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Abstract. An extended fuzzy automaton is introduced as a fuzzy automaton, where sets of states and inputs, respectively, are the sets of all fuzzy subsets $F(S)$ and $F(\Lambda)$, respectively, for some finite sets S and Λ. A category of these extended fuzzy automata is introduced and a functor between this category and a category of fuzzy automata is investigated. A relationship between output functions of fuzzy automaton and extended fuzzy automaton are derived.

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1 Introduction

In papers [2],[3] and [4] we investigated some properties of a category of automata and its generalization — a category of fuzzy automata. We also derived some functors between these categories which enable us to approximate the behaviour of a fuzzy automaton by an appropriate classical automaton. Namely, if $\mathbf{A} = (S, \Lambda, F, p, G)$ is a fuzzy automaton with a set of states S, an input alphabet Λ, a transition function $F \underset{\sim}{\subset} S \times \Lambda \times S$, an initial fuzzy state $p \underset{\sim}{\subset} S$ and final fuzzy state $G \underset{\sim}{\subset} S$ and if $f_{\mathbf{A}}$ is its output function (i.e. $f_{\mathbf{A}}$ is a fuzzy set in a free monoid Λ^* such that $f_{\mathbf{A}} = p \circ F^*(\Lambda) \circ G$) then for any positive real number $\epsilon < 1$ there exists a classical Moore automaton $\mathbf{B}$ over the same input alphabet Λ with an output function $g_{\mathbf{B}}$ such that

$$(\forall \lambda \in \Lambda) f_{\mathbf{A}}(\lambda) > \epsilon \Leftrightarrow g_{\mathbf{B}}(\lambda) = 1.$$

In this paper we deal with some special kind of fuzzy automaton. We introduce the so called extended fuzzy automaton which is a fuzzy automaton where sets of states and inputs, respectively, are the sets of all fuzzy subsets $F(S)$ and $F(\Lambda)$, respectively, for some finite sets S and Λ. We define a category of these extended fuzzy automata and we prove that there exists a functor M from a category of fuzzy automaton into a category of this extended fuzzy automaton. By using this functor we show that there exists some relationship between output function $f_{\mathbf{A}}$ of a fuzzy automaton $\mathbf{A}$ and the output function $f_{M(\mathbf{A})}$ of a corresponding extended fuzzy automaton $M(\mathbf{A})$. Moreover,we prove that an analogy of the above mentioned approximation theorem can be derived for extended fuzzy automata as well.

We recall firstly some basic properties of fuzzy automaton and their category.

Definition 1.1 (see [2]) *A (Moore) type of fuzzy automaton is a system* $\mathbf{A} = (S, \Lambda, p, F, G)$ *where* S *is a set of states,* Λ *is a set of inputs,* $p \underset{\sim}{\subseteq} S$ *is a fuzzy set called a fuzzy initial state,* $F : S \times \Lambda \times S \to [0,1]$ *is a fuzzy transition function i. e.* $F(\lambda) \underset{\sim}{\subseteq} S \times S$, $G \underset{\sim}{\subseteq} S$ *is a fuzzy set called a fuzzy final state.*

If S and Λ are finite then $\mathbf{A}$ is called a finite fuzzy automaton. If we denote by Λ^* the free monoid generated by Λ then F can be extended to a fuzzy transition function $F^* : S \times \Lambda^* \times S \to [0,1]$.
A principal identification of a fuzzy automaton is provided by its output function $f_{\mathbf{A}} : \Lambda^* \to [0,1]$ such that for $\lambda = \lambda_1 \lambda_2 \dots \lambda_n \in \Lambda^*$

$$f_{\mathbf{A}}(\lambda) = p \circ F^*(\lambda) \circ G = \bigvee_{s \in S} (\bigvee_{z \in S} (p(z) \wedge F^*(\lambda)(z,s)) \wedge G(s)).$$

Recall that by a category Φ we understand a category of Moore type automata (see [2]), the objects of which are finite automata $\mathbf{B} = (S, \Lambda, p, d, G)$ and morphisms from $(S_1, \Lambda_1, p_1, d_1, G_1)$ to $(S_2, \Lambda_2, p_2, d_2, G_2)$ are pairs (α, β), where $\alpha : S_1 \to S_2$, $\beta : \Lambda_1 \to \Lambda_2$ are maps such that

1. $\alpha(p_1) = p_2$,
2. $\alpha(G_1) \subseteq G_2$,
3. $\alpha(d_1(s, \lambda)) = d_2(\alpha(s), \beta(\lambda))$ for any $s \in S$, $\lambda \in \Lambda$.

