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On completely dense sequences

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Abstract: We define completely density of real sequences as a generalization of the completely uniformly distribution of real sequences. There are given sufficient conditions for the completely density of multiplicative arithmetic function and the topological properties of completely dense sequences are studied.

Key Words: sequences, multiplicative arithmetical function, density, residual set

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The concept of completely uniform distribution mod 1 was introduced in [2] by N. M. Korobov (see also [3]). The sequence (x_n) of real numbers is called completely uniformly distributed mod1 if for every $s = 1, 2, \dots$ the s-dimensional sequence $((x_{n+1}, x_{n+2}, \dots x_{n+s}))$ is uniformly distributed mod1. This suggest the following definition.

Definition. The sequence (x_n) of real numbers is completely dense mod1 if for every $s = 1, 2, \dots$ the s-dimensional sequence $((x_{n+1}, x_{n+2}, \dots, x_{n+s}))$ is dense mod1 in $[0, 1]^s$. Analogously, the sequence (x_n) is completely dense in an interval $\mathcal{I}$ if for every $s = 1, 2, \dots$ the s-dimensional sequence $((x_{n+1}, x_{n+2}, \dots, x_{n+s}))$ is dense in $\mathcal{I}^s$

Trivially, if a sequence is completely uniformly distributed mod1 then it is completely dense mod1. Examples of completely uniformly distributed sequences mod1 was constructed by L. P. Starčenko [5].

In this paper we give sufficient conditions for the complete density of multiplicative arithmetic functions and show, that from topological point of view the complete density is a typical property of sequences.

Let p_n denote the n-th prime number. We have

Theorem 1. *Let $f(n)$ be a positive multiplicative arithmetical function satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} f(p_n)^n = 1$

(ii) $\prod\limits_{\substack{p_n \\ f(p_n)>1}} f(p_n) = +\infty, \qquad \prod\limits_{\substack{p_n \\ f(p_n)<1}} f(p_n) = 0.$

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Then the sequence $(f(n))_{n=1}^{\infty}$ is completely dense in the set of positive real numbers.

Proof. It is sufficient to prove, that for arbitrary $\varepsilon > 0$ and arbitrary $a_1, a_2, \ldots, a_l > 0$ $(l \geq 1)$ there exists an $k \in \mathbb{N}$ such that

$$|a_i - f(k+i)| < \varepsilon, \quad (i = 1, 2, \ldots, l).$$

Let $\varepsilon > 0$ and $a_1, a_2, \ldots, a_l$ be positive real numbers. The condition *(i)* gives $\lim_{n\to\infty} f(p_n) = 1$. From this and the condition *(ii)* follows that there exist $A_1, A_2, \ldots$ $\ldots, A_l$ pairwise disjoint sets of natural numbers such that

$$|a_i - f(i) \prod_{j \in A_i} f(p_j)| < \frac{\varepsilon}{2}, \quad (i = 1, 2, \ldots, l) \tag{1}$$

and the minimum element of $A = \bigcup_{i=1}^{l} A_i$ is greater than l.

Let us consider the system of congruences

$$\begin{aligned} x &\equiv 0 \pmod{(l!)^2} \\ x &\equiv 0 \pmod{\prod_{\substack{j \notin A \\ p_j > l,\ j < m}} p_j} \\ x + i &\equiv p_j \pmod{p_j^2},\ j \in A_i, \quad (i = 1, 2, \ldots, l), \end{aligned} \tag{2}$$

where m is a sufficiently large natural number. By the Chinese Remainder Theorem system (2) has an $x = k$ solution satisfying $0 < k < y$, where

$$y = (l!)^2 \prod_{j \in A} p_j^2 . \prod_{\substack{j \notin A \\ p_j > l,\ j < m}} p_j .$$

It is known that every term of the arithmetical progression $(k + by)_{b=1}^{\infty}$ is a solution of (2).

