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Fast growing sequences of partial denominators

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Abstract. It is common knowledge that numbers with a fast growing sequence of partial denominators are transcendental. Several versions of this fact have been used repeatedly in the past. We give a rather general one which can serve as a convenient technical tool.

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1 Preliminaries and a Theorem

We will use the following notations: If $a_0 \in \mathbb{Z}$ and $a_1, a_2, a_3, \ldots \in \mathbb{N}$, then

$$\alpha = [a_0; a_1, a_2, a_3, \ldots]$$

serves as a short notation for

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}}.$$

As usual, we set $\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$ for $n \geq 0$, where $(p_n, , q_n) = 1$. Then the recurrence relations $p_{n+1} = a_{n+1}p_n + p_{n-1}$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$ are valid for $n \geq 1$. Our proofs will heavily depend on the following theorem.

Theorem 1 (A. Baker[1]). *Let $\alpha \in \mathbb{C}$, $\kappa > 2$ and let K be an algebraic number field. Suppose there exists a sequence $(\xi_j)_{j\geq 1}$ of pairwise different elements in K such that*

$$|\alpha - \xi_j| < H(\xi_j)^{-\kappa}$$

for all $j \geq 1$, where $H(\xi_j)$ denotes the height of ξ_j. Then α is transcendental. If further

$$\limsup_{j\to\infty} \frac{\log H(\xi_{j+1})}{\log H(\xi_j)} < \infty,$$

then α is a S- or T-number according to Mahler's classification.

This is a generalization of the celebrated Thue-Siegel-Roth Theorem. The following Lemma is a basic fact from the theory of continued fractions. We include it for convinience.

Lemma 2. *Let l, $n \in \mathbb{N}$. Then*

$$a_{n+1}a_{n+2}\dots a_{n+l} \geq \tfrac{q_{n+l}}{q_n} \geq (a_{n+1}+1)(a_{n+2}+1)\dots(a_{n+l}+1)$$
$$\geq 2^l a_{n+1}a_{n+2}\dots a_{n+l}$$

PROOF: Use induction on l and the above mentioned recursion formulae. □

We now proceed to our main technical lemma.

Lemma 3. *Let $K \geq 1$. Assume that $q_n \leq a_{n+l}^k$ and $2^{l(l-1)/2} \leq a_{n+l}^k$. Then at least one of the following inequalities holds:*

$$a_{n+1}^{K(l+1)} \geq q_n,\ a_{n+2}^{K(l+1)} \geq q_{n+1},\ \dots,\ a_{n+l}^{K(l+1)} \geq q_{n+l-1}.$$

PROOF: Lemma 2 implies

$$\begin{aligned} q_nq_{n+1}\cdots q_{n+l-1} &\leq q_n\cdot(2a_{n+1}q_n)\cdot(2^2a_{n+2}a_{n+1}q_n)\cdot\ldots\cdot(2^{l-1}a_{n+l-1}\dots a_{n+1}q_n \\ &= 2^{1+2+\ldots+(l-1)}q_n^l a_{n+1}^{l-1}a_{n+2}^{l-2}\dots a_{n+l-1} \\ &\leq 2^{l(l-1)/2}a_{n+1}^{l-1}a_{n+2}^{l-2}\dots a_{n+l-1}a_{n+l}^{Kl} \\ &\leq a_{n+1}^{l-1}a_{n+2}^{l-2}\dots a_{n+l-1}a_{n+l}^{K(l+1)} \\ &\leq (a_{n+1}a_{n+2}\dots a_{n+l-1}a_{n+l})^{K(l+1)}. \end{aligned}$$

Assuming

$$a_{n+1}^{K(l+1)} < q_n,\quad a_{n+2}^{K(l+1)} < q_{n+1},\quad \dots,\quad a_{n+l}^{K(l+1)} < q_{n+l-1}$$

leads to

$$(a_{n+1}a_{n+2}\dots a_{n+l})^{K(l+1)} < q_nq_{n+1}\cdots q_{n+l-1},$$

a contradiction. □

The announced theorem is now proved by appealing to Roth's theorem.

Thorem 4. *Let $\alpha = [0; a_1, a_2, a_3, \dots]$. Suppose there exists $l \in \mathbb{N}$, $K > 0$ and a strictly increasing sequence $(n_j)_{j\geq 1}$ of positive integers with the property $q_{n_j} \leq a_{n_j+l}^K$. Then α is transcendental.*