Any Moore type automaton $\mathbf{B}$ provides an output function $g_{\mathbf{B}} : \Lambda^* \to \{0,1\}$ such that for $\lambda = \lambda_1 \dots \lambda_n \in \Lambda^*$

$$g_{\mathbf{B}}(\lambda) = 1 \quad \text{iff} \quad d^*(p, \lambda) := d(d^*(p, \lambda_1 \dots \lambda_{n-1}), \lambda_n) \in G$$

where d^* is an extension of a function d onto Λ^*. A category of fuzzy automata was introduced in [2], the objects of which are fuzzy automata and morphisms are defined as follows. We recall the following definitions.

Definition 1.2 (see [2]) *Let* A *and* B *be sets and* f *and* g *be fuzzy sets in* A *and* B*, respectively. Then for a map* $v : A \to B$ *the following diagram*

$$\begin{array}{ccc} A & \xrightarrow{v} & B \\ f \downarrow & & \downarrow g \\ [0,1] & = & [0,1] \end{array}$$

is said to fuzzy commutes if

$$(\forall a \in A) \quad f(a) \leq g \cdot v(a) \tag{1}$$
$$(\forall b \in v(A)) \quad g(b) \leq \bigvee_{v(a)=b} f(a) \tag{2}$$

Definition 1.3 (see [2]) *Let $\mathbf{A}_i = (S_i, \Lambda_i, p_i, F_i, G_i)$ be (in general nonfinite) fuzzy automata for $i = 1, 2$. Then a couple $(\alpha, \beta) : \mathbf{A}_1 \to \mathbf{A}_2$ is called a fuzzy morphism if $\alpha : S_1 \to S_2$, $\beta : \Lambda_1 \to \Lambda_2$ are maps such that the following diagrams fuzzy commute.*

$$\begin{tikzcd} S_1 \arrow[r, "\alpha"] \arrow[d, "p_1"'] & S_2 \arrow[d, "p_2"] \\ {[0,1]} \arrow[r, equal] & {[0,1]} \end{tikzcd} \qquad \begin{tikzcd} S_1 \arrow[r, "\alpha"] \arrow[d, "G_1"'] & S_2 \arrow[d, "G_2"] \\ {[0,1]} \arrow[r, equal] & {[0,1]} \end{tikzcd}$$

$$\begin{tikzcd} S_1 \times \Lambda_1 \times S_1 \arrow[r, "\alpha\times\beta\times\alpha"] \arrow[d, "F_1"'] & S_2 \times \Lambda_2 \times S_2 \arrow[d, "F_2"] \\ {[0,1]} \arrow[r, equal] & {[0,1]} \end{tikzcd}$$

Let Ψ be a category of (in general nonfinite) fuzzy automata with morphisms defined above. By Ψ_c we denote a subcategory of Ψ the objects of which are finite fuzzy automata and a morphism between two fuzzy automata is a pair of functions (α, β) such that the above mentioned diagrammes commute (and not only fuzzy commute).

2 Category of extended fuzzy automata

An *extended fuzzy automaton* is a fuzzy automaton where a set of states is the set $F(S)$ of all fuzzy sets over some finite set S and a set of inputs is the set $F(\Lambda)$ of all fuzzy sets over some finite set Λ.
It means that extended fuzzy automata are some special objects of a category Ψ. The full subcategory of these objects will be denoted by $\Psi_e \subseteq \Psi$.
It is a natural question if from any finite fuzzy automaton from Ψ can be constructed an extended fuzzy automaton and, moreover, whether this construction $M : Ob(\Psi_c) \to Ob(\Psi_e)$ (which will be called *an extension function*) is a functor $\Psi_c \to \Psi_e$.
We will use the following notation. Let $f : X_1 \times \ldots \times X_n \to Y$ be a function. By $\tilde{f}$ we denote a function $F(X_1) \times \ldots \times F(X_n) \to F(Y)$ obtained by using the extension principle applied on f.
An *extension function* is then any function $M : Ob(\Psi_c) \to Ob(\Psi_e)$ such that

$$M(S, \Lambda, p, F, G) = (F(S), F(\Lambda), M(p), M(F), M(G))$$

where $M(p) \underset{\sim}{\subset} F(S)$, $M(F) \underset{\sim}{\subset} F(S) \times F(\Lambda) \times F(S)$ and $M(G) \underset{\sim}{\subset} F(S)$.
We present firstly an example of this extension function. Let S and Λ be finite sets and let $p \underset{\sim}{\subset} S$, $G \underset{\sim}{\subset} S$, $F \underset{\sim}{\subset} S \times \Lambda \times S$, $u \underset{\sim}{\subset} \Lambda$ and $\delta \in [0,1]$.