We will show, that in this arithmetical progression there is a term for which in the factorings

$$k + i + by = i . \prod_{j \in A_i} p_j . u_i, \qquad (u_i, y) = 1 \tag{3}$$

and each u_i $(i = 1, 2, \ldots, l)$ is square-free.

The number of terms in the sequence $b = 1, 2, \ldots, y$ for which at least one term in the arithmetic progressions $(k+i+by)_{b=1}^{y}$ has in the factoring into prime numbers a prime greater or equal to p_m the exponent at least 2, is not greater than $l . \sum_{j \geq m} \lceil \frac{y}{p_j^2} \rceil$ for sufficiently large m. If we denote the number of primes that do not exceed x by $\pi(x)$ we have

$$l . \sum_{j \geq m} \lceil \frac{y}{p_j^2} \rceil \leq y . \left(l . \sum_{j \geq m} \frac{1}{j^2} + \frac{l . \pi(\sqrt{y})}{y} \right)$$

Since

$$l.\sum_{j\geq m}\frac{1}{j^2}+\frac{l.\pi(\sqrt{y})}{y}\to 0 \quad \text{for} \quad m\to\infty$$

for sufficiently large m there is a solution $k+by$ of (2), where $0\leq b\leq y$ and in the factorings (3) each u_i is square-free ($i=1,2,\dots,l$).

Further, the condition *(i)* guarantees such a choosing of m that $|f(p_m)-1|\geq |f(p_n)-1|$ for arbitrary positive integer $n\geq m$, moreover

$$|f(p_m)^{2m}-1|<\frac{\varepsilon}{4.\sum_{i=1}^{l}(a_i+1)} \tag{4}$$

and

$$|(2-f(p_m))^{2m}-1|<\frac{\varepsilon}{4.\sum_{i=1}^{l}(a_i+1)} \tag{5}$$

Let the factoring of u_i be of the form $u_i=\prod_{j=1}^{r}q_j$. Notice that $q_j\geq p_m$ ($j=1,2,\dots,r$). The combination of $u_i^2\leq y^2$ and $p_m^m>y$ yields $r\leq 2m$.

Using the inequality $f(q_j)\leq f(p_m)$ in the case $f(p_m)>1$ and the inequality $f(q_j)\leq 2-f(p_m)$ in the case $f(p_m)<1$ together with (3), (4) and (5) we get

$$f(k+i+by)=f(i).\prod_{j\in A_i}f(p_j)\prod_{j=1}^{r}f(q_j)<(a_i+\frac{\varepsilon}{2})(1+\frac{\varepsilon}{4.\sum_{i=1}^{l}(a_i+1)})<a_i+\varepsilon.$$

Using the same method, we can prove that $f(k+i+by)>a_i-\varepsilon,\ i=1,2,\dots,l$. The proof is complete. □

Remark. The multiplicative arithmetic function

$$f(n)=\prod_{\substack{p_r|n\\ r\text{ is even}}}(1+\frac{1}{p_r})\prod_{\substack{p_r|n\\ r\text{ is odd}}}(1-\frac{1}{p_r})$$

is completely dense in the set of positive real numbers.

From the proof of Theorem 1 follows that the sequences $(\frac{\sigma(n)}{n})_{n=1}^{\infty}$, $(\frac{n}{\varphi(n)})_{n=1}^{\infty}$, $(\frac{\sigma(n)}{\varphi(n)})_{n=1}^{\infty}$ are completely dense in $(1,\infty)$ generalizing results in [1], [4].

In the next part we will study the topological properties of completely dense sequences.

