Some remarks on the discrepancy of the sequence $(\alpha\sqrt{n})$

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Abstract: Let $\alpha > 0$ and $\alpha^2 \in \mathbb{Q}$. We describe a way of calculating $\underline{\lim}_{N\to\infty} N^{-1/2} D_N^+(\alpha)$ where $D_N^+(\alpha)$ is a quantity related to the discrepancy of the uniformly distributed sequence $(\alpha\sqrt{n})_{n\geq 1}$.

Key Words: Uniformly distributed sequence, discrepancy

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For any $\alpha > 0$ the sequence $(\alpha\sqrt{n})_{n\geq 1}$ is uniformly distributed modulo 1. The discrepancies

$$D_N^*(\alpha) = \sup_{0\leq x<1} \left| \sum_{n=1}^{N} c_{[0,x)}(\{\alpha\sqrt{n}\}) - Nx \right|$$

and

$$D_N(\alpha) = \sup_{0\leq x<y\leq 1} \left| \sum_{n=1}^{N} c_{[x,y)}(\{\alpha\sqrt{n}\}) - N(y-x) \right|$$

are used to study this fact from a quantitative point of view. (Here c_A denotes the characteristic function of the set A and $\{x\} = x - [x]$ is the fractional part of the real number x.) They are related to the auxiliary quantities

$$D_N^+(\alpha) = \sup_{0\leq x<1} \Big(\sum_{n=1}^{N} c_{[0,x)}(\{\alpha\sqrt{n}\}) - Nx \Big)$$

and

$$D_N^-(\alpha) = \sup_{0\leq x<1} \Big(Nx - \sum_{n=1}^{N} c_{[0,x)}(\{\alpha\sqrt{n}\}) \Big)$$

via $D_N^*(\alpha) = \max\{D_N^+(\alpha), D_N^-(\alpha)\}$ and $D_N(\alpha) = D_N^+(\alpha) + D_N^-(\alpha)$. If $\alpha^2 \notin \mathbb{Q}$ J. Schoißengeier [4] proved

$$\overline{\lim_{N\to\infty}} \tfrac{1}{\sqrt{N}} D_N^+(\alpha) = \overline{\lim_{N\to\infty}} \tfrac{1}{\sqrt{N}} D_N^-(\alpha) = \overline{\lim_{N\to\infty}} \tfrac{1}{\sqrt{N}} D_N^*(\alpha) = \lim_{N\to\infty} \tfrac{1}{\sqrt{N}} D_N(\alpha) = \frac{1}{4\alpha},$$

$$\underline{\lim}_{N\to\infty} \tfrac{1}{\sqrt{N}} D_N^+(\alpha) = \underline{\lim}_{N\to\infty} \tfrac{1}{\sqrt{N}} D_N^-(\alpha) = 0 \quad \text{and} \quad \underline{\lim}_{N\to\infty} \tfrac{1}{\sqrt{N}} D_N^*(\alpha) = \frac{1}{8\alpha}.$$

The much more difficult case $\alpha^2 \in \mathbb{Q}$ was tackled recently by C. Baxa and J. Schoißengeier [2] who described a way of calculating

$$\overline{\lim_{N\to\infty}} N^{-1/2} D_N^+(\alpha) \quad \text{and} \quad \overline{\lim_{N\to\infty}} N^{-1/2} D_N^-(\alpha) \quad \text{and thus} \quad \overline{\lim_{N\to\infty}} N^{-1/2} D_N^*(\alpha).$$

An analogous result for

$$\overline{\lim_{N\to\infty}} N^{-1/2} D_N(\alpha)$$

was proved by C. Baxa in a follow-up paper [1].

It is the purpose of this note to describe an analogous result for

$$\underline{\lim_{N\to\infty}} N^{-1/2} D_N^+(\alpha)$$

and to discuss the limitations of the method used.