PROOF: As $q_{n_j} \leq a_{n_j+l}^K \leq a_{n_j+l}$ whenever $0 < K < 1$ we may restrict ourselves to $K \geq 1$. From $\lim_{j\to\infty} q_{n_j} = \infty$ we see that $2^{l(l-1)/2} \leq q_{n_j} \leq a_{n_j+l}^K$ for sufficiently large j. Thus, we may assume $2^{l(l_1)/2} \leq a_{n_j+l}^K$ for all $j \in \mathbb{N}$ without loss of generality. By virtue of Lemma 3 there exist infinitely many $m_j \in \mathbb{N}$ ($j = 1, 2, 3, \dots$) such that $a_{m_j+1}^{K(l+1)} \geq q_{m_j}$. The assertion is a consequence of

$$\left|\alpha - \frac{p_{m_j}}{q_{m_j}}\right| < \frac{1}{a_{m_j+1}q_{m_j}^2} \leq q_{m_j}^{-2-1/(Kl+K)}$$

and Theorem 1. □

Remarks and Further Results

1. The condition $q_{n_j} \leq a_{n_j+l}^K$ for a $K > 0$ and all $j \geq 1$ is just another way of expressing

$$\limsup_{n\to\infty} \frac{\log a_{n+l}}{\log q_n} > 0.$$

If α is an algebraic irrational then

$$\lim_{n\to\infty} \frac{\log a_{n+l}}{\log q_n} = \limsup_{n\to\infty} \frac{\log a_{n+l}}{\log q_n} = 0$$

for all $l \in \mathbb{N}$.

2. Let α be an algebraic irrational. Then for any positive integer l and any positive real number K there exist just finitely many indices n such that $a_{n+l}^K \geq q_n$. Using results of [3] or their sharpenings in [2] and [5] a bound for the number of n can be given.

Theorem 5. *Let α be an algebraic irrational of degree $\leq d$ and $K > 1$. Let $h(\alpha)$ denote the absolute height of α. The number of n, for which $a_{n+1}^K \geq q_n$ is satisfied, is bounded by*

$$\frac{1}{\log(1+1/K)} \log^+ \log h(\alpha) + 2 \cdot 10^5 \cdot K^5 (\log d)^2 \log(200K^2 \log d),$$

where $\log^+ x = \max\{\log x, 0\}$ *for* $x > 0$.

PROOF: Each such n renders a solution (p, q) of the inequality $|q\alpha - p| < q^{-1-1/K}$. The bound follows from Theorem 3 in [5]. □

Note that this is not the same height function as above. Whereas H is the field height as defined in [1], h denotes the absolute height as used in [2] and [5]. The bound δ_0 which is employed in Theorem 3 in [5] may be replaced by $\min\{1, 6/\sqrt{28}\} = 1$ by virtue of Theorem 2 in [2] and a remark to be found a few lines above.

Corollary 6. *Let α be an algebraic irrational of degree $\leq d$ and $K(l+1) > 1$. Let $h(\alpha)$ denote the absolute height of α. The number of n, for which $a_{n+l}^K \geq q_n$ is satisfied, is bounded by*

$$\frac{1}{\log(1+1/(Kl+K))} l \log^+ \log h(\alpha)$$
$$+2 \cdot 10^5 \cdot l(l+1)^5 K^5 (\log d)^2 \log(200K^2(l+1)^2 \log d).$$

PROOF: Each index n, for which $a_{n+l}^K \geq q_n$ is satisfied, gives a solution of the inequality $|q\alpha - p| < q^{-1-1/(Kl+K)}$ by virtue of Lemma 3. As above, a bound for this number follows from Theorem 3 in [5]. The number of n which are related to a pair (p, q) is bounded by l. □

3. Let p_{m_j}, q_{m_j} be as in the proof of Theorem 4. The second part of Theorem 1 yields that

$$\limsup_{j\to\infty} \frac{\log q_{m_j+1}}{\log q_{m_j}} < \infty$$

implies that α is a S- or T-number.

4. Among the numbers to which Theorem 4 applies are also Liouville-numbers and therefore U-numbers (choose e. g. $a_{n+1} \geq q_n^n$ for $n \geq 1$).

5. Recently J. L. Davison and J. O. Shallit [4] proved the transcendency of Cahen's constant C by exploring its continued fraction expansion. Cahen's constant can be defined as follows. Let $S_0 = 2$ and $S_{n+1} = S_n^2 - S_n + 1$ for $n \geq 0$ then

$$C = \sum_{j=0}^{\infty} \frac{(-1)^j}{S_j - 1}.$$

Let $a_0 = 0$, $a_1 = 1$ and $a_{n+2} = q_n^2$ for $n \geq 0$, then $C = [0;\, a_1,\, a_2\, a_3,\, \ldots]$ as was shown in [4]. Obviously $q_n \leq a_{n+2}$ and the transcendency of C is an immendiate consequence of Theorem 4. According to Lemma 3 at least one of the inequalities $a_{n+1}^3 \geq q_n$ and $a_{n+2}^3 \geq q_{n+1}$ holds for all n. As

$$\log q_{m_{j+1}} \leq \log q_{m_{j+2}} \leq \log 4 + 5 \log q_{m_j}$$

the number C is a S- or T-number.

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