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A Landesman-Lazer Type Condition and the Long Time Behaviour of Floating Plates

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Abstract. A dynamical plate theory model is shown to be dissipative in the sense of Levinson and eventually globally oscillatory

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(Dedicated to the memory of Svatopluk Fučík)

In the Kirchhoff model, the small transversal vibrations of a thin plate with the reference configuration $\Omega \subset \mathbb{R}^2$ are described by means of a function $u = u(x, t)$, $x = (x_1, x_2) \in \Omega$, $t \in \mathbb{R}^1$ satisfying the equation

$$\rho_s h u_{tt} + D\Delta^2 u = \mathcal{F}(t, u) \text{ on } \Omega \times \mathbb{R}^1 \tag{KE}$$

where ρ_s is the material density, h denotes the thickness, D the flexural rigidity and the operator $\mathcal{F}$ stands for external forces to be specified below.

We suppose that the plate is floating freely in a liquid so that there are no additional geometrical constraints for u along the boundary Γ of Ω. Hence any admissible solution has to comply with the natural boundary conditions

$$-\frac{\partial}{\partial \nu}\Delta u + (1-\sigma)\frac{\partial}{\partial \tau}\left[\frac{\partial^2 u}{\partial x_1^2}\nu_1\nu_2 - \frac{\partial^2 u}{\partial x_1 \partial x_2}(\nu_1^2 - \nu_2^2) - \frac{\partial^2 u}{\partial x_2^2}\nu_1\nu_2\right] = 0, \tag{B}$$

$$\sigma\Delta u + (1-\sigma)\frac{\partial^2 u}{\partial \nu^2} = 0 \text{ on } \Gamma \times \mathbb{R}^1$$

where $\sigma \in (0, \frac{1}{2})$ is the Poisson ratio and ν, τ are respectively the normal and tangent vector to Γ.

The external force density $\mathcal{F} = \mathcal{F}_g + \mathcal{F}_b + \mathcal{F}_f + \mathcal{F}_e$ results from the competition of the gravity component $\mathcal{F}_g$, the buoyancy component $\mathcal{F}_b$ and the external friction $\mathcal{F}_f$. The remaining part $\mathcal{F}_e$ represents an external load or inertial forces caused e.g. by waves on the liquid surface.

Denoting by g the gravity constant we have

$$\mathcal{F}_g = -g\rho_s h. \tag{P$_1$}$$

If the zero level of the vertical coordinate coincides with the liquid surface, we obtain

$$\mathcal{F}_b = g\rho_\ell \min\{h, (h/2 - u)^+\} \tag{P$_2$}$$

with ρ_ℓ the liquid density and $v^+ = \max\{v, 0\}$.

The external friction term is given by

$$\mathcal{F}_f = -d\Big(\int_\Omega u \,\mathrm{d}x\Big) u_t. \tag{P$_3$}$$

The rather awkward from of (P$_3$) should correspond to the obvious fact that the friction coefficient varies when passing from the air to the liquid. A typical situation is

$$d(z) \begin{cases} = d_2 & \text{for } z \le -z_0 < 0, \\ \in [d_1, d_2] & \text{for } z \in (-z_0, z_0), \\ = d_1 & \text{for } z \ge z_0 > 0 \end{cases} \tag{P$_4$}$$

where d_1, d_2 are strictly positive but generally distinct constants, and d is a Lipschitz continuous function.

The external forces are of the form

$$\mathcal{F}_e = \rho_s h q(x, t). \tag{P$_5$}$$

Finally, we determine the initial state

$$u(\cdot, t_0) = u_0, \quad u_t(\cdot, t_0) = u_1 \tag{I}$$

to obtain an evolution problem (KE), (B), (I) we shall deal with.

As we show, or rather recall, in Section 1, the problem is well posed and generates a process on an appropriate energy space. Consequently, the first and quite natural conjecture would be that the process is dissipative in the sense of Levinson because of the presence of the damping term $\mathcal{F}_f$. More specifically, any trajectory ends in a fixed bounded subset of the phase space regardless the size of the initial state.

