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Tadeusz Dłotko

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Some records on second order differential equations

TADEUSZ DŁOTKO

Abstract. Solutions of special Neumann boundary value problems will be find.

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In spite of more than 150 years of investigations on asymptotic behaviour of ordinary differential equations (see [8]) the theory of second order ordinary differential equations still has interesting but insufficiently investigated areas. This has been confirmed again by a recent paper of Jean Mawhin [6], even though quite a lot of papers were devoted to such problems (especially [1], [2], [3], [4], [6], [7]).

Our purpose is to find a solution to the Neumann boundary value problem

$$\begin{cases} u'' + g(u') = f(t), & t \in [a, b] \\ u'(a) = u'(b) = 0 \end{cases} \tag{1}$$

or periodic problem with additional conditions

$$u(a) - u(b) = 0, \quad u'(a) - u'(b) = 0.$$

In the paper [3] it was demonstrated that for a bounded $g \in C^{(1)}$ and f such that $f = \overline{f} + \tilde{f}$, $\overline{f} = (b-a)^{-1} \int_a^b f(s)ds$, for every $\tilde{f}$ there exists a unique $\overline{f}$ such that (1) has a solution.

Another form of the above existence theorem, different from the typical form of such theorems in the theory of differential equations, was given by S. Fučik [3].

This theorem assures the existence of solutions only for some functions f. It can happen that for $\hat{f}$ the problem (1) has a solution and for $\hat{\hat{f}}$ such that $0 < |\hat{f} - \hat{\hat{f}}| < \varepsilon$ the problem

$$u'' + g(t, u') = f, \qquad u'(a) = u'(b) = 0, \qquad t \in [a, b] \tag{2}$$

has no solution.

An explicit example of such a situation is provided by the problem

$$u''(t) = \varphi(t)u'(t) + k\varphi(t), \qquad u'(0) = u'(1) = 0, \qquad t \in [0,1],\ k > 0, \tag{3}$$

where φ is a given continuous function.

Substituting $u' = v$ we have

$$|k + v(t)| = k \exp\left(\int_0^t \varphi(s)ds\right) \tag{4}$$

and for $t = 0$ and $t = 1$ we get $k = k \exp\left(\int_0^1 \varphi(s)ds\right)$.

When $k > 0$ and $\int_0^1 \varphi(s)ds \neq 0$, then (3) has no solutions. When k is arbitrary and $\int_0^1 \varphi(s)ds = 0$, then (3) has solutions. These troubles are consequences of the absence of a Green function for the problem

$$\left\{\begin{array}{lcl} u'(t) & = & v(t) \qquad t \in [a,b] \\ v'(t) & = & g(t,v(t)) + f_1(t), \qquad v(a) = v(b) = 0, \end{array}\right. \tag{5}$$

So it is impossible to transform (1) to an integral equation over a suitable Green function. Interestingly enough, also the method of "a priori" estimates for boundary value problems in ordinary differential equations, so excellently described in [4], cannot be applied to systems (1).

J. Mawhin [6] proposed to use the integral form of (1) instead of the Green function. Thus we consider

$$\begin{aligned} v(t) \; = \; & \int_0^t [g(s,v(s)) - (b-a)^{-1} \int_a^b g(t,v(t)dt + f(s) - \\ & -(b-a)^{-1} \int_a^b f(t)]ds. \end{aligned} \tag{6}$$

It is clear that the solution $v(t)$ of the last equation satisfies (5) with $f_1 = f + (b-a)^{-1} \int_a^b [g(t,v(t)) + f(t)]dt$ and

$$u(t) = \int_a^t v(s)ds + c, \qquad c \in \mathbb{R}^n$$

satisfies (1).

An open problem is to establish the density of such functions in $L^1([a,b],\mathbb{R}^n)$.

The observed situation of existence of solutions to (1) is in accordance with the above example (3).

Our aim is to generalize J. Mawhin's results using the theory of completely continuous vector fields in Banach spaces (see [5]) instead of Schauder fixed point theorem.

