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On the Hilbert–Ackermann Theorem in Fuzzy Logic

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Abstract. We deal with fuzzy logic in narrow sense. Our aim is to prepare the background for the resolution in fuzzy logic. The first step is to prove the analogue of the classical Hilbert–Ackerman's consistency theorem which is done in this paper.

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1 Introduction

In this paper, we continue the development of the theory of fuzzy logic in narrow sense. This concept has been studied in many papers and the term "fuzzy logic in narrow sense" has been introduced also by several authors. Recall that we mean by it a special many-valued logic stemming from Lukasiewicz one in which all the truth values are of equal importance and both syntax as well as semantics are graded. This logic plays an important role in the development of fuzzy logic in broader sense (approximate reasoning) as well as in fuzzy set theory. We claim this logic to become the language of the latter.

In the literature, many attempts to develop the resolution for fuzzy logic have been presented. Among the first ones were the papers by C. T. Lee. However, the attempts have not been too successful for various reasons, among them the most important is the fact that the authors sticked closely on the

original proposal of L. A. Zadeh for fuzzy logic which has only three basic connectives, namely $\wedge$, $\vee$ and $\neg$ which are interpreted in the interval $[0, 1]$ by the operations $\wedge, \vee$ and $1 - x$ respectively. However, such logic is too weak and to be quite candid, even not fulfilling all the intuitive requirements to be a logic suitable for modeling of the vagueness phenomenon. Of course, this does not diminish the seminal importance of the L. A. Zadeh's work. Several systems of fuzzy logic appeared during the years. In this paper, we continue the development of one of the most universal systems whose properties have been discussed in many papers. Among them, recall especially that this fuzzy logic is direct but non-trivial generalization of classical one preserving many of its important properties, and the possibility to introduce additional connectives which may be used for any purpose *without harming* the main properties of fuzzy logic. Hence, most of the results in other systems of logic as well as in fuzzy set theory can be considered to be part of our logical system.

One of the reasons why most attempts for resolution principle failed was the lack of sufficiently strong theorems justifying it. The best and most deeply penetrating work into the problem is the work of S. Lehmke [5] in which he developed the resolution principle for the propositional fuzzy logic. He introduced some unnecessary modifications, however, some of his results can be directly applied to the fuzzy logic in narrow sense presented below. To extend the resolution principle to the first-order fuzzy logic, we need some analogue of Herbrand theorem assuring us that the resolution method which also gets rid of quantifiers, is valid. This paper is a first step to this goal as we will present the analogue of the classical Hilbert–Ackermann's consistency theorem for fuzzy logic in narrow sense. To do this, we have to extend the language of fuzzy logic by the equality relation.

2 Preliminaries

We will recall only few basic notions of fuzzy logic in narrow sense. The reader may find the precise definitions and full proofs of theorems (if missing) in the cited papers.

The set of truth values forms a residuated lattice

$$\mathcal{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, 1, 0\rangle$$

where L is either a *finite chain*, or $L = [0,1]$, $\rightarrow$ is the Lukasiewicz implication and $\otimes$ is the Lukasiewicz product. We will use the operation of Lukasiewicz sum defined by

$$a \oplus b = \neg(\neg a \otimes \neg b) \qquad a, b \in L.$$

Furthermore, we introduce the following symbols:

$$a^n = \underbrace{a \otimes \cdots \otimes a}_{n-\text{times}}$$
$$na = \underbrace{a \oplus \cdots \oplus a}_{n-\text{times}}.$$

The *language* of first-order fuzzy logic consists of variables, constants, n-ary functional and predicate symbols, binary predicate symbol $=$ (equality sign), symbols for truth values $\boldsymbol{a}$, $a \in L$[†)] binary connective $\Rightarrow$and general quantifier $\forall$.

Terms and formulas are defined as usual with the exception that all symbols for truth values are atomic formulas.

[†)] This is only auxiliary. As presented in [13], we can get rid of them.

A set of all the terms of a language J is denoted by M_J and a set of all the well-formed formulas by F_J. The set of all terms without variables is denoted by M_V.

The common abbreviations of formulas $\neg A$, $A \vee B$, $A \wedge B$, $A \,\&\, B$, $A \Leftrightarrow B$, $(\exists x)A$, A^k are introduced (see [6, 7, 12, 15]). Moreover, we will use also the abbreviation $A \nabla B$ defined by

$$A \nabla B := \neg(\neg A \,\&\, \neg B)$$

and call it Lukasiewicz disjunction.

As explained in these works, syntax of fuzzy logic is evaluated by *syntactic truth values*.

An *evaluated formula* is a couple

$$[A;\ a]$$

where $A \in F_J$ and $a \in L$. The (syntactic) truth value a is an evaluation of the formula A in the syntax of fuzzy logic.

In fuzzy logic, we deal with fuzzy sets of axioms. The propositional logical axioms are those of Rose and Rosser [18], and also

(T1) $\models (\boldsymbol{a} \Rightarrow \boldsymbol{b}) \Leftrightarrow (\overline{\boldsymbol{a} \rightarrow \boldsymbol{b}})$

where $\overline{\boldsymbol{a} \rightarrow \boldsymbol{b}}$ denotes the symbol (atomic formula) for the truth value $a \rightarrow b$ if a and b are given.

(T2) $\models (\forall x)A \Rightarrow A_x[t]$

for any term t.

(T3) $\models (\forall x)(A \Rightarrow B) \Leftrightarrow (A \Rightarrow (\forall x)B)$

provided that x is not free in A.