Example. We define $M(p), M(G)$ and $M(F)$ such that

$$M(p)(T) := \bigvee_{\substack{s \in S \\ p(s) \geq \delta}} T(s) \quad , \qquad T \in F(S)$$

$$M(G)(T) := \bigvee_{\substack{s \in S \\ G(s) \geq \delta}} T(s) \quad , \qquad T \in F(S)$$

$$M(F)(T, A, T') := \bigvee_{\substack{(s,\lambda,s') \in S \times \Lambda \times S \\ F(s,\lambda,s') \geq \delta}} (T(s) \wedge A(\lambda) \wedge T'(s')) \quad ,$$

$$T, T' \in F(S), A \in F(\Lambda).$$

Moreover, if $u : \Lambda \to [0,1]$ is a map then $M(u) : F(\Lambda) \to [0,1]$ is defined such that

$$M(u)(A) := \bigvee_{\substack{\lambda \in \Lambda \\ u(\lambda) \geq \delta}} A(\lambda) \quad , \qquad A \in F(\Lambda).$$

Theorem 2.1 *The extension function M from the Example is a functor $M : \Psi_c \to \Psi_e$, where for a morphism $(\alpha, \beta) : \mathbf{A}_1 \to \mathbf{A}_2$ in Ψ_c, $M(\alpha, \beta) = (\tilde{\alpha}, \tilde{\beta})$.*

Proof: Let $(\alpha, \beta) : \mathbf{A}_1 \to \mathbf{A}_2$ be a morphism in Ψ_c. At first, we have to prove that the following diagram fuzzy commutes.

$$\begin{array}{ccc}
F(S_1) & \xrightarrow{\tilde{\alpha}} & F(S_2) \\
M(p_1) \downarrow & & \downarrow M(p_2) \\
[0,1] & = & [0,1]
\end{array}$$

i.e.

$$\begin{aligned}
M(p_1)(T) &\leq M(p_2) \cdot \tilde{\alpha}(T) && (3) \\
M(p_2)(T') &\leq \bigvee_{\tilde{\alpha}(T) = T'} M(p_1)(T) && (4)
\end{aligned}$$

We have

$$M(p_1)(T) = \bigvee_{\substack{s_1 \in S_1 \\ p_1(s_1) \geq \delta}} T(s_1).$$

Then

$$M(p_2)(\tilde{\alpha}(T)) = \bigvee_{\substack{s_2 \in S_2 \\ p_2(s_2) \geq \delta}} \tilde{\alpha}(T)(s_2) \geq \tilde{\alpha}(T)(\alpha(s_1)) \geq T(s_1)$$

for any element $s_1 \in S_1$ such that $p_1(s_1) \geq \delta$.

In the last inequality we use the fact that for morphisms in a category Ψ_c we have $p_1(s_1) = p_2(\alpha(s_1))$. Then

$$M(p_2)(\tilde{\alpha}(T)) \geq \bigvee_{\substack{s_1 \in S_1 \\ p_1(s_1) \geq \delta}} T(s_1) = M(p_1)(T)$$

and the inequality (3) holds.

Let $T' \in F(S_2)$ and $\tilde{\alpha}(T) = T'$ for some $T \in F(S_1)$. Then there exists $s_2^0 \in S_2$ such that $p_2(s_2^0) \geq \delta$ and

$$M(p_2)(T') = \bigvee_{\substack{s_2 \in S_2 \\ p_2(s_2) \geq \delta}} T'(s_2) = T'(s_2^0) = \tilde{\alpha}(T)(s_2^0) = \bigvee_{\substack{s_1 \in S_1 \\ \alpha(s_1) = s_2^0}} T(s_1).$$

Since

$$\{s_1 \in S_1, \alpha(s_1) = s_2^0\} \subseteq \{s_1 \in S_1, p_1(s_1) \geq \delta\}$$

it follows that

$$M(p_1)(T) = \bigvee_{\substack{s_1 \in S_1 \\ p_1(s_1) \geq \delta}} T(s_1) \geq \bigvee_{\substack{s_1 \in S_1 \\ \alpha(s_1) = s_2^0}} T(s_1) = M(p_2)(T').$$

Hence,

$$M(p_1)(T) \geq M(p_2)(T')$$

and the inequality (4) holds.

