condition (18) (or by condition (19)).*

Then the set M *is t-uniformly globally asymptotically stable with respect to problem* (1), (2), (3).

Corollary 5. *Let the conditions of Theorem 5 hold, condition 3 being replaced by the condition (18) (or by condition (19)).*

Then the set M *is* α*-uniformly globally asymptotically stable with respect to problem* (1), (2), (3).

Theorem 6. *Let the following conditions hold:*

1. *Conditions H1-H9 are met.*
2. *The function* $V \in V_0$ *is such that*
   (i) $d(x, M(t)) \leq V(t,x)$ *for* $(t,x) \in [t_0, \infty) \times R^n$;
   (ii) $V(t_0, \varphi_0(t_0)) \leq K(\alpha, \eta) d(\varphi_0(t_0), M_0(t_0))$, $\alpha > 0$, $\eta > 0$, $\varphi_0(t_0) \in \overline{S_\alpha} \bigcap \overline{M_0(t_0, \eta)}$.
3. *The inequality*

$$D_- V(t, x(t)) \leq -cV(t, x(t)), \ t \neq \tau_k, \ k = 1, 2 \ldots$$

is valid for any $t > t_0$, $x \in \Omega, V \in V_0$ *and* $c = const > 0$.

Then the set M *is exponentially globally asymptotically stable with respect to problem* (1), (2), (3).

Proof. Let $\alpha > 0$, $\eta > 0$, $\varphi_0 \in \overline{S_\alpha}(C_0) \bigcap \overline{M_0(t, \eta)}$. From condition 3 of Theore 6 and (4) we obtain

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0, \varphi_0(t_0)) exp[-c(t - t_0)], \quad t > t_0.$$

Then from condition 2 of Theorem 6 it follows that

$$\begin{aligned} d(x(t; t_0, \varphi_0), M(t)) &\leq V(t, x(t; t_0, \varphi_0)) \leq \\ &\leq V(t_0, \varphi_0(t_0)) exp[-c(t - t_0)] \leq \\ &\leq K(\alpha, \eta) d(\varphi_0(t_0), M_0(t_0)) exp[-c(t - t_0)] \leq \\ &\leq K(\alpha, \eta) d_0(\varphi_0(t), M_0(t)) exp[-c(t - t_0)], \ t > t_0 \end{aligned}$$

which proves that the set M is exponentially globally asymptotically stable with respect to problem (1), (2), (3). □

4. An example

Consider the problem

$$\begin{cases} \dot{x}(t) = \begin{cases} A(t)x(t) + B(t)x(t-h), \ x(t) > 0, \ t \neq \tau_k, \ t > 0, \\ \quad 0, \qquad x(t) \leq 0, \ t \neq \tau_k, \ t > 0, \end{cases} \\ x(t) = \varphi_1(t), t \in [-h, 0], \\ \Delta x(t) = \begin{cases} I_k(x(t)), \ x(t) > 0, \ t = \tau_k, \ t > 0, \\ 0, \qquad x(t) \leq 0, \ t = \tau_k, \ t > 0, \end{cases} \end{cases} \tag{20}$$

where $x \in PC[R_+, R^n]$; A(t) and B(t) are $(n \times n)$ matrix- valued continuous functions, $B(t)$ is diagonal and $A(t)$ is skew-symmetric; $\varphi_1 \in C[[-h, 0], R^n]$; $I_k(x)$, $k = 1, 2, \ldots$ are continuous in R^n and such that $x + I_k(x) > 0$ and $|x + I_k(x)| \leq |x|$ for $x > 0$ (Here $x > 0$ (respectively $x \leq 0$) means that $x_k > 0$ ($x_k \leq 0$) for $k = 1, 2, \ldots, n$, where x_k is the k-th component of the vector $x \in R^n$).

The impulse moments $\{\tau_k\}_{k=1}^{\infty}$ form a strictly increasing sequence, i.e. $0 < \tau_1 < < \tau_2 < \cdots < \tau_k < \ldots, \lim\limits_{k\to\infty} \tau_k = \infty$.

Let $M = [-h, \infty) \times \{x \in R^n : x \leq 0\}$.

Consider the function

$$V(t,x) = \begin{cases} <x,x>, & for\ x > 0, \\ 0, & for\ x \leq 0. \end{cases}$$

Then the set Ω is defined by the equality

$$\Omega = \{x \in PC[R_+, R^n] : <x(s), x(s)> \leq \\ \leq <x(t), x(t)>,\ t-h \leq s \leq t,\ t \in R_+\}.$$

For $t > 0$ and $x \in \Omega$ we have

$$D_-V(t, x(t)) = \begin{cases} 2 <x(t), B(t)x(t-h)>, & x(t) > 0,\ t \neq \tau_k, \\ 0, & x(t) \leq 0, t \neq \tau_k \end{cases}$$

$$\leq \begin{cases} 2 <x(t), B(t)x(t)>, & x(t) > 0,\ t \neq \tau_k, \\ 0, & x(t) \leq 0, t \neq \tau_k. \end{cases}$$

$$V(t+0, x + I_k(x)) = \begin{cases} |x + I_k(x)|^2, & for\ x > 0,\ t = \tau_k, \\ 0, & for\ \ x(t) \leq 0,\ t = \tau_k. \end{cases}$$

Hence

$$V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k))) \leq V(\tau_k, x(\tau_k)),\ k = 1, 2, \ldots$$

Let $B(t) = diag(b_1(t), \ldots, b_n(t))$ and $b_k(t) \leq -\gamma_k < 0$ for $k = 1, 2, \ldots, n$ and $t > 0$.

Since $d(x, M(t)) = |x|$ for $t > 0$ and $x > 0$, then the conditions of Theorem 3 are satisfied.

Hence the set M is uniformly globally asymptotically stable with respect to problem (20).

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