Acta Mathematica et Informatica Universitatis Ostraviensis

Stéphane Louboutin; M. F. Newman
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Acta Mathematica et Informatica Universitatis Ostraviensis, Vol. 6 (1998), No. 1, 155--158

Persistent URL: http://dml.cz/dmlcz/120528

On the diophantine equation $XY + YZ + ZX = D$

S. Louboutin
M. F. Newman

Abstract: We characterize the positive integers d such that the diophantine equation $xy + yz + zx = d$ has no solution in positive integers and deduce that there are only finitely many d's such that this equation has no solution in positive integers.

Key Words: ternary quadratic forms, diophantine equations, ideal class groups (of imaginary quadratic fields)

Mathematics Subject Classification: Primary 11D09; Secondary 11E04, 11R11 and 11R29

Let $d \geq 1$ denote a positive integer. Then, the diophantine equation

$$E_m(d) \qquad \sum_{1 \leq i < j \leq m} x_i x_j = d$$

has at least one solution in positive integers provided that $m \geq 4$ and $d \geq 136m^2$ (see [Kov]). L.J. Mordell discussed the solutions of $E_3(d)$ in non-negative integers. Here, we prove the following result on $E_3(d)$ which is much more satisfactory than the one proved by T. Cai and more enlightening that the remarks made in the beginings of [HBP] and [Kov] (note also the misprint in [HBP] for $E_3(d)$ has no solution for d =2, 6 and 10):

Theorem 1. *Let $d \geq 1$ be a positive integer. The diophantine equation*

$$E_3(d) \qquad xy + yz + zx = d$$

has no solution in positive integers if and only if either $d \in \{1, 4, 18\}$, or $d \equiv 2 \pmod 4$ is square-free and such that the ideal class group of the imaginary quadratic number field $\mathbf{Q}(\sqrt{-d})$ of discriminant $-4d$ has exponent at most 2. Hence, assuming the generalized Riemann hypothesis, there are exactly 18 positive integers d such that the diophantine equation $E_3(d)$ has no solution in positive integers, namely

$$d \in \{1, 2, 4, 6, 10, 18, 22, 30, 42, 58, 70, 78, 102, 130, 190, 210, 330, 462\}.$$

Moreover, without assuming this hypothesis there is at most one more $d \geq 1$ such that the diophantine equation $E_3(d)$ has no solution in positive integers. In particular, there are only finitely many d's such $E_3(d)$ has no solution in positive integers.

The last part of Theorem 1 follows readily from its first part and the results of [Lou] and [Tat]. The proof of its first part will be divided into three Lemmas. To start with, let us recall that a (binary quadratic) positive definite form $aX^2+bXY+cY^2$ is primitive if $\gcd(a,b,c)=1$, and a (binary quadratic) form $aX^2+bXY+cY^2$ is reduced if it is positive, definite and such that $|b| \leq a \leq c$, and $b \geq 0$ if either $|b|=a$ or $a=c$. If $D<0$, we let $h(D)$ denote the number of classes of primitive positive definite forms of discriminant D. Then, $h(D)$ is equal to the number of reduced forms of discriminant D (see [Cox, Th. 2.13])

Lemma 2. *Assume that $E_3(d)$ has no solution in positive integers. Then either $d \in \{1,4,18\}$, or $d \equiv 2 \pmod 4$ is square-free.*

Proof. First, if $d>1$ is odd then $d = xy+yz+zx$ with

$$(x,y,z) = (1,1,(d-1)/2).$$

Second, if 4 divides $d>4$ then $d=xy+yz+zx$ with

$$(x,y,z) = (2,2,(d/4)-1).$$

Hence $d \in \{1,4\}$ or $d \equiv 2 \pmod 4$. Third, assume that $d \equiv 2 \pmod 4$ is not square-free. Let p be a prime whose square divides d and write $d = p^2(a-1)$ with $a \geq 3$ odd. If $3 \leq a < p$ then we write $p = ac+b$ with $1 \leq b \leq a-1$ and notice that

$$(x,y,z) = (b, a-b, p^2 - c(p+b) - b)$$

is a solution of $E_3(d)$ in positive integers. Therefore, $a \geq p$. If $a>p$ then

$$(x,y,z) = (p, p^2-p, a-p)$$

is a solution of $E_3(d)$ in positive integers. Therefore, $a=p$ and $d=p^2(p-1)$. If $p>3$ then

$$(x,y,z) = (6, p-3, p^2-4p+6)$$

is a solution of $E_3(d)$ in positive integers. Therefore, $p=3$ and $d=18$. (We are indebted to [Hal, proof of Th. II] for all these solutions.) •

Lemma 3. *Let $d \equiv 2 \pmod 4$ be a positive square-free positive integer. If $1 \leq x \leq y \leq z$ are positive integers such that $d = xy+yz+zx$, then*

$$Q(X,Y) = (x+y)X^2 + 2xXY + (x+z)Y^2$$

is reduced of discriminant $-4d$. Conversely, if $Q(X,Y) = aX^2+bXY+cY^2$ is a reduced form of discriminant $-4d$ with $b \geq 1$ then

$$(x,y,z) = (\frac{b}{2}, a-\frac{b}{2}, c-\frac{b}{2})$$

are positive integers such that $d = xy + yz + zx$ *and* $1 \leq x \leq y \leq z$.