To this end let us consider the Banach space

$$X =: \{x : \; x \in C([a,b],\mathbb{R}^n), \; x' \in L^1([a,b],\mathbb{R}^n), \; x(a) = x(b) = 0\} \tag{7}$$

with the norm

$$|x|_X =: \max_{t\in[a,b]} |x(t)| + \int_a^b |x'(t)|dt. \tag{8}$$

Let us consider a more general form of the problem (1), namely

$$\begin{cases} u''(t) = A(t)u'(t) + g(t, u(h(t)), u'(k(t))), & t \in [a,b] \\ u(a) = u_0, \quad u'(a) = u'(b) = 0, & \end{cases} \tag{9}$$

Here $A(t)$ is a given $n \times n$ matrix, u and $g(t,u,v)$ are n-vectors,

$$A(t)u' : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n, \; g(t,u,v) : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

and satisfies Caratheodory conditions. The given continuous functions $h(t), k(t)$ denote the deviations of the argument t

$$h, k : [a,b] \to [a,b], \quad h(a) = a, \quad u_0 \in \mathbb{R}^n \quad \text{is given.}$$

The solution u and the derivative u' are absolutely continuous in $[a,b]$.
Together with (9) let us consider

$$\begin{cases} v'(t) = A(t)v(t) + g(t, u_0 + \int_a^{h(t)} v(s)ds, v(k(t))), \\ v(a) = v(b) = 0, \quad t \in [a,b]. \end{cases} \tag{9'}$$

We assume that there exists such an $R_0 > 0$ that

$$\begin{aligned} &\sup_{v \in X,\ |v|_X = R_0} \Big| \int_a^t [g(\tau, u_0 + \int_a^{h(\tau)} v(s)ds, v(k(\tau))) - \\ &- \tfrac{t-a}{b-a} \int_a^b g(t, u_0 + \int_a^{h(t)} v(s)ds, v(k(t)))dt]d\tau \Big|_X < \\ &< \inf_{v \in X,\ |v| = R_0} \Big| v(t) - \int_a^t [A(\tau)v(\tau) - (b-a)^{-1} \int_a^b A(t)v(t)dt]d\tau \Big|_X \end{aligned} \tag{10}$$

and the problem

$$v(t) = \int_a^t [A(\tau)v(\tau) - (b-a)^{-1} \int_a^b A(t)v(t)dt]d\tau, \quad v(a) = v(b) = 0 \tag{11}$$

has only the $v = 0$ solution.

Theorem. *Assume that the functions A, g, h, k in the equation (9') satisfy the conditions (10) and (11).*
Then the integral equation

$$v(t) = \int_a^t [(Fv)(\tau) - (b-a)^{-1} \int_a^b (Fv)(s)ds]d\tau \tag{12}$$

where

$$(Fv)(t) =: A(t)v(t) + g(t, u_0 + \int_a^{h(t)} v(s)ds, v(k(t)))$$

has at least one solution in X. The function $w(t) =: u_0 + \int_a^t v(s)ds$ *satisfies*

$$\begin{aligned} w''(t) &= A(t)w'(t) + g(t, w(h(t)), w'(k(t))) - (b-a)^{-1} \int_a^b [A(t)w'(t) + \\ &+ g(t, w(h(t)), w'(k(t)))]dt, \quad w(a) = u_0, \; w'(a) = w'(b) = 0, \\ &\text{for a.e.} \quad t \in [a, b]. \end{aligned}$$

Proof: It is clear that the solution v of (12) satisfies $v(a) = v(b) = 0$,

$$v'(t) = (Fv)(t) - (b-a)^{-1} \int_a^b (Fv)(s)ds \quad \text{and} \quad w(a) = u_0, \tag{13}$$

where $w(t) = u_0 + \int_a^t v(s)ds$. As a consequence of the earlier assumptions the solution v of (12) is absolutely continuous in $[a, b]$.