From the more physical point of view, however, such a result calls for additional restrictions concerning the data. If any solution is to remain bounded for all times, we should have

$$\rho_\ell > \rho_s. \tag{P$_6$}$$

Moreover, the external force $\mathcal{F}_\ell$ is to be dominated by $\mathcal{F}_g$, $\mathcal{F}_b$ in order to eliminate large oscillations due to resonance phenomena. Analytically, it leads to a Landesman-Lazer type condition well known from the theory of boundary value problems (see Section 2, and also Fučík [3]).

Adopting the above stipulations we are able to prove that the process in question is dissipative (see Section 2, Theorem 1).

A rough statement of our ultimate goal is as follows: Once a solution is in the absorbing set, it oscillates around the rest position. To prove this, even more restrictions imposed on the function q are necessary (see Section 2, Theorem 2). Indeed, one easily imagines the situation when, for instance, q is positive and so is u for all times.

Our final remarks concern the reference material. To begin with, the model equation is probably the simplest one we could choose. A more detailed treatment of the dynamical plate theory may be found in Lagnese-Lions [5]. The underlying idea we adopt is that the problem is essentialy linear, both geometrically and physically. From this point of view, there seems to be no stumbling block to generalize the results to the equations containing the rotational inertia term along with internal damping etc. (cf. [5]).

The nonlinear theory (see e.g. Antman [1], Ball [2]), however, would require, vaguely speaking, a more sophisticated approach.

The one dimensional case of a floating beam has been treated by Lazer-Mc Kenna [7], [8]. Attacking the problem both analytically and numerically they obtained a lot of interesting results concerning the time-periodic solutions which are of particular relevance to Section 2 of the present paper.

Landesman-Lazer type problems have been discussed at length by many authors. Originated by the paper of Landesman-Lazer [6] there have appeared a considerable amount of literature, a complete list of which lies beyond the scope of our paper. Note, however, that a vast majority of authors addresses the boundary value problems whether evolutinary, i.e. the existence of periodic orbits to Hamiltonian systems, or stationary. In our context, the conditioon determines the asymptotic behaviour of an *evolution* problem.

1 Weak formulation and preliminary results

In what follows, all the physical parameters are supposed to be constant. To simplify the writing, we rescale the equation (KE) to a more concise form

$$u_{tt} + d\Big(\int_\Omega u \,\mathrm{d}x\Big)u_t + \Delta^2 u + f(u) = q \text{ on } \Omega \times \mathbb{R}^1. \tag{E}$$

In agreement with the hypotheses (P_1)–(P_5), we assume

$$d,\, f : \mathbb{R}^1 \to \mathbb{R}^1 \text{ are globally Lipschitz continuous,} \tag{A$_1$}$$

$$q \in C^1(\overline{\Omega} \times \mathbb{R}^1). \tag{A$_2$}$$

We introduce a bilinear form

$$b(u,\, v) = \int \frac{\partial^2 u}{\partial x_1^2}\Big(\frac{\partial^2 v}{\partial x_1^2} + \sigma \frac{\partial^2 v}{\partial x_2^2}\Big) + 2(1-\sigma)\frac{\partial^2 u}{\partial x_1 \partial x_2}\,\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2}\Big(\frac{\partial^2 v}{\partial x_2^2} + \sigma \frac{\partial^2 v}{\partial x_1^2}\Big)\,\mathrm{d}x.$$

The Green formula yields

$$b(u,\, v) = \int_\Omega \Delta^2 u v \,\mathrm{d}x$$

for u, v smooth, u satisfying (B).

Assume that the boundary Γ is Lipschitz. We construct a space $H^2(\Omega)$ via completion the set of all smooth functions on $\overline{\Omega}$ with respect to the norm

$$\|v\| = (b(v,\, v) + |v|^2)^{\frac{1}{2}}$$

where $|v|^2 = \int_\Omega v^2 \,\mathrm{d}x$ denotes the norm on $L_2(\Omega)$.