Furthermore, we will introduce also the equality predicate fulfilling the following (common) logical axioms:

(E1) $\vdash x = x$

(E2) $\vdash (x_1 = y_1) \Rightarrow \ldots \Rightarrow (x_n = y_n) \Rightarrow (p(x_1, \ldots, x_n) \Rightarrow p(y_1, \ldots, y_n))$

(E3) $\vdash (x_1 = y_1) \Rightarrow \ldots \Rightarrow (x_n = y_n) \Rightarrow (f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n))$

for every n-ary functional symbol f and predicate symbol p.

A special kind of fuzzy equality is the sharp one defined by

$$\mathcal{D}(t = s) = \begin{cases} 1 & \text{if } \mathcal{D}(t) = \mathcal{D}(s) \\ 0 & \text{otherwise} \end{cases}$$

in every model $\mathcal{D}$.

A *theory* T in the language J of first-order fuzzy logic (a *fuzzy theory*) is a triple

$$T = \langle A_L, A_S, R\rangle$$

where $A_L \underset{\sim}{\subset} F_J$, and $A_S \underset{\sim}{\subset} F_J$ are fuzzy sets of logical and special axioms, respectively and R is a set of inference rules containing, at least, the rules R_{MP}, r_G and r_{Rb}, $b \in L$ (cf. [6, 7, 14]).

An *evaluated proof* (or shortly, a proof) of a formula A from a fuzzy set A_S of special axioms is a sequence of evaluated formulas which are logical or special axioms or they are derived using some many-valued inference rule. The provability degree in the fuzzy theory $T = \langle A_L, A_S, R\rangle$ is given by

$$(C^{syn}A_S)A = \bigvee\{\mathrm{Val}_T(w) \mid w \text{ is a proof of } A \text{ from } A_S \underset{\sim}{\subset} F_J\}$$

where $\mathrm{Val}_T(w)$ is a value of the proof w in the fuzzy theory T. If $(C^{syn}A_S)A = a$, $A \in F_J$ then we write

$$T \vdash_a A.$$

A formula $A \in F_J$ is *true* in the degree a in the fuzzy theory T if

$$(C^{sem}A_S)A = \bigwedge\{\mathcal{D}(A) \mid \mathcal{D} \models T\}.$$

We write then

$$T \models_a A.$$

Note that

$$\neg a^m \oplus \neg b^n = a^m \to \neg b^n = a^m \to (b^n \to 0) = (a^m \otimes b^n) \to 0 = \neg(a^m \otimes b^n).$$

Therefore

$$\models \neg(A_1^{m_1} \,\&\, \cdots \,\&\, A_n^{m_n}) \Leftrightarrow (\neg A_1^{m_1} \nabla \cdots \nabla \neg A_n^{m_n})$$

and it may be demonstrated that this equivalence is also theorem in the degree 1.

A fuzzy theory is *consistent* if

$$\mathrm{Val}_T(w_A) \otimes \mathrm{Val}_T(w_{\neg A}) = 0$$

holds for every formula $A \in F_{J(T)}$ and all proofs w_A of A and $w_{\neg A}$ $\neg A$, respectively.

The following theorems will be used in the sequel.

Theorem 1 (validity) *If $T \vdash_a A$ and $T \models_b A$ then $a \leq b$ holds for every formula A.*

Theorem 2 (closure) *Let $A \in F_{J(T)}$ and A' be its closure. Then*

$$T \vdash_a A \quad \textit{iff} \quad T \vdash_a A'.$$

Theorem 3 *A theory T is contradictory iff $T \vdash A$ holds for every formula $A \in F_{J(T)}$.*

This theorem has been proved in [7]. It follows from it that, if T is contradictory then for every formula there always is its proof w such that $\mathrm{Val}(w) = 1$.

A fuzzy theory T is *Henkin* if Henkin axioms

$$A_S(A_x[\boldsymbol{r}] \Rightarrow (\forall x)A(x)) = 1 \tag{1}$$

are added to the fuzzy set of special axioms where $\boldsymbol{r}$ is a special constant for the formula $(\forall x)A(x)$.

Theorem 4 *Let T be a consistent theory, K a set of special constants for all the closed formulas $(\forall x)A$ and let A_H be a fuzzy set of Henkin axioms (1) where $A_H(C) = 0$ if C is not a Henkin axiom. Then the theory*

$$T_H = T \cup A_H$$

is a conservative extension of the theory T.

Theorem 5 (deduction) *Let A be a closed formula and $T' = T \cup \{\mathbf{1}/A\}$. Then to every B there is n such that*

$$T \vdash_a A^n \Rightarrow B \quad \text{iff} \quad T' \vdash_a B.$$

Theorem 6 (completeness II) *A theory T is consistent iff it has a model.*

3 Further extensions

In this section, we present several lemmas and theorems demonstrating some properties of fuzzy theories and provability. We will refer to logical axioms and formal theorems in the degree 1 presented in [7, 14]. They are denoted by (T1) – (T11) and (D1) – (D24). Most of these (schemes of) formulas can be found also in [6], Chapter 4.

Let $A(x_1, \dots, x_n)$ be a formula. Then its instance is a formula $A_{x_1,\dots,x_n}[t_1, \dots, t_n]$ where $t_1, \dots, t_n$ are terms.

Lemma 1 *Let $A \in F_{J(T)}$ be a formula and A' its instance. Then:*

a) If $T \vdash_a A$ then $T \vdash_b A'$ where $a \le b$.

b) If $T \models_a A$ then $T \models_b A'$ where $a \le b$.