Analogously we can prove that the following diagram fuzzy commutes.

$$\begin{tikzcd}
F(S_1) \arrow[r, "\tilde{\alpha}"] \arrow[d, "M(G_1)"'] & F(S_2) \arrow[d, "M(G_2)"] \\
{[0,1]} \arrow[r, equal] & {[0,1]}
\end{tikzcd}$$

Finally, we prove that the following diagram fuzzy commutes.

$$\begin{tikzcd}
F(S_1) \times F(\Lambda_1) \times F(S_1) \arrow[r, "\tilde{\alpha}\times\tilde{\beta}\times\tilde{\alpha}"] \arrow[d, "M(F_1)"'] & F(S_2) \times F(\Lambda_2) \times F(S_2) \arrow[d, "M(F_2)"] \\
{[0,1]} \arrow[r, equal] & {[0,1]}
\end{tikzcd}$$

i. e.

$$M(F_1)(T_1, A_1, T_1') \leq M(F_2) \cdot (\tilde{\alpha} \times \tilde{\beta} \times \tilde{\alpha})(T_1, A_1, T_1') \tag{5}$$

$$M(F_2)(T_2, A_2, T_2') \leq \bigvee_{(\tilde{\alpha} \times \tilde{\beta} \times \tilde{\alpha})(T_1, A_1, T_1') = (T_2, A_2, T_2')} M(F_1)(T_1, A_1, T_1') \tag{6}$$

In fact, we have

$$M(F_1)(T, A, T') = \bigvee_{\substack{(s_1, \lambda_1, s_1') \in S_1 \times \Lambda_1 \times S_1 \\ F_1(s_1, \lambda_1, s_1') \geq \delta}} (T(s_1) \wedge A(\lambda_1) \wedge T'(s_1')).$$

Then

$$M(F_2)(\tilde{\alpha}(T), \tilde{\beta}(A), \tilde{\alpha}(T')) =$$

$$= \bigvee_{\substack{(s_2, \lambda_2, s_2') \in S_2 \times \Lambda_2 \times S_2 \\ F_2(s_2, \lambda_2, s_2') \geq \delta}} (\tilde{\alpha}(T)(s_2) \wedge \tilde{\beta}(A)(\lambda_2) \wedge \tilde{\alpha}(T')(s_2')) \geq$$

$$\geq \tilde{\alpha}(T)(\alpha(s_1)) \wedge \tilde{\beta}(A)(\beta(\lambda_1)) \wedge \tilde{\alpha}(T')(\alpha(s_1')) \geq T(s_1) \wedge A(\lambda_1) \wedge T'(s_1')$$

for any $(s_1, \lambda_1, s_1') \in S_1 \times \Lambda_1 \times S_1$ such that $F_1(s_1, \lambda_1, s_1') \geq \delta$.
In the last inequality we use again the fact that for morphisms in category Ψ_c we have $F_1(s_1, \lambda_1, s_1') = F_2(\alpha(s_1), \beta(\lambda_1), \alpha(s_1'))$.
Then

$$M(F_2)(\tilde{\alpha}(T), \tilde{\beta}(A), \tilde{\alpha}(T')) \geq \bigvee_{\substack{(s_1, \lambda_1, s_1') \in S_1 \times \Lambda_1 \times S_1 \\ F_1(s_1, \lambda_1, s_1') \geq \delta}} (T(s_1) \wedge A(\lambda_1) \wedge T'(s_1')) =$$

$$= M(F_1)(T, A, T')$$

and (5) holds.
Finally, let $(\tilde{\alpha} \times \tilde{\beta} \times \tilde{\alpha})(T_1, A_1, T_1') = (T_2, A_2, T_2')$. Since S_2, Λ_2 are finite, then there exists $(s_2^0, \lambda_2^0, s_2^{0'}) \in S_2 \times \Lambda_2 \times S_2$ such that $F_2(s_2^0, \lambda_2^0, s_2^{0'}) \geq \delta$ and