It is easy to observe that $H^2(\Omega)$ coincides with the standard Sobolev space of functions whose generalized derivatives up to the order two belong to $L_2(\Omega)$. As a consequence of the well known embedding theorems, we get

$$H^2(\Omega) \subset C(\overline{\Omega}). \tag{1.1}$$

As a next step, we define a self-adjoint operator

$$A : D(A) \subset L_2(\Omega) \to L_2(\Omega),$$

$$\begin{gathered} D(A) = \{v \mid v \in H^2(\Omega),\ \text{there is } \zeta \in L_2(\Omega) \text{ such that} \\ b(v,\, w) + \int_\Omega vw \,\mathrm{d}x = \int_\Omega \zeta w \,\mathrm{d}x \text{ for all } w \in H^2(\Omega)\}, \\ Av = \zeta - v \end{gathered} \tag{1.2}$$

with the null space

$$N(A) = \operatorname{span}\{1,\, x_1,\, x_2\}. \tag{1.3}$$

It is classical result of the linear semigroup theory that the solution operator to the abstract problem

$$u_{tt} + Au + u = 0, \quad u(0) = u_0, \quad u_t(0) = u_1 \tag{L}$$

generates a group $\{T_t\}$ of linear isometries on the enregy space $H^2(\Omega) \times L_2(\Omega)$, specifically,

$$\begin{gathered} (u(t),\, u_t(t)) = T_t(u_0,\, u_1), \quad t \in \mathbb{R}^1, \quad T_t \in \mathcal{L}(H^2 \times L_2), \\ |u_t(t)|^2 + \|u(t)\|^2 = \text{const for all } t \in \mathbb{R}^1 \end{gathered}$$

(see Lions-Magenes [9]).

By virtue of (A_1), (A_2), the problem (E), (B), (I) may be viewed as a semilinear Lipschitz perturbation of (L). Consequently, the variation-of-constants formula combined with the Banach fixed point theorem yields local existence and uniqueness. As the nonlinearities are globally Lipschitz, the Gronwall lemma guarantees that any local solution may be continued to solve the problem on $\mathbb{R}_1$.

Let us sum up what has been achieved.

Proposition 1. *Let the hypotheses* (A_1), (A_2) *hold. Then for any pair* $u_0 \in H^2(\Omega)$, $u_1 \in L_2(\Omega)$, *and any* $t_0 \in \mathbb{R}^1$ *there exists a unique solution* u,

$$u \in C(\mathbb{R}^1,\, H^2(\Omega)) \cap C^1(\mathbb{R}^1,\, L_2(\Omega)),$$

of the problem (E), (H), (I), *i. e.* u *satisfies* (I) *along with the integral identity*

$$\int_{-\infty}^{\infty}\int_{\Omega} -u_t\varphi_t + (d(\int_{\Omega} u\,\mathrm{d}x)u_t + f(u) - q)\varphi\,\mathrm{d}x + b(u,\,\varphi)\,\mathrm{d}t = 0 \tag{1.4}$$

for any test function $\varphi \in C^{\infty}(\overline{\Omega}\times \mathbb{R}^1)$ *with a compact support in* $\overline{\Omega}\times\mathbb{R}^1$.

2 Main results

To achieve our goal, further restrictions concerning the data are needed:

$$0 < d_1 \le d(z) \le d_2 < \infty \text{ for all } z \in \mathbb{R}^1, \tag{A$_3$}$$

$$|g(x,\,t)| \le c_1 \text{ for all } x,\,t, \tag{A$_4$}$$

$$-\infty < f_{-\infty} = \lim_{z\to-\infty} f(z) < 0 < \lim_{z\to\infty} f(z) = f_{\infty} < \infty \tag{A$_5$}$$

Note that (A_3) corresponds to (P_4) while (A_5) agrees with (P_6). Here (and always) the symbols c_i, $i = 1,\,2,\,\ldots$ stand for positive constants.

Next, it is convenient to have $f(0) = 0$. To this end, we shift u in the vertical direction as the case may be. Bearing the last agreement in mind we postulate

$$\text{there is a continuous function } k : [0,\,\infty) \to (0,\,\infty) \tag{A$_6$}$$

$$\text{such that}$$

$$f(z)z \ge k(|z|)z^2 \text{ for all } z \in \mathbb{R}^1.$$

Note that (A_6) is in full agreement with (P_1), (P_2) and that $k(z) \to 0$ as $z \to \infty$ by (A_5).