PROOF: a) This was proved as Lemma 14 in [7].

b) This follows immediately from the definition of the model using axiom (T9) (substitution) of [7]. □

Lemma 2 *a) Let $T \vdash_a A$ where $a > 0$. Then there is a proof w of A such that* $\mathrm{Val}(w) > 0$.

b) Let $T \vdash_0 A \wedge B$. Then $T \vdash_a A$ and $T \vdash_b B$ and $a \wedge b = 0$.

c) Let T be consistent and $T \vdash_a A$. Then $T \vdash_b A^m$ for every $m \geq 1$ where $b \leq a$.

PROOF: a) If such a proof does not exist the $\mathrm{Val}(w) = 0$ holds for every proof w of A which gives $T \vdash_0 A$ — a contradiction.

b) Let $a \wedge b > 0$. Then $a > 0$ as well as $b > 0$. Using Theorem (D5) from [7], we get $T \vdash_c A \wedge B$ where $a \otimes (a \rightarrow b) \geq c$ — a contradiction.

c) Let $b > a$. Using (D15) we obtain $T \vdash_c A$ where $c \geq b > a$ which is a contradiction with the assumption. □

Lemma 3 *Let $T \vdash_a A$. Then $T' = T \cup \{ {}^a/A \}$ is a conservative extension of T.*

PROOF: Let $T \vdash_b B$ where B is some formula. Using the induction on the length of proof we will demonstrate that $\mathrm{Val}_{T'}(w'_B) \leq b$ holds for every proof w'_B of B in T'. Then we will obtain $b \leq \bigvee_{w'_B} \mathrm{Val}_{T'}(w'_B) \leq b = \bigvee_{w_B} \mathrm{Val}_T(w_B)$ by the definition of provability and, at the same, by the fact that T' is extension of T.

a) Let B be an axiom (logical or special).

First, we assume that $B := A$[†]. Then

$$w'_B := [A;\ a]_{SA}\,.$$

But since $T \vdash_a A$, we have $a = \mathrm{Val}_{T'}(w'_B) \leq \bigvee_{w_A} \mathrm{Val}_T(w_A) = a$.

If B is not A then $w'_B := [B;\ b']_{SA[LA]}$ and we have

$$b' = \mathrm{Val}_{T'}(w'_B) \leq \bigvee_{w_B} \mathrm{Val}_T(w_B)$$

by the definition of provability and the fact that B is also the axiom of T.

b) Let

$$w'_B := [C;\ c]_{w_C}\,, [C \Rightarrow B;\ d]_w\,, [B;\ c \otimes d]_{r_{MP}}\,.$$

[†] The symbol := should be read as "is defined as" or simply "is".

By the inductive assumption, $c \leq \bigvee_{w_C} \mathrm{Val}_T(w_C)$, $d \leq \bigvee_w \mathrm{Val}_T(w)$ and so

$$\mathrm{Val}_{T'}(w'_B) = c \otimes d \leq \bigvee_{w_C, w} (\mathrm{Val}_T(w_C) \otimes \mathrm{Val}_T(w)) \leq \bigvee_{w_B} \mathrm{Val}_T(w_B) = b.$$

The case when the inference rule r_G is used follows immediately from the semicontinuity of the semantic part of the rule and the inductive assumption. □

Lemma 4 *Let T be consistent and $T \vdash_a A$. Then $T \vdash_b \neg A$ where $b \leq \neg a$.*

PROOF: Let $b > \neg a$. We will write the proof

$$w := [A;\, a']_{w_A}, [\neg A;\, b']_{w_{\neg A}}, \ldots, [A \mathbin{\&} \neg A;\, a' \otimes b']_{r_{MP}}$$

where $b = \bigvee_{w_{\neg A}} b'$ which follows that

$$\bigvee_{w_A, w_{\neg A}} \mathrm{Val}(w) = a \otimes b$$

and thus, $T \vdash_c A \mathbin{\&} \neg A$ where $c \geq a \otimes b > 0$. Therefore, T is contradictory by Lemma 2 a) — a contradiction. □

Lemma 5 *Let every formula closed $A \in F_J$ be tied with some closed formula A^* in such a way that $(A \Rightarrow B)^*$ is $A^* \Rightarrow B^*$. Then from*

$$\mathcal{C}^{sem}\left(\{\, {}^{a_i}/_{A_i} \mid i \in I\}\right) B \geq b$$

it follows that

$$\mathcal{C}^{sem}\left(\{\, {}^{a_i}/_{A_i^*} \mid i \in I\}\right) B^* \geq b$$

where I is some index set (possibly empty).

PROOF: Let $\mathcal{D} \in \mathcal{C}^{sem}\left(\{\, {}^{a_i}/_{A_i} \mid i \in I\}\right)$ and put $\mathcal{D}'(A) = \mathcal{D}(A^*)$ for every closed atomic formula A. Then $\mathcal{D}'(C) = \mathcal{D}(C^*)$ holds for every formula $C \in F_J$. Let $\mathcal{D} \in \mathcal{C}^{sem}\left(\{\, {}^{a_i}/_{A_i^*} \mid i \in I\}\right)$. Then $\mathcal{D}(A^*) \geq a_i$, $i \in I$ gives $\mathcal{D}'(A_i) \geq a_i$, i.e. $\mathcal{D}' \in \mathcal{C}^{sem}\left(\{\, {}^{a_i}/_{A_i} \mid i \in I\}\right)$, and so $\mathcal{D}'(B) \geq b$ which implies $\mathcal{D}(B^*) \geq b$. □

Lemma 6 *a) If $T \vdash_a A \Rightarrow B$ and $T \vdash_b B \Leftrightarrow B'$ then $T \vdash_c A \Rightarrow B'$ where $c \geq a \otimes b$.*

b) If $T \vdash_a A \Leftrightarrow A'$ then $T \vdash_c (A \Rightarrow B) \Leftrightarrow (A' \Rightarrow B)$ where $c \geq a$.