$$M(F_2)(T_2, A_2, T_2') = \bigvee_{\substack{(s_2, \lambda_2, s_2') \in S_2 \times \Lambda_2 \times S_2 \\ F_2(s_2, \lambda_2, s_2') \geq \delta}} (T_2(s_2) \wedge A_2(\lambda_2) \wedge T_2'(s_2')) =$$

$$= T_2(s_2^0) \wedge A_2(\lambda_2^0) \wedge T_2'(s_2^{0'}) = \tilde{\alpha}(T_1)(s_2^0) \wedge \tilde{\beta}(A_1)(\lambda_2^0) \wedge \tilde{\alpha}(T_1')(s_2^{0'}) =$$

$$= \bigvee_{\substack{(s_1, \lambda_1, s_1') \in S_1 \times \Lambda_1 \times S_1 \\ \alpha(s_1) = s_2^0 \\ \beta(\lambda_1) = \lambda_2^0 \\ \alpha(s_1') = s_2^{0'}}} (T_1(s_1) \wedge A_1(\lambda_1) \wedge T_1'(s_1')).$$

Since

$$\{(s_1, \lambda_1, s_1') \in S_1 \times \Lambda_1 \times S_1, \alpha(s_1) = s_2^0, \beta(\lambda_1) = \lambda_2^0, \alpha(s_1') = s_2^{0'}\} \subseteq$$
$$\subseteq \{(s_1, \lambda_1, s_1') \in S_1 \times \Lambda_1 \times S_1, F_1(s_1, \lambda_1, s_1') \geq \delta\}$$

we have

$$M(F_1)(T_1, A_1, T_1') = \bigvee_{\substack{(s_1,\lambda_1,s_1') \in S_1 \times \Lambda_1 \times S_1 \\ F_1(s_1,\lambda_1,s_1') \geq \delta}} (T_1(s_1) \wedge A_1(\lambda_1) \wedge T_1'(s_1')) \geq$$

$$\geq \bigvee_{\substack{(s_1,\lambda_1,s_1') \in S_1 \times \Lambda_1 \times S_1 \\ \alpha(s_1) = s_2^0 \\ \beta(\lambda_1) = \lambda_2^0 \\ \alpha(s_1') = s_2^{0'}}} (T_1(s_1) \wedge A_1(\lambda_1) \wedge T_1'(s_1')) = M(F_2)(T_2, A_2, T_2')$$

and it follows that

$$M(F_1)(T_1, A_1, T_1') \geq M(F_2)(T_2, A_2, T_2')$$

for any $(T_1, A_1, T_1') \in F(S_1) \times F(\Lambda_1) \times F(S_1)$ such that
$(\tilde{\alpha} \times \tilde{\beta} \times \tilde{\alpha})(T_1, A_1, T_1') = (T_2, A_2, T_2')$.
Hence,

$$\bigvee_{(\tilde{\alpha} \times \tilde{\beta} \times \tilde{\alpha})(T_1,A_1,T_1') = (T_2,A_2,T_2')} \tilde{F}_1(T_1, A_1, T_1') \geq \tilde{F}_2(T_2, A_2, T_2')$$

and the inequality (8) holds.
Hence, $(\tilde{\alpha}, \tilde{\beta}) : M(\mathbf{A}_1) \to M(\mathbf{A}_2)$ is the morphism in Ψ_e and M is a functor (as it may be proved easily too). □

Now we will study an output function of an extended fuzzy automaton.
Let $M : \Psi_c \to \Psi_e$ be a functor which is constructed from some extended function $M : Ob(\Psi_c) \to Ob(\Psi_e)$.
For any fuzzy automaton $\mathbf{A} \in \Psi_c$ we then have an output function of an extended fuzzy automaton $M(\mathbf{A})$

$$f_{M(\mathbf{A})} : F(\Lambda)^* \to [0,1].$$

Moreover, by using some construction of a function $M(u) : F(\Lambda) \to [0,1]$ from a function $u : \Lambda \to [0,1]$ (see Example) we obtain another output function $M(f_{\mathbf{A}}/\Lambda) : F(\Lambda) \to [0,1]$ where $f_{\mathbf{A}} : \Lambda^* \to [0,1]$ is an output function of a fuzzy automaton $\mathbf{A}$.
In the following proposition we show some relationship between functions $M(f_{\mathbf{A}}/\Lambda)$ and $f_{M(\mathbf{A})}/F(\Lambda)$.