To conclude with, we postulate a Lndesman-Lazeı type condition

$$f_{-\infty} + \varepsilon \le \frac{1}{m(\Omega)}\int_{\Omega} q(x,\,t)\,\mathrm{d}x \le f_{\infty} - \varepsilon \quad (\varepsilon > 0) \text{ for all } t \tag{A$_7$}$$

where m denotes the 2-dimensional Lebegue measure.

The main difficulty encountered when looking for bounded solutins is that neither is the operator A coercive nor the nonlinearity f strong enough to prevent the system from possibly growing oscillations. Thus the most delicate question is to estimate the compnent of u belonging to the null space $N(A)$. To this end, it is desirable to restrict the class of admissible solutions to a set S such that

$$N(A) \cap S = \{c\}_{c\in\mathbb{R}^1} \text{ - the space of constants.} \tag{2.1}$$

The usual way to achieve (2.1) is to consider symmetric functions. Say that $\Omega = [-a,\,a] \times [-b,\,b]$ is a rectangle. We set

$$S = \{v \,|\, v(x_1,\,x_2) = v(-x_1,\,x_2) = v(x_1,\,-x_2) \text{ for all } x \in \Omega\}.$$

Then we have

$$A : D(A) \cap S \to L_2(\Omega) \cap S \tag{S_1}$$

and, what is more important,

$$\text{for any } v \in S, \text{ we have } f(v) \in S. \tag{S_2}$$

Finally, one easily observes that (2.1) holds.

Note that a similar result may be achieved when considering a circle in $\mathbb{R}^2$ and taking radially symmetric functions etc.

Having completed the preliminary discussion, we proceed to the statement of the main result.

Theorem 1. *Let the data satisfy the hypotheses* (A_1)–(A_7). *Suppose that the geometry of* Ω *admits the existence of a subspace* $S \subset L_2(\Omega)$ *such that* (S_1), (S_2) *along with (2.1) hold. Finally, let* q *satisfy*

$$q(\cdot, t) \in S \text{ for any } t \in \mathbb{R}^1. \tag{2.2}$$

Then there exists R *such that any solution* u *of* (E), (H), (I) *starting from the initial data* $u(\cdot, t_0) = u_0 \in H^2(\Omega) \cap S$, $u_t(\cdot, t_0) = u_1 \in L_2(\Omega) \cap S$ *satisfies*

$$\limsup_{t \to \infty} \|u(t)\| \le R, \quad \limsup_{t \to \infty} |u(t)| \le R. \tag{2.3}$$

Corollary 1. *The constant* R *may be chosen of the form*

$$R = c_2 \max_{x,\, t} |q(x, t)|. \tag{2.4}$$

Corollary 2. *Under the hypotheses of Theorem 1, any solution of the problem* (E), (B), (I) *with* $q = 0$ *tends to zero.*

Remark. If q is independent of t, Theorem 1 can be proved more easily using a Lyapunov function. In that case, no symmetry assumption are necessary.

Proof of Corollary 1: Once a solution enters the absorbing set, its supremum norm is bounded due to (1.1). According to (A_6), we have

$$f(u(t))u(t) \ge c_3 u^2(t)$$

for all t large enough. Consequently, the standard energy estimates for coercive systems imply (2.4). □

The proof of Theorem 1 will be postponed to Section 3. Our eventual goal is to obtain more information about the behavior of u for large times. We shall make further assumptions concerning the driving force q:

$$\left| \int_t^s \int_\Omega q(x, \tau)\, \mathrm{d}x\, \mathrm{d}\tau \right| \le c_4 \text{ for all } s, t \in \mathbb{R}^1, \tag{A_8}$$

$$\begin{array}{l} \text{there is a triple } K,\ \ell,\ \delta \text{ of strictly positive constants such that for any interval} \\ \quad I \subset \mathbb{R}^1,\ \text{length}\,(I) \geq \ell \text{ there are points} \\ \quad t_1^+,\ t_2^+,\ t_1^-,\ t_2^- \in I,\ t_2^+ - t_1^+ = t_2^- - t_1^- = \delta \text{ such that} \\ \quad \int_\Omega q(x,\, t)\, \mathrm{d}x \geq K \text{ for all } t \in [t_1^+,\, t_2^+], \\ \quad \int_\Omega q(x,\, t)\, \mathrm{d}x \leq K \text{ for all } t \in [t_1^-,\, t_2^-]. \end{array} \tag{A$_9$}$$

Remark. Observe that (A_8), (A_9) hold if, say, q is t-periodic with a period T, $q \not\equiv 0$ and

$$\int_0^T \int_\Omega q(x,\, \tau)\, \mathrm{d}x\, \mathrm{d}\tau = 0.$$

Theorem 2. *In addition to the hypotheses of Theorem 1 assume that q satisfies* (A_8), (A_9).