PROOF: a) Write down the proof

$$w := [A\Rightarrow B;\ a']_{w_1}, [B\Leftrightarrow B';\ b']_{w_2}, [(B\Leftrightarrow B')\Rightarrow(B\Rightarrow B';\ 1]_{LA}, \dots,$$
$$[A\Rightarrow B';\ a'\otimes b']_{r_{MP}},$$

which gives the proposition.

b) Using the logical axiom $\vdash (A\Rightarrow A')\Rightarrow((A'\Rightarrow B)\Rightarrow(A\Rightarrow B))$ we obtain the proofs w' of $(A'\Rightarrow B)\Rightarrow(A\Rightarrow B)$ and w'' of $(A\Rightarrow B)\Rightarrow(A'\Rightarrow B)$, both with the value $a' \leq a$. From them, we can find a proof of the formula $(A\Rightarrow B)\Leftrightarrow(A'\Rightarrow B)$ with the value a'. □

The following theorem holds for every fuzzy equality.

Theorem 7 (equality) *Let $T \vdash_{a_i} t_i = s_i,\ i = 1, \dots, n$. Then there are $m_1, \dots, m_n$ such that*

$$T \vdash_b A\Leftrightarrow A' \qquad b \geq a_1^{m_1} \otimes \cdots \otimes a_n^{m_n}$$

where A' is a formula which is a result of replacing of the terms t_i by the term s_i in A, respectively.

PROOF: By induction on the length of the formula.

If $A := p(t_1, \dots, t_n)$ where p is an n-ary predicate symbol then the proposition follows from the equality axiom (E2) and modus ponens.

We show only the induction step for implication as the proof is analogous to the corresponding classical proof (cf. [20]).

Let $A := B\Rightarrow C$ and

$$T \vdash_b B\Leftrightarrow B' \qquad b \geq a_1^{m_1} \otimes \cdots \otimes a_n^{m_n}$$
$$T \vdash_c C\Leftrightarrow C' \qquad c \geq a_1^{m'_1} \otimes \cdots \otimes a_n^{m'_n}.$$

where B', C' are formulas in which the replacements have been done. By Lemma 6 we have

$$T \vdash_d (B\Rightarrow C)\Leftrightarrow(B'\Rightarrow C) \qquad d \geq b$$
$$T \vdash_{d'} (B'\Rightarrow C)\Leftrightarrow(B'\Rightarrow C') \qquad d' \geq c,$$

i.e. $T \vdash_{e'} (B\Rightarrow C)\Rightarrow(B'\Rightarrow C')$, $e' \geq d \otimes d'$. Analogously we prove the opposite implication. □

Let us remark that this theorem is quite weak. For small a_i it is practically trivial and becomes interesting only for a_i close to 1. The numbers m_i depend on the complexity of the given formula. The magnitude of b depends furthermore on the number of replacements of t_i by s_i. Quite analogously we may prove the equivalence theorem.

Theorem 8 (equivalence) *Let A be a formula and $B_1, \ldots, B_n$ some of its subformulas. Let $T \vdash_{a_i} B_i \Leftrightarrow B'_i$, $i = 1, \ldots, n$. Then there are $m_1, \ldots, m_n$ such that*

$$T \vdash_b A \Leftrightarrow A' \qquad b \geq a_1^{m_1} \otimes \cdots \otimes a_n^{m_n}$$

where A' is a formula which is a result of replacing of the formulas $B_1, \ldots, B_n$ in A by $B'_1, \ldots, B'_n$.

Let Γ be a fuzzy set of formulas. By $\mathrm{Supp}(\Gamma)$ we denote its support, i.e. $A \in \mathrm{Supp}(\Gamma)$ if $\Gamma(A) > 0$.

Theorem 9 (reduction for the consistency) *A theory $T' = T \cup \Gamma$ is contradictory iff there are $m_1, \ldots, m_n$ and $A_1, \ldots, A_n \in \mathrm{Supp}(\Gamma)$ such that*

$$T \vdash_c \neg A_1^{m_1} \nabla \cdots \nabla \neg A_n^{m_n}$$

where $a_i = \Gamma(A_i)$, $i = 1, \ldots, n$ and $c > \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n})$ or $c = 1$ if the right-hand side is equal to 1.

PROOF: Let T be contradictory. Then T' is also contradictory and, therefore, $T' \vdash \mathbf{0}$. Let $A_1, \ldots, A_n \in \mathrm{Supp}(\Gamma)$ be formulas occurring in a proof w_0 for which $\mathrm{Val}_{T'}(w_0) > \neg(a_1^{m_1} \otimes \cdots \otimes a_n^{m_n})$ (or $\mathrm{Val}_{T'}(w_0) = 1$ if the sharp inequality is impossible; such a proof always exists). Then the theory

$$T'' = T \cup \{\, {}^{a_i}/A_i \mid i = 1, \ldots, n\}$$

is contradictory and the theory T' is its extension. By repeated application of the deduction theorem we may find a $m_1, \ldots, m_n$ and a proof w of the formula $A_1^{m_1} \Rightarrow (\cdots \Rightarrow (A_n^{m_n} \Rightarrow \mathbf{0}) \cdots)$ such that

$$\mathrm{Val}_T(w) \geq \mathrm{Val}_{T'}(w_0).$$

Using the formal theorem (T3) we get

$$T \vdash_c \neg A_1^{m_1} \nabla \cdots \nabla \neg A_n^{m_n}$$

where $c \geq \mathrm{Val}(w_0)$.

Vice-versa, by the equivalence theorem we have

$$T \vdash_c \neg(A_1^{m_1} \mathbin{\&} \cdots \mathbin{\&} A_n^{m_n}).$$

But then there is a proof

$$w := [A_1;\, a_1]_{SA}, \ldots, [A_n;\, a_n]_{SA}, \ldots, [A_1^{m_1} \mathbin{\&} \cdots \mathbin{\&} A_n^{m_n};\, a_1^{m_1} \otimes \cdots \otimes a_n^{m_n}]_{r_{MP}}$$

in T' which follows that

$$T \vdash_d (A_1^{m_1} \mathbin{\&} \cdots \mathbin{\&} A_n^{m_n}) \mathbin{\&} \neg(A_1^{m_1} \mathbin{\&} \cdots \mathbin{\&} A_n^{m_n})$$

for some $d \geq c \otimes (a_1^{m_1} \otimes \cdots \otimes a_n^{m_n}) > 0$, i.e. T' is contradictory. □

Corollary 1 *A theory* $T' = T \cup \{ {}^{\neg a}/\neg A\}$ *is contradictory iff* $T \vdash_b mA$ *for some* m *and* $b > ma$ *or* $b = 1$ *if* $ma = 1$.

PROOF: Then $T \vdash_b \neg(\neg A)^m$, $b > \neg(\neg a)^m$, but $\neg(\neg a)^m = \neg(\neg ma) = ma$ which gives also $\neg(\neg A)^m \Leftrightarrow mA$. □

4 Open fuzzy theories

Most of the results presented before and in the other papers concern closed formulas. They can be extended to open ones using the closure theorem. However, characterization of theories given by open axioms may also be interesting. We will follow the classical way as we want fuzzy logic to be developed, besides other, in parallel with classical one to demonstrate many classical results to be special cases of ours. However, methods in the proofs quite often differ as we have to use weaker properties. In what follows, we will deal with Henkin fuzzy theories as they can easily be obtained by conservative extension of the non-Henkin ones.

Let T be a fuzzy theory and T_H its conservative Henkin extension. Let $\mathbf{r}$ be the special constant for $(\forall x)A$. We say that a formula A is in relation with a special constant $\mathbf{r}$ if it is either of the formulas

$$A_x[\mathbf{r}] \Rightarrow (\forall x)A$$

or

$$(\forall x)A \Rightarrow A_x[t]$$

where t is a term without variables.

Given a Henkin fuzzy theory T we will define, analogously as in classical logic, the fuzzy set of formulas $\Delta(T)$ as follows.

$$\Delta(T)(A) = \begin{cases} 1 & \text{if } A := Q \\ A_S(B) \vee A_L(B) & \text{if } A := B_{x_1 \dots x_n}[t_1, \dots, t_n] \end{cases}$$

where Q is any of the formulas

$$A_x[\mathbf{r}] \Rightarrow (\forall x)A$$
$$(\forall x)A \Rightarrow A_x[t]$$
$$t = t$$
$$(t_1 = s_1) \Rightarrow (\cdots \Rightarrow ((t_n = s_n) \Rightarrow f(t_1, \dots, t_n) = f(s_1, \dots, s_n)) \dots)$$
$$(t_1 = s_1) \Rightarrow (\cdots \Rightarrow ((t_n = s_n) \Rightarrow p(t_1, \dots, t_n) \Rightarrow p(s_1, \dots, s_n)) \dots)$$

and t_i, s_i, $i = \dots, n$ are terms without variables, $\mathbf{r}$ is Henkin constant for $(\forall x)A$ and B is a formula not being any of the previous cases.

Lemma 7 *Let T be a fuzzy theory. Then*

$$\mathcal{D} \models T_H \quad \textit{iff} \quad \Delta(T_H) \leq \mathcal{D}$$

holds for every structure $\mathcal{D}$ for the language $L(T_H)$.

PROOF: Let $\mathcal{D} \models T_H$. Then $A_S, A_L \leq \mathcal{D}$ and $\mathcal{D}(A_H) = 1$ for every Henkin axiom A_H. If B is a special axiom then

$$A_S(B) = \Delta(T_H)(B_{x_1 \ldots x_n}[t_1, \ldots, t_n]) \leq \mathcal{D}(B).$$

Using the substitution axiom and the definition of the closure we obtain

$$\mathcal{D}(B) \leq \mathcal{D}(B_{x_1 \ldots x_n}[t_1, \ldots, t_n]).$$

The opposite implication follows immediately from the definition of $\Delta(T_H)$. □

Corollary 2

$$\mathcal{C}^{sem}(A_S)B = \mathcal{C}^{sem}(\Delta(T_H))B.$$

holds for every formula $B \in F_{L(T)}$.

We say that a formula B is a *tautological consequence* of the formulas $A_1^{p_1}, \ldots, A_n^{p_n}$ in the degree b, $p_i \geq 1$, $i = 1 \ldots, n$ if

$$\models_b A_1^{p_1} \Rightarrow (\ldots \Rightarrow (A_n^{p_n} \Rightarrow B) \ldots). \tag{2}$$

Note that (2) is equivalent with

$$\models_b (A_1^{p_1} \mathbin{\&} \cdots \mathbin{\&} A_n^{p_n}) \Rightarrow B.$$

If $b = 1$ then we say that B is a tautological consequence of $A_1^{p_1}, \ldots, A_n^{p_n}$.