Denote by s the space of all real sequences with the Frèchet metric

$$\rho(\mathbf{a},\mathbf{b})=\sum_{n=1}^{\infty}2^{-n}\frac{|a_n-b_n|}{1+|a_n-b_n|},$$

where $\mathbf{a}=(a_n)_{n=1}^{\infty}\in \mathbf{s}$ and $\mathbf{b}=(b_n)_{n=1}^{\infty}\in\mathbf{s}$. Denote by $\mathcal{A}$ the set of all real sequences $(a_n)_{n=1}^{\infty}$ with the property: for arbitrary $\varepsilon>0$, arbitrary positive integer l and arbitrary $b_1,b_2,\dots,b_l$ real numbers there exist infinitely many k such that

$$|a_{k+i}-b_i|<\varepsilon,\quad (i=1,2,\dots,l).$$

We have

Theorem 2. *The set $\mathcal{A}$ is residual in the metric space* $(\mathbf{s}, \rho)$.

Proof. Let $l \in \mathbb{N}$. In the sequel we need some auxiliary sets. For $\varepsilon > 0$ and $\mathbf{b} = (b_1, b_2, \ldots, b_l) \in \mathbb{R}^l$ denote by $A(l, \mathbf{b}, \varepsilon)$ the set of all real sequences satisfying the property: there exist only finitely many k such that $|a_{k+i} - b_i| < \varepsilon$, $(i = 1, 2, \ldots, l)$. Then $A(l, \mathbf{b}, \varepsilon) = \cup_{m=1}^{\infty} A_m(l, \mathbf{b}, \varepsilon)$, where $A_m(l, \mathbf{b}, \varepsilon)$ is the set of all real sequences $(a_n)_{n=1}^{\infty}$ such that there exist at most m solutions of the system of inequalities $|a_{k+i} - b_i| < \varepsilon$, $(i = 1, 2, \ldots, l)$ for k. Taking into account that a $\mathbf{x}^{(r)} = (x_n^{(r)})_{n=1}^{\infty}$, $r = 1, 2, \ldots$ sequence of real sequences converges by Frèchet metrics to $\mathbf{x} = (x_n)_{n=1}^{\infty}$ (for $r \to \infty$) if and only if $\lim_{r\to\infty} x_n^{(r)} = x_n$ for every $n = 1, 2, \ldots$, it is easy to see that the set $A_m(l, \mathbf{b}, \varepsilon)$ is closed in $(\mathbf{s}, \rho)$. Therefore $A(l, \mathbf{b}, \varepsilon)$ is an F_σ set in $\mathbf{s}$.

We prove,that the set $\mathbf{s} \setminus A(l, \mathbf{b}, \varepsilon)$ is dense in $\mathbf{s}$. Let $\mathbf{a} = (a_n)_{n=1}^{\infty} \in A(l, \mathbf{b}, \varepsilon)$ and $\eta > 0$. Then there exists an $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0+1}^{\infty} \frac{1}{2^n} < \eta$. Define the sequence $\mathbf{c} = (c_n)_{n=1}^{\infty}$:

$$c_n = \begin{cases} a_n, & \text{if } n \le n_0 \\ b_i, & \text{if } n = n_0 + k.l + i, \quad \text{where } k \ge 0 \text{ and } i = 1, 2, \ldots, l. \end{cases}$$

Then evidently $\mathbf{c} \in \mathbf{s} \setminus A(l, \mathbf{b}, \varepsilon)$ and we have

$$\rho(\mathbf{a}, \mathbf{c}) = \sum_{n=n_0+1}^{\infty} \frac{1}{2^n} \frac{|a_n - c_n|}{1 + |a_n - c_n|} < \sum_{n=n_0+1}^{\infty} \frac{1}{2^n} < \eta.$$

Therefore, the set $A(l, \mathbf{b}, \varepsilon)$ is of the first category in $\mathbf{s}$.

Denote by $\mathbb{Q}$ the set of all rational numbers. Using

$$\bigcap_{l=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{\mathbf{b} \in \mathbb{Q}^l} \mathbf{s} \setminus A(l, \mathbf{b}, \frac{1}{n}) \subset \mathcal{A},$$

together with the pervious facts we get that the set $\mathcal{A}$ is residual in $(\mathbf{s}, \rho)$. □

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