Then there exists a number $j > 0$ such that for any solution u of (E), (B), (I) *(with the initial value in S) there is a time T such that*

$$\begin{aligned} &m\{(x,\, t) \mid x \in \Omega,\, t \in J,\, u(x,\, t) > 0\} > 0, \\ &m\{(x,\, t) \mid x \in \Omega,\, t \in J,\, u(x,\, t) < 0\} > 0 \end{aligned} \tag{2.5}$$

for any interval $J \subset [T,\, \infty)$, length $(J) \geq j$.

The proof of Theorem 2 will be given in section 4.

3 The proof of Theorem 1

(A) Under the hypotheses of Theorem 1 we have

$$u(t) \in H^2(\Omega) \cap S, \quad u_t(t) \in L_2(\Omega) \cap S \text{for all } t \in \mathbb{R}^1.$$

Consider the orthogonal projection

$$P : L_2(\Omega) \cap S \to N(A) \cap S = \{c\}_{c \in \mathbb{R}^1}.$$

Any solution u may be decomposed as the sum

$$u(t) = v(t) \oplus w(t), \quad v = Pu$$

where v is, in fact, a scalar function of t and

$$b(w(t),\, w(t)) \geq \lambda \|w(t)\|^2, \quad \lambda > 0. \tag{3.1}$$

(B) The component w solves a linear problem

$$w_{tt} + d(t)w_t + Aw = r(t) \tag{3.2}$$

where, by virtue of (A_3),

$$0 < d_1 \le d(t) \le d_2 \text{ for all } t \tag{3.3}$$

and $r(t) = (Id - P)(q(t) - f(u(t)))$.

As the functions q, f are bounded, we deduce

$$|r(t)|^2 \le |q(t) - f(u(t))|^2 \le c_5 \text{ for all } t. \tag{3.4}$$

The point is that the linear operator A in (3.2) is coercive (cf. (3.1)) so that the standard technique of energy a priori estimates for damped hyperbolic problems may be used to obtain the estimate

$$\begin{aligned} \|w(t)\|^2 + |w_t(t)|^2 \le c_6[\exp(-c_7(t-t_0))(\|w(t_0)\|^2 + \\ + |w_t(t_0)|^2) + c_5] \text{ for all } t \ge t_0. \end{aligned} \tag{3.5}$$

The relation (3.5) can be obtained formally via multiplying (3.2) by $w_t + \varepsilon_1 w$, $\varepsilon_1 > 0$ small, and rigorously using a regularization technique (we refer to Lions-Magenes [9] or Haraux [4] for details).

Consequently, there is a time T, the magnitude of which depends solely on the norm of the initial data u_0, u_1 such that

$$\|w(t)\|^2 + |w_t(t)|^2 \le c_8 \text{ for all } t \ge T. \tag{3.6}$$

The constant c_8 is, of course, independent of the initial state. (C) The component v satisfies a scalar differential equation

$$v_{tt} + d(v)v_t + \int_\Omega f(w(x,\cdot) + v) - q(x,\cdot)\,\mathrm{d}x = 0 \tag{3.7}$$

where we should have written, strictly speaking, $\dot{v}$ instead of v_t. Using the standard regularity theorems one observes that v is classical solution so that the formal obstacles encountered in part (B) do not occure here. Note that w is continuous and the supremum of w is uniformly bounded as soon as time reaches T.

To begin with, observe that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}v_t^2 + d_1 v_t^2 \le c_9,$$

which leads immediately to the estimate

$$|v_t(t)| \le c_{10} \text{ for all } t \ge T. \tag{3.8}$$

To complete the proof of Theorem 1, we have to estimate $v(t)$.