We need a few notations which will be in force throughout the paper: Let $\alpha^2 = q/p$ where p, q are positive integers and $\gcd(p,q) = 1$, $q = q_1^2 q_2$ and q_2 squarefree,

$$f(x,\beta) = \beta(1-\beta) - |x-\beta| + (x-\beta)^2 \quad \text{for} \quad 0 \le x, \beta < 1,$$

$$B_1(x) = \{x\} - 1/2, \qquad B_2(x) = \{x\}^2 - \{x\} + 1/6,$$

$$M = \{(a,u,v) \in \mathbb{Z}^3 \mid 0 \le a < 2p, \quad 0 \le u < v, \quad \gcd(u,v) = 1\},$$

$$x(a,u,v) = \frac{1}{2p}(a + \frac{u}{v})$$

and

$$S^+(a,u,v) = \tfrac{1}{2} \sup_{0<|\kappa|\le 1} \tfrac{1}{\kappa} \sum_{k=1}^{q} \Big(B_2\big(\tfrac{vp}{q}(k + x(a,u,v))^2\big) - B_2\big(\tfrac{vp}{q}(k + x(a,u,v))^2 + \kappa\big)\Big)$$

for $(a,u,v) \in M$.

Lemma 1. *As* $N \to \infty$

$$\tfrac{1}{\sqrt{N}} D_N^+(\alpha) = \tfrac{1}{\sqrt{pq_2}} + \tfrac{1}{\sqrt{pq}} \sum_{k=0}^{q-1} B_1(\tfrac{p}{q}k^2)$$
$$+ \sup_{(a,u,v)\in M} \Big(\sqrt{\tfrac{p}{q}} f(x(a,u,v), \{\alpha\sqrt{N}\}) + \tfrac{1}{v\sqrt{pq}} S^+(a,u,v)\Big) + O(N^{-1/4}\log^2 N).$$

Proof. This is part of Lemma 5 of [2]. □

Lemma 2.

$$\varlimsup_{N\to\infty} \tfrac{1}{\sqrt{N}} D_N^+(\alpha) = \tfrac{1}{\sqrt{pq_2}}$$
$$+ \tfrac{1}{\sqrt{pq}} \sum_{k=0}^{q-1} B_1(\tfrac{pk^2}{q}) + \sqrt{\tfrac{p}{q}} \sup_{(a,u,v)\in M} \Big(x(a,u,v)\big(x(a,u,v)-1\big) + \tfrac{1}{pv} S^+(a,u,v)\Big).$$

Proof. Set $N_\mu = pq\mu^2$ for $\mu \geq 1$. Then $\{\alpha\sqrt{N_\mu}\} = 0$ and by Lemma 1

$$\varlimsup_{N\to\infty} \tfrac{1}{\sqrt{N}} D_N^+(\alpha) \leq \lim_{\mu\to\infty} \tfrac{1}{\sqrt{N_\mu}} D_{N_\mu}^+(\alpha)$$
$$= \lim_{\mu\to\infty} \Big(\tfrac{1}{\sqrt{pq_2}} + \tfrac{1}{\sqrt{pq}} \sum_{k=0}^{q-1} B_1(\tfrac{pk^2}{q}) + \sqrt{\tfrac{p}{q}} \sup_{(a,u,v)\in M} \Big(f\big(x(a,u,v),0\big) + \tfrac{1}{vp} S^+(a,u,v)\Big)\Big)$$
$$= \tfrac{1}{\sqrt{pq_2}} + \tfrac{1}{\sqrt{pq}} \sum_{k=0}^{q-1} B_1(\tfrac{pk^2}{q})$$
$$+ \sqrt{\tfrac{p}{q}} \sup_{(a,u,v)\in M} \Big(x(a,u,v)\big(x(a,u,v)-1\big) + \tfrac{1}{vp} S^+(a,u,v)\Big).$$

It is easy to check that $f(x,\beta) \geq f(x,0)$ for $0 \leq x, \beta < 1$ and the converse inequality follows from Lemma 1. □

Theorem.

$$\varlimsup_{N\to\infty} \tfrac{1}{\sqrt{N}} D_N^+(\alpha)$$
$$= \tfrac{1}{\sqrt{pq_2}} + \tfrac{1}{\sqrt{pq}} \sum_{k=0}^{q-1} B_1(\tfrac{pk^2}{q}) + \sqrt{\tfrac{p}{q}} \sup_{(a,u,v)\in B} \Big(x(a,u,v)\big(x(a,u,v)-1\big) + \tfrac{1}{vp} S^+(a,u,v)\Big)$$

where $B = \{\, (a,u,v) \in \mathbb{Z}^3 | 0 \leq a < 2p, \quad 0 \leq u < v < 16pq^3, \quad \gcd(u,v) = 1 \,\}$.