Proof. The only non trivial point is the one which asserts that $Q(X,Y) = (x+y)X^2+2xXY+(x+z)Y^2$ is primitive. Since $\gcd(x,y,z) = 1$ then $\gcd(x+y,2x,x+z)$ divides $\gcd(2x+2y,2x,2x+2z) = \gcd(2x,2y,2z) = 2$. Moreover, $\gcd(x+y,2x,x+z) = 2$ if and only if $x \equiv y \equiv z \pmod 2$. Since $\gcd(x,y,z) = 1$ then $\gcd(x+y,2x,x+z) = 2$ if and only if $x \equiv y \equiv z \equiv 1 \pmod 2$. But this would imply $d = xy + yz + zx$ odd. A contradiction. •

Lemma 4. *Let* $d \equiv 2 \pmod 4$ *be a positive integer and let* t_d *be the number of odd primes dividing d. Then, a reduced form* $Q(X,Y) = aX^2 + bXY + cY^2$ *of discriminant* $-4d$ *has order at most* 2 *if and only if* $b = 0$. *Hence, there are* 2^{t_d} *reduced forms of discriminant* $-4d$ *and order at most* 2, *and the* 2*-rank of the form class group* $C(-4d)$ *is equal to* t_d.

Proof. Since $Q(X,Y)$ has discriminant $-4d = b^2 - 4ac$ then $b = 2B$ is even. According to [Cox, Lemma 3.10], the form $Q(X,Y)$ has order at most 2 if and only if $b = 0$, $b = a$ or $a = c$. Now, we cannot have $b = a$ since it would imply

$$d = ac - B^2 = 2Bc - B^2 = c^2 - (c-B)^2 \equiv 0,\ 1 \text{ or } 3 \pmod 4.$$

In the same way, we cannot have $a = c$ since it would imply

$$d = ac - B^2 = a^2 - B^2 \equiv 0,\ 1 \text{ or } 3 \pmod 4. \bullet$$

According to Lemmas 2 and 3 the equation $xy + yz + zx = d$ has no solutions in positive integers if and only if $d \in \{1,4,18\}$, or $d \equiv 2 \pmod 4$ is square-free and $b = 0$ for all the reduced form $Q(X,Y) = aX^2 + bXY + cY^2$ of discriminant $-4d$. Acording to Lemma 4, the first part of Theorem 1 is proved. We conclude this paper by giving a formula for the number of solutions in positive integers of $E_3(d)$ whenever $d \equiv 2 \pmod 4$ is square-free:

Theorem 5. *Let* $d \equiv 2 \pmod 4$ *be a positive square-free integer, and let* t_d *be the number of odd primes dividing d. The number* $n_3(d)$ *of solutions* (x,y,z) *in positive integers (with* $1 \leq x \leq y \leq z$*) of the diophantine equation* $E_3(d)$ *is equal to*

$$n_3(d) = (h(-4d) - 2^{t_d})/2.$$

Proof. Acording to Lemma 3 we have a bijective correspondence between the solutions (x,y,z) of $E_3(d)$ with $1 \leq x \leq y \leq z$ and the reduced forms $Q(X,Y) = aX^2+bXY+cY^2$ of discriminant $-4d$ with $b \geq 1$. According to Lemma 4, there are 2^{t_d} reduced forms of discriminant $-4d$ with $b = 0$. Finally, since the $h(-4d) - 2^{t_d}$ reduced forms $Q(X,Y) = aX^2 + bXY + cY^2$ of discriminant $-4d$ with $b \neq 0$ come in pairs $(aX^2 + bXY + cY^2, aX^2 - bXY + cY^2)$, we do get the desired result. •

Remark. If $E_3(d)$ has a solution in positive integers (x, y, z), we may assume that $1 \le x \le y \le z$. Hence, we get

$$1 \le x \le \sqrt{d/3}.$$

Moreover, if $xy + yz + zx = d$ with $x \ge 1$ and $y, z \ge 0$, then $y \le z$ if and only if $d \le xy + y^2 + xy = (x + y)^2 - x^2$, hence if and only if

$$0 \le y \le \sqrt{d + x^2} - x.$$

Note that $x \le \sqrt{d/3}$ implies $x \le \sqrt{d + x^2} - x$. These remarks make it easy to compute the number of solutions in positive integers (x, y, z) of $E_3(d)$ with $1 \le x \le y \le z$.

References

[Cai] T. Cai, *On the diophantine equation* $xy+yz+zx = m$, Publ. Math. Debrecen **45** (1994), 131-132.

[Cox] D. Cox, *Primes of the form* $x^2 + ny^2$, John Wiley & Sons (1989).

[Hal] N. A. Hall, *Binary quadratic discriminants with a single class of forms in each gennus,* Math. Zeit. **44** (1938), 85-90.

[HBP] Al-Zaid Hassan, B. Brindza and Á. Pintér, *On positive integer solutions of the equation* $xy + yz + xz = n$, Canad. Math. Bull. **39** (1996), 199-202.

[Kov] K. Kovács, *About some positive solutions of the diophantine equation* $\sum_{1 \le i < j \le n} a_i a_j = m$, Publ. Math. Debrecen **40** (1992), 207-210.

[Lou] S. Louboutin, *Minorations (sous l'hypothèse de Riemann généralisée) des nombres de classes des corps quadratiques imaginaires. Application,* C. R. Acad. Sci. Paris **310** (1990), 795-800.

[Mor] L.J. Mordell, *Diophantine equations,* Chapter 30, Section 2 : The equation $xy + yz + zx = d$, Academic Press (1969).

[Tat] T. Tatuzawa, *On a theorem of Siegel,* Japan J. Math. **21** (1951), 163-178.

Author's address: Stéphane Louboutin, Université de Caen,UFR Science, Département de Mathématiques, Esplanade de la Paix,
14032 Caen Cedex, FRANCE
Mike Newman, School of Mathematical Sciences, Australian National University, Canberra 0200, AUSTRALIA

E-mail: `louboutiQmath.unicaen.fr`
`newmanQmaths.anu.edu.au`

Received: November 17, 1997