Let us define two completely continuous vector fields (see [5]) in the Banach space X by setting

$$(\phi v)(t) =: v(t) - \int_a^t [A(\tau)v(\tau) - (b-a)^{-1} \int_a^b A(s)v(s)ds]d\tau, \quad x \in X \tag{14}$$

and

$$(\psi v)(t) =: v(t) - \int_a^t [(Fv)(\tau) - (b-a)^{-1} \int_a^b (Fv)(s)ds]d\tau, \quad x \in X. \tag{15}$$

It is important that $\phi, \psi : X \to X$. In particular

$$(\phi v)(a) = (\phi v)(b) = (\psi v)(a) = (\psi v)(b) = 0.$$

From (11) it follows that the vector field ϕ on spheres

$$S_R =: \{x : \; x \in X, |x|_X = R > 0\} \tag{16}$$

is correctly defined and

$$\phi(-v) = -\phi(v), \quad v \in S_R. \tag{17}$$

Therefore the rotation $\gamma(\phi, S_R)$ of the completely continuous vector field ϕ on the sphere S_R is nonzero.

Now let us consider the difference

$$\begin{aligned} (\phi v - \psi v)(t) &= \int_a^t [g(\tau, u_0 + \int_a^{h(\tau)} v(s)ds, v(k(\tau))) - \\ -(t-a)(b-a)^{-1} &\int_a^b g(t, u_0 + \int_a^{h(t)} v(s)ds, v(k(t)))dt]d\tau. \end{aligned} \tag{18}$$

We have

$$\inf_{v\in X,\ |v|_X=R} |(\phi v)(t)|_X = R \inf_{v\in X,\ |v|_X=1} |(\phi v)(t)|_X \to \infty, \quad \text{when } R\to\infty. \tag{19}$$

Now using (11) we have for $v \in S_{R_0}$

$$|\phi v - \psi v|_X < \inf_{v\in X,\ |v|=R_0} |\phi v|_X \leq |\phi v|_X. \tag{20}$$

It follows that the rotation

$$(\psi, S_{R_0}) \neq 0. \tag{21}$$

The last inequality is sufficient for the existence of at least one solution $\overline{v}$ of the problem (13) in X. Moreover, it satisfies $|\overline{v}|_X \leq R_0$. □

Remarks.

1 *When $A = 0$ and $g(t, \int_a^{h(t)} v(s)ds,\ v(k(t))) \equiv g(t, v(t))$, then equation (12) is of the form (3) considered in [6].*

2 *Condition (10) is satisfied, for instance, when the function g is bounded. The same is true, when the growth of the left hand side of (10) is sublinear with respect to the radius of the sphere S_{R_0}.*

Similarly, the method of completely continuous vector fields in Banach space can be applied to the following problems. First, we can apply it to gradient systems of differential equation

$$\begin{cases} u''(t) = \nabla G(t, u'(t)) + f(t, u(h(t)), u'(k(t))), & t\in[a,b] \\ u'(a) = u'(b) = 0, \end{cases} \tag{22}$$

and second, also to periodic solutions of systems

$$\begin{cases} u''(t) = A(t)u'(t) + g(t, u(h(t)), u'(k(t))), & t\in[a,b], \\ u(a) - u(b) = 0, \quad u'(a) - u'(b) = 0. \end{cases} \tag{23}$$

Applying the above considerations it is enough to examine the following integral equations:

for the gradient equation (22)

$$\begin{aligned} v(t) \quad = \quad & \int_a^t [\nabla G(\tau, u'(\tau)) - (b-a)^{-1} \int_a^b \nabla G(t, u'(t))dt + \\ & + f(\tau, \int_a^{h(\tau)} v(s)ds, v(k(\tau))) - \\ & - (b-a)^{-1} \int_a^b f(t, \int_a^{h(t)} v(s)ds, v(k(t)))dt]d\tau \end{aligned} \tag{24}$$

and

$$v(t) = \int_a^t I(\tau)d\tau - (b-a)^{-1}\int_a^b I(\tau)d\tau \tag{25}$$

where

$$\begin{aligned} I(t) \quad = \quad & A(t)v(t) - (b-a)^{-1}\int_a^b A(s)v(s)ds + g(t, \int_a^{h(t)} v(s)ds, v(k(t)))- \\ & -(b-a)^{-1}\int_a^b\int_a^t g(\tau, \int_a^{h(\tau)} v(s)ds, v(k(\tau)))d\tau dt \end{aligned}$$

for the periodic systems (23).

As in the above theorem one can prove existence results for the problems (22) and (23). Unfortunately, the results hold only for special types of homogeneity terms.

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