Theorem 2.2 *Let M be a functor from Theorem 2.1. Then*

$$(f_{M(\mathbf{A})}/F(\Lambda))(A) \geq M(f_{\mathbf{A}}/\Lambda)(A) \quad \textit{for any } A \in F(\Lambda).$$

PROOF: For simplicity we set

$$h_{\mathbf{A}} := f_{\mathbf{A}}/\Lambda, \quad h_{M(\mathbf{A})} := f_{M(\mathbf{A})}/F(\Lambda).$$

Let $\lambda_0 \in \Lambda$ be such that $M(h_{\mathbf{A}})(A) = \bigvee_{\substack{\lambda\in\Lambda \\ h_{\mathbf{A}}(\lambda)\geq\delta}} A(\lambda) = A(\lambda_0)$ where $h_{\mathbf{A}}(\lambda_0) = p(z_0) \wedge F(z_0, \lambda_0, s_0) \wedge G(s_0) \geq \delta$. Then

$$h_{M(\mathbf{A})}(A) = M(p) \circ M(F)(A) \circ M(G) =$$

$$= \bigvee_{T\in F(S)} (\bigvee_{T'\in F(S)} (M(p)(T') \wedge M(F)(A)(T', T)) \wedge M(G)(T)) =$$

$$= \bigvee_{T\in F(S)} (\bigvee_{T'\in F(S)} (\bigvee_{p(s')\geq\delta} T'(s') \wedge \bigvee_{F(s'',\lambda,s''')\geq\delta} (T'(s'') \wedge A(\lambda) \wedge T(s''')))\wedge$$

$$\wedge \bigvee_{G(s)\geq\delta} T(s))) \geq \bigvee_{p(s')\geq\delta} \{z_0\}(s') \wedge \bigvee_{F(s'',\lambda,s''')\geq\delta} (\{z_0\}(s'') \wedge A(\lambda) \wedge \{s_0\}(s'''))\wedge$$

$$\wedge \bigvee_{G(s)\geq\delta} \{s_0\}(s) \geq \{z_0\}(z_0) \wedge \{z_0\}(z_0) \wedge A(\lambda_0) \wedge \{s_0\}(s_0) \wedge \{s_0\}(s_0) = A(\lambda_0)$$

Then

$$h_{M(\mathbf{A})}(A) \geq \bigvee_{\substack{\lambda\in\Lambda \\ h_{\mathbf{A}}(\lambda)\geq\delta}} A(\lambda) = M(h_{\mathbf{A}})(A)$$

□

In [2] it was proved that there exists a functor $F_\epsilon : \Psi_c \to \Phi$ which transforms any finite fuzzy automaton onto a finite automaton (with required precision $\epsilon > 0$). In the following theorem we prove an analogical result for the category Ψ of (in general nonfinite) fuzzy automata. In this case the resulting functor is H_ϵ from Ψ into a category Φ_n of (in general nonfinite) classical Moore automata.

Theorem 2.3 *Let $\epsilon > 0$. Then there exists a functor $H = H_\epsilon : \Psi \to \Phi_n$ such that for every fuzzy automaton $\mathbf{A} \in \Psi$ and every word $\lambda \in \Lambda^*$,*

$$f_{\mathbf{A}}(\lambda) > \epsilon \Leftrightarrow g_{H_\epsilon(\mathbf{A})}(\lambda) = 1.$$

PROOF: Let $\mathbf{A} = (S, \Lambda, p, F, G) \in \Psi$. We set

$$H_\epsilon(\mathbf{A}) = (2^S, \Lambda, p_\epsilon, d_\epsilon, G_\epsilon),$$

where 2^S is the set of all subsets of S, $p_\epsilon = \{s \in S, p(s) > \epsilon\} \in 2^S$,
$G_\epsilon = \{X \subseteq S : \exists x \in X, G(x) > \epsilon\} \subseteq 2^S$ and $d_\epsilon : 2^S \times \Lambda \to 2^S$ is defined such that

$$d_\epsilon(X, \lambda) = \{s \in S : \exists x \in X, F(x, \lambda, s) > \epsilon\} \in 2^S.$$

Then if $(\alpha, \beta) : \mathbf{A}_1 \to \mathbf{A}_2$ is a morphism in Ψ then $H_\epsilon(\alpha, \beta) = (u, v) : H_\epsilon(\mathbf{A}_1) \to H_\epsilon(\mathbf{A}_2)$ is defined such that

$$u(X) := \alpha(X) \subseteq S_2, \quad v := \beta.$$

We can prove that (u, v) is a morphism in Φ_n, i. e.

1. $u(p_{1,\epsilon}) = p_{2,\epsilon}$,
2. $u(G_{1,\epsilon}) \subseteq G_{2,\epsilon}$,
3. $\alpha(d_{1,\epsilon}(X, \lambda)) = d_{2,\epsilon}(\alpha(X), \beta(\lambda))$.

Let $s \in u(p_{1,\epsilon}) = \alpha(p_{1,\epsilon}) = \{\alpha(s) : s \in S_1, p_1(s) > \epsilon\}$. Then $\epsilon < p_1(s) \leq p_2(\alpha(s))$, i. e. $\alpha(s) \in p_{2,\epsilon}$ and $u(p_{1,\epsilon}) \subseteq (p_{2,\epsilon})$.
Now let $s \in p_{2,\epsilon}$. Then $\epsilon < p_2(s) \leq \bigvee_{\substack{t \in S_1 \\ \alpha(t) = s}} p_1(t)$ holds. Then there exists $t \in S_1$ such that $\alpha(t) = s, p_1(t) > \epsilon$. Then $t \in p_{1,\epsilon}$ and $s = \alpha(t) \in \alpha(p_{1,\epsilon})$. Hence $p_{2,\epsilon} \subseteq \alpha(p_{1,\epsilon}) = u(p_{1,\epsilon})$ and (1) holds.
Let $T \in u(G_{1,\epsilon}) = \{u(X), X \subseteq S_1 : \exists x \in X, G_1(x) > \epsilon\}$. Then exists $t \in T$ such that $\epsilon < G_1(t) \leq G_2(\alpha(t))$ and $\alpha(T) = u(T) \in G_{2,\epsilon}$, i. e. (2) holds.
Finally, let $s \in \alpha(d_{1,\epsilon}(X, \lambda)) = \{\alpha(s), s \in S_1 : \exists x \in X, F_1(x, \lambda, s) > \epsilon\}$. Then $\epsilon < F_1(x, \lambda, s) \leq F_2(\alpha(x), \beta(\lambda), \alpha(s))$, i. e. $\alpha(s) \in d_{2,\epsilon}(\alpha(X), \beta(\lambda))$ and $\alpha(d_{1,\epsilon}(X, \lambda)) \subseteq d_{2,\epsilon}(\alpha(X), \beta(\lambda))$.
Now let $s \in d_{2,\epsilon}(\alpha(X), \beta(\lambda))$. Then

$$\epsilon < F_2(x', \lambda', s') \leq \bigvee_{(\alpha \times \beta \times \alpha)(x, \lambda, s) = (x', \lambda', s')} F_1(x, \lambda, s).$$

Then exists $(x, \lambda, s) \in S_1 \times \Lambda_1 \times S_1$ such that $(\alpha \times \beta \times \alpha)(x, \lambda, s) = (x', \lambda', s')$ and $F_1(x, \lambda, s,) > \epsilon$. Then $s \in d_{1,\epsilon}(X, \lambda)$ and hence $\alpha(s) \in \alpha(d_{1,\epsilon}(X, \lambda))$. Then $d_{2,\epsilon}(\alpha(X), \beta(\lambda)) \subseteq \alpha(d_{1,\epsilon}(X, \lambda))$ and (3) holds.
Hence, H_ϵ is a functor (as it may be proved easily too).
Now let $d_\epsilon(\lambda, p_\epsilon) \in G_\epsilon$, i. e. $g_{H_\epsilon(\mathbf{A})}(\lambda) = 1$. Hence there exists $s \in d_\epsilon(\lambda, p_\epsilon)$ such that $G(s) > \epsilon$ and an $x \in S$ such that $p(x) > \epsilon$ and $F(x, \lambda, s) > \epsilon$.
Thus,

$$f_{\mathbf{A}}(\lambda) = \bigvee_{s \in S} (\bigvee_{x \in S} (p(x) \wedge F(x, \lambda, s)) \wedge G(s)) > \epsilon.$$

The converse implication can be proved analogously. □

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