Lemma 8 *Let w be a proof of A in T, $\mathrm{Val}_T(w) = a$ and A' be a closed instance $A' \in L(T_H)$ where T_H is a Henkin extension of T. Then there are formulas $C_1, \ldots, C_m \in \Delta(T)$ such that A' is their tautological consequence and $\Delta(T)C_1 \otimes \cdots \Delta(T)C_m \leq a$.*

PROOF: Consider the proof

$$w := [A_1;\ a_1]_{w_1}, \ldots [A_n := A;\ a_n]_{w_n}.$$

We prove by induction on the length of the proof that there always is the desired tautology. In what follows, we will denote $\Delta(T)C$ by the corresponding small letter c, possibly with the subscript.

a) Let $A := \boldsymbol{a}$. Then $A' := \boldsymbol{a}$, $\boldsymbol{a} \in \Delta(T)$ and $\models \boldsymbol{a} \Rightarrow \boldsymbol{a}$ where $a \leq a$.

b) Let A' be instance of special axiom. Then $A' \in \Delta(T)$ and $\models A' \Rightarrow A'$. By the assumption, $\Delta(T)(A') = A_S(A) \leq a$.

c) If A' is an instance of Rose-Rosser's axioms (R1) – (R4) then A' is tautological consequence of the empty set of formulas.

d) Let $A := (\forall x)B \Rightarrow B$. Then $A' := (\forall x)B' \Rightarrow B'_x[t'] \in \Delta(T)$ and we have $\models A' \Rightarrow A'$. Similarly for the identity and equality axioms.

e) Let $A := (\forall x)(C \Rightarrow D) \Rightarrow (C \Rightarrow (\forall x)D)$. Then

$$A' := (\forall x)(C' \Rightarrow D') \Rightarrow (C' \Rightarrow (\forall x)D')$$

where C', D' are closed instances of the formulas C, D. Furthermore, let $\boldsymbol{r}$ be a special constant for $(\forall x)D'$. Then $(\forall x)(C' \Rightarrow D') \Rightarrow (C' \Rightarrow D'_x[\boldsymbol{r}]) \in \Delta(T)$ and $D'_x[\boldsymbol{r}] \Rightarrow (\forall x)D' \in \Delta(T)$. The desired tautology is then

$$\begin{aligned}\models\ & ((\forall x)(C' \Rightarrow D') \Rightarrow (C' \Rightarrow D'_x[\boldsymbol{r}])) \Rightarrow ((D'_x[\boldsymbol{r}] \Rightarrow (\forall x)D') \Rightarrow \\ & ((\forall x)(C' \Rightarrow D') \Rightarrow (C' \Rightarrow (\forall x)D'))).\end{aligned}$$

In cases c) – e), $a = 1$.

Let the induction assumption holds.

f) Consider the proof

$$[B;\, b]_{w_1}, [B \Rightarrow A; .e]_{w_2}, [A;\, b \otimes e]_{r_{MP}}.$$

By the inductive assumption

$$\begin{aligned}&\models (C_1 \,\&\, \cdots \,\&\, C_m) \Rightarrow B' \\ &\models (D_1 \,\&\, \cdots \,\&\, D_m) \Rightarrow (B' \Rightarrow A')\end{aligned}$$

where $c_1 \otimes \cdots c_m \leq b$ and $d_1 \otimes \cdots d_n \leq e$ (recall that $c_1 = \Delta(T)C_i, d_1 = \Delta(T)D_i$). Then

$$\begin{aligned}\mathcal{D}(C_1) \otimes \cdots \otimes \mathcal{D}(C_m) &\leq \mathcal{D}(B') \\ \mathcal{D}(D_1) \otimes \cdots \otimes \mathcal{D}(D_n) \otimes \mathcal{D}(B') &\leq \mathcal{D}(A')\end{aligned}$$

for every $\mathcal{D}$, which gives

$$\models ((C_1 \,\&\, \cdots \,\&\, C_m) \,\&\, (D_1 \,\&\, \cdots \,\&\, D_m)) \Rightarrow A'$$

and $c_1 \otimes \cdots c_m \otimes d_1 \otimes \cdots d_n \leq b \otimes e$.

g) Let $A := (\forall x)B$ and consider the proof

$$[B;\, b]_{w_1}, [(\forall x)B;\, b]_{r_G}.$$

By the induction assumption

$$\models (C_1 \,\&\, \cdots \,\&\, C_m) \Rightarrow B'_x[\boldsymbol{r}]$$

where $\boldsymbol{r}$ is a special constant for $(\forall x)B'$, i. e.

$$\mathcal{D}(C_1) \otimes \cdots \otimes \mathcal{D}(C_m) \le \mathcal{D}(B'_x[\boldsymbol{r}])$$

for every structure $\mathcal{D}$. However, the formula $(B'_x[\boldsymbol{r}] \Rightarrow (\forall x)B') \in \Delta(T)$ and we have

$$\mathcal{D}(B'_x[\boldsymbol{r}]) \otimes \mathcal{D}(B'_x[\boldsymbol{r}] \Rightarrow (\forall x)B') \le \mathcal{D}((\forall x)B')$$

for every structure $\mathcal{D}$ which follows

$$\models ((C_1 \,\&\, \cdots \,\&\, C_m) \,\&\, (B'_x[\boldsymbol{r}] \Rightarrow (\forall x)B')) \Rightarrow (\forall x)B'.$$

□

We will call the sequence of formulas $A_1^{m_1}, \ldots, A_n^{m_n}$ *special* where $m_i \ge 1$, $i = 1, \ldots, n$ if

$$\models \neg A_1^{m_1} \triangledown \cdots \triangledown \neg A_n^{m_n}$$

i. e.

$$\models \neg(A_1^{m_1} \,\&\, \cdots \,\&\, A_n^{m_n}).$$

Lemma 9 *Let $A_1^{m_1}, \ldots, A_n^{m_n}$ be a special sequence. Then*

$$\mathcal{D}(A_1^{m'_1}) \otimes \cdots \otimes \mathcal{D}(A_n^{m'_n}) = 0$$

holds true for all structures $\mathcal{D}$ and $m'_i \ge m_i$, $i = 1, \ldots, n$.

PROOF: Obvious. □

Corollary 3 *If $A_1^{m_1}, \ldots, A_n^{m_n}$ is a special sequence then $A_1^{m_1}, \ldots, A_n^{m_n}$, $B_1^{p_1}, \ldots, B_q^{p_q}$ is a special sequence.*

The *order* of the special constant $\boldsymbol{r}$ for $(\forall x)A$ is the number of occurrences of $\forall$ in the latter. Analogously as in the classical logic, we define the fuzzy set $\Delta_m(T) \subseteq \Delta(T)$ which is obtained from $\Delta(T)$ by omitting all the formulas A from its support which are in relation to a special constants with the order greater than m.

Lemma 10 *Let T be a Henkin theory, $m > 0$ and let there be a special sequence $A_1^{m_1}, \ldots, A_n^{m_n} \in \Delta_m(T)$. Then there is a special sequence $B_1^{s_1}, \ldots, A_p^{s_p} \in \Delta_{m-1}(T)$.*

PROOF: Analogously as in the classical proof (cf. [20], Lemma 2 in Section 4.3) we will consider a special sequence consisting either of formulas $A_1^{m_1}, \ldots, A_p^{m_p} \in \Delta_{m-1}(T)$ or those being in relation to special constants $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s$ with the same order as a special constant $\boldsymbol{r}$ with the highest order. Let the remaining formulas in the considered special sequence be

$$\begin{aligned} & B_x[\boldsymbol{r}] \Rightarrow (\forall x)B \\ & (\forall x)B \Rightarrow B_x[t_i], \qquad i = 1, \ldots, q. \end{aligned}$$

Let $A_i := C_y[\boldsymbol{s}] \Rightarrow (\forall x)C$ or $A_i := (\forall x)C \Rightarrow C_y[t]$. Since the order of $\boldsymbol{s}$ is smaller than or equal to that of $\boldsymbol{r}$, $(\forall x)C$ can occur neither in $C_y[t]$ nor in $C_y[\boldsymbol{s}]$. Furthermore, $(\forall x)B \neq (\forall x)C$ since $\boldsymbol{s} \neq \boldsymbol{r}$. Hence, no A_i contains $(\forall x)B$. Hence, we have the special sequence

$$A_1^{m_1}, \ldots, A_p^{m_p}, (B_x[\boldsymbol{r}] \Rightarrow (\forall x)B)^m, ((\forall x)B \Rightarrow B_x[t_1])^{m_{p1}}, \ldots ((\forall x)B \Rightarrow B_x[t_q])^{m_{pq}},$$

which means that

$$\begin{aligned} & \mathcal{D}(A_1)^{m_1} \otimes \cdots \otimes \mathcal{D}(A_p)^{m_p} \otimes \mathcal{D}(B_x[\boldsymbol{r}] \Rightarrow (\forall x)B)^m \otimes \\ & \otimes \mathcal{D}((\forall x)B \Rightarrow B_x[t_1])^{m_{p1}} \otimes \cdots \otimes \mathcal{D}((\forall x)B \Rightarrow B_x[t_q])^{m_{pq}} = 0 \end{aligned}$$

holds for every $\mathcal{D}$. Furthermore, the value of the Lukasiewicz conjunction of the formulas

$$A_1^{m_1}, \ldots, A_p^{m_p}, (B_x[\boldsymbol{r}] \Rightarrow (\forall x)B)^m, (\forall x)B \Rightarrow (B_x[t_1]^{m_{p1}} \mathbin{\&} \cdots \mathbin{\&} B_x[t_q]^{m_{pq}}) \tag{3}$$

is equal to 0 for every $\mathcal{D}$ and thus, it is a special sequence as well.

Let now, as the first step, $(\forall x)B$ be replaced by $B_x[\boldsymbol{r}]$ and, as the second step, the latter by $B_x[t_i]$, $i = 1, \ldots, q$. Furthermore, we will replace all the occurrences of $\boldsymbol{r}$ in all the formulas by the term t_i. Then we obtain two special sequences

$$A_1^{m_1}, \ldots, A_p^{m_p}, (B_x[\boldsymbol{r}] \Rightarrow B_x[t_1]^{m_{p1}} \mathbin{\&} \cdots \mathbin{\&} B_x[t_q])^{m_{pq}} \tag{4}$$

$$(A_1^{(i)})^{m_1}, \ldots, (A_p^{(i)})^{m_p}, (B_x[t_i] \Rightarrow B_x^{(i)}[t_1]^{m_{p1}} \mathbin{\&} \cdots \mathbin{\&} B_x^{(i)}[t_q])^{m_{pq}}, \tag{5}$$

$i = 1, \ldots, q$. We want to demonstrate that

$$A_1^{m'_1}, \ldots, A_p^{m'_p}, (A_1^{(1)})^{m'_1}, \ldots, (A_p^{(1)})^{m'_p}, (A_1^{(q)})^{m'_1}, \ldots, (A_p^{(q)})^{m'_p}, \tag{6}$$

is a special sequence for some m'_j, $j = 1, \ldots, p$.

Assume the opposite, i. e. for every m'_j, m'_{p_j} there exists $\mathcal{D}$ such that

$$\bigotimes_{j=1}^{p} \mathcal{D}(A_j)^{m'_j} \otimes \bigotimes_{\substack{j=1,\ldots,p \\ i=1,\ldots,q}} \mathcal{D}(A_j^{(i)})^{m'_j} > 0. \tag{7}$$

Let us find m'_{p_j} such that

$$\mathcal{D}(B_x[t_i]) \rightarrow (\mathcal{D}(B_x^{(i)}[t_1])^{m'_{p1}} \otimes \cdots \otimes \mathcal{D}(B_x^{(i)}[t_q])^{m'_{pq}}) = 0$$

(this is possible due to nilpotency of $\otimes$ and the inequality (7)). This may hold true only if $\mathcal{D}(B_x[t_i]) = 1$ and $\mathcal{D}(B_x^{(i)}[t_1])^{m'_{p_1}} \otimes \cdots \otimes \mathcal{D}(B_x^{(i)}[t_q]))^{m'_{p_q}} = 0$. But then $\mathcal{D}(B_x[\boldsymbol{r}] \Rightarrow (B_x[t_1]^{m_{p_1}} \,\&\, \cdots \,\&\, B_x[t_q]^{m_{p_q}}) = 1$, from which follows that (4) is not special — a contradiction.

Finally, analogously as in the classical proof, we may demonstrate that each $A_j^{(i)}$ belongs to $\Delta_{m-1}(T)$ or is in relation with some of the constants $\boldsymbol{r}_1, \boldsymbol{r}_s$. Then the previous procedure can be repeated to get rid of all the special constants with the order greater than $m-1$. □

We say that a formula A is a *fuzzy quasitautology* in the degree a if

$$\models_a B_1 \,\&\, \cdots \,\&\, B_k \Rightarrow A$$

where B_i are closed instances of the equality axioms. Formally, we will write

$$\models_a^Q A.$$

The following is a fuzzy analogy the the famous Hilbert–Ackermann's consistency theorem. However, as we have no direct proof of the tautology theorem (saying that every tautology is a theorem), we are forced to use the completeness theorem in its proof.

Theorem 10 (consistency) *Open theory T is contradictory iff there are $p_1, \ldots, p_n$ and special axioms $A_1, \ldots, A_n$ of the theory T such that*

$$\models_b^Q \neg \bar{A}_1^{p_1} \nabla \cdots \nabla \neg \bar{A}_n^{p_n}$$

where $\bar{A}_i$ are instances of the special axioms and $b > \neg(a_1^{p_1} \otimes \cdots \otimes a_n^{p_n})$ where $a_i = A_S(A_i)$, $i = 1, \ldots, n$.

PROOF: Let T be contradictory. Then $T \vdash x \neq x$ and $\boldsymbol{r} \neq \boldsymbol{r}$ is instance of this formula. Hence, by Lemma 8 and the fact that there is a proof of $\boldsymbol{r} \neq \boldsymbol{r}$ with the value 1, there are formulas $A_1, \ldots, A_{n-1} \in \mathrm{Supp}(\Delta(T))$ such that

$$\models_b A_1 \,\&\, \cdots \,\&\, A_{n-1} \Rightarrow \boldsymbol{r} \neq \boldsymbol{r}.$$

for b fulfilling the above condition.

As $\boldsymbol{r} \neq \boldsymbol{r} \in \Delta(T)$, we conclude that

$$\models_b \neg A_1 \nabla \cdots \nabla \neg A_n$$

where $A_1, \ldots, A_n \in \Delta(T)$. By Lemma 10, there are $p_1, \ldots, p_n$ and a special sequence $A_1^{p_1}, \ldots, A_m^{p_m}$ of formulas from $\Delta_0(T)$. Then

$$\models_b B_1 \,\&\, \cdots \,\&\, B_k \Rightarrow (\neg A_1^{p_1} \nabla \cdots \nabla \neg A_n^{p_n}) \tag{8}$$

where $B_1, \ldots, B_k$ are instances of the equality axioms occurring in $A_1^{p_1}, \ldots, A_m^{p_m}$ (the exponents at B_i are equal to 1 as all these instances are theorems in the degree 1), i.e. $\neg A_1^{p_1} \nabla \cdots \nabla \neg A_n^{p_n}$ is the required quasitautology.

Vice-versa, from Lemma 1a) we obtain

$$T \vdash_a \bar{A}_1^{p_1} \& \cdots \& \bar{A}_n^{p_2} \quad a \geq a_1^{p_1} \otimes \cdots \otimes a_n^{p_n},$$

because if $a_i = A_S(A_i)$, then $T \vdash_{a_i} A_i$ which follows $T \vdash_{\bar{a}_i} \bar{A}_i$ where $a \geq \bar{a}_i \geq a_i$ and we may use formal theorem (D8).

Let $\mathcal{D} \models T$. Then $\mathcal{D}(\bar{A}_1^{p_1} \& \cdots \& \bar{A}_n^{p_2}) \geq a$. But $\mathcal{D} \in \mathcal{C}^{sem}(\Delta(T))$ which follows

$$\mathcal{D}(\neg \bar{A}_1^{p_1} \nabla \cdots \& \neg \bar{A}_n^{p_2}) = \mathcal{D}(\neg(\bar{A}_1^{p_1} \& \cdots \& \bar{A}_n^{p_2})) \geq b$$

and $a \otimes b > 0$. But no such structure $\mathcal{D}$ may exist and thus, T is contradictory by the completeness theorem. □

5 Conclusion

This paper is a continuation of the development of the theory of fuzzy logic in narrow sense. Our goal was to prove some properties of open fuzzy theories and especially, to prove analogy of the classical Hilbert–Ackermann's consistency theorem. This theorem is stated in the previous section as Theorem 10.

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