Proof. Suppose that

$$x(a,u,v)\big(x(a,u,v)-1\big) + \tfrac{1}{vp} S^+(a,u,v)$$
$$> x(0,1,2q)\big(x(0,1,2q)-1\big) + \tfrac{1}{2pq} S^+(0,1,2q) = \tfrac{1}{32p^2q^2}$$

where we made use of a way of calculating $S^+(0,1,2q)$ described in [2]. Using the trivial estimate $S^+(0,1,2q) \leq q/2$ yields $\frac{1}{4}(1 - \frac{av+u}{pv})^2 - \frac{1}{4} + \frac{q}{2pv} > \frac{1}{32p^2q^2}$ and therefore $\big(1 - 2x(a,u,v)\big)^2 + \frac{2q}{pv} > 1 + \frac{1}{8p^2q^2}$. As $|1 - 2x(a,u,v)| \leq 1$ we get $v < 8p^2q^2 \cdot 2q/p = 16pq^3$. Finally note that $(0,1,2q) \in B$. □

Remark. The sum $\sum_{k=0}^{q-1} B_1(pk^2/q)$ was studied by M. Lerch [3] in the case $2 \nmid q$. If q is an odd prime his result is reduced to

$$\sum_{k=0}^{q-1} B_1(\tfrac{pk^2}{q}) = \begin{cases} -\frac{1}{2} - \left(\frac{p}{q}\right)\frac{2h(-q)}{w} & \text{if } q \equiv 3 \pmod 4 \\ -\frac{1}{2} & \text{if } q \equiv 1 \pmod 4. \end{cases}$$

Here $h(-q)$ denotes the class number and w the order of the unit group of the imaginary quadratic field $\mathbb{Q}(\sqrt{-q})$.

Corollary.

$$\varliminf_{N\to\infty} \frac{1}{\sqrt{N}} D_N^+\left(\sqrt{\tfrac{q}{p}}\right) = \begin{cases} \frac{1}{\sqrt{p}} & \textit{if } q = 1 \\ (1 + \frac{1}{8p})\frac{1}{\sqrt{2p}} & \textit{if } q = 2 \textit{ (and thus } 2 \nmid p\textit{)} \\ \frac{1}{\sqrt{3p}} & \textit{if } q = 3 \textit{ and } p \equiv 1 \pmod 3 \\ (\frac{3}{2} + \frac{1}{8p})\frac{1}{\sqrt{3p}} & \textit{if } q = 3 \textit{ and } p \equiv 2 \pmod 3. \end{cases}$$

Proof. Only the case $q = 1$ will be proved. The other assumptions can be proved along the same lines. If

$$x(a,u,v)\big(x(a,u,v)-1\big)+\tfrac{1}{vp}S^+(a,u,v) > x(0,0,1)\big(x(0,0,1)-1\big)+\tfrac{1}{p}S^+(0,0,1) = \tfrac{1}{2p}$$

then $\frac{1}{2p} < \frac{1}{4}(1 - \frac{av+u}{pv})^2 - \frac{1}{4} + \frac{1}{2pv} \le \frac{1}{2pv}$ and thus $v < 1$, which is impossible. □

Remarks.

1. The papers [2] and [1] contain tables of values of $\varlimsup_{N\to\infty} N^{-1/2}D_N^+(\sqrt{q/p})$, $\varlimsup_{N\to\infty} N^{-1/2}D_N^-(\sqrt{q/p})$ and $\varlimsup_{N\to\infty} N^{-1/2}D_N(\sqrt{q/p})$ which exhibit very little regularity.
   In comparison the Corollar suggests that the values of
   $$\varliminf_{N\to\infty} N^{-1/2}D_N^+(\sqrt{q/p})$$
   should obey a fairly simple law. However, the method used so far seems to be unsuitable to prove a respective theorem. This is due to the heavily increasing amount of computation necessary for larger values of q, one reason for which is that the estimate $S^+(a,u,v) \le q/2$ is rather weak for larger q.
2. Lemma 5 of [2] also contains a description of $N^{-1/2}D_N^-(\sqrt{q/p})$ analogous to that of Lemma 1. Therefore, the reader might be surprised by the absence of a formula for $\varliminf_{N\to\infty} N^{-1/2}D_N^-(\sqrt{q/p})$, but there is no easy way of replacing Lemma 2 as there is no $\beta_0 \in [0,1)$ such that $f(x,\beta) \le f(x,\beta_0)$ for all $x \in [0,1)$.

References

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