By virtue of the Landesman-Lazer condition (A_7), the function

$$F(t,v) = \int_\Omega f(w(x,t) + v) - q(x,t)\,\mathrm{d}x$$

satisfies

$$F(t, v) \geq K_1 > 0, \quad F(t, -v) \leq -K_1 \text{ for all } t \geq T, v \geq \tilde{v}_0. \tag{3.9}$$

Neither $\tilde{v}_0$ nor K_1 depend on w.

Lemma 1. *There is a sequence $t_n \to \infty$ such that*

$$|v(t_n)| \leq \tilde{v}_0 \textit{ for } n = 1, 2, \ldots \tag{3.10}$$

PROOF OF LEMMA 1: Suppose, for instance, that

$$v(s) > \tilde{v}_0 \text{ for all } s \geq T_1 \geq T. \tag{3.11}$$

We integrate (3.7) to obtain

$$v_t(t) - v_t(s) + \hat{D}(v(t)) - \hat{D}(v(s)) + \int_s^t F(\tau, v(\tau))\,\mathrm{d}\tau = 0 \tag{3.12}$$

where $\frac{\mathrm{d}}{\mathrm{d}z}\hat{D}(z) = d(z)$.

Combining (3.3), (3.8) and (3.9) we deduce the estimate

$$d_1 v(t) \leq 2c_{10} - (t-s)K_1 + d_2|v(s)|$$

for any $t \geq s \geq T_1$ which contradicts to (3.11) for $t - s$ large.

In case $v(s) \leq -v_0$ for all large s, we get a contradiction in a similar way. □

Let $\tilde{s} \in [t_n, t_{n+1}]$ be a point where

$$v(\tilde{s}) = \max_{\tau \in [t_n, t_{n+1}]} |v(\tau)|.$$

Then either

$$|v(\tilde{s})| \leq \tilde{v}_0. \tag{3.13}$$

or

$$|v(\tilde{s})| > \tilde{v}_0, \quad \tilde{s} \in (t_n, t_{n+1}), \quad v_t(\tilde{s}) = 0. \tag{3.14}$$

Suppose that in the latter case $v(\tilde{s}) > \tilde{v}_0$. We pick up $s \in [t_n, \tilde{s})$ such that

$$v(s) = \tilde{v}_0, \quad v|_{[s, \tilde{s}]} \geq \tilde{v}_0. \tag{3.15}$$

Inserting $t = \tilde{s}$ in (3.12) and using (3.3), (3.8) and (3.9) we obtain

$$d_1 v(\tilde{s}) \leq c_{10} + d_2 v_0. \tag{3.16}$$

If $v(s) < -\tilde{v}_0$, the same arguments may be used to get

$$d_1 v(\tilde{s}) \geq -c_{10} - d_2 v_0. \tag{3.17}$$

Combining (3.10) together with (3.13), (3.16), (3.17) we obtain

$$|v(t)| \leq c_{11} \text{ for all } t \geq T. \tag{3.18}$$

Finally, the relations (3.6), (3.8), (3.18) complete the proof of Theorem 1.

4 The proof of Theorem 2

The conclusion of Theorem 1 gives a time T such that

$$\|u(t)\| \leq 2R, \quad |u_t(t)| \leq 2R \text{ for all } t \geq T. \tag{4.1}$$

Assume that

$$u \geq 0 \text{ a.e. on } \Omega \times J \tag{4.2}$$

where length $(J) = j$, $J \subset [T, \infty)$.

Similarly as in Section 3, we project the equation (E) onto $N(A)$ to obtain the scalar equation

$$v_{tt} + d(v)v_t + \int_\Omega f(u(x, t))\,\mathrm{d}x = \int_\Omega q(x, t)\,\mathrm{d}x \tag{4.3}$$

where the projection v satisfies

$$v \geq 0 \text{ on } J. \tag{4.4}$$

In what follows, we adopt to certain extent some ideas from the qualitative analysis of ordinary differential equations (see e.g. Reissig-Sansone-Conti [10]).

Integrating (4.3) for $s, t \in J$ we get

$$\begin{gathered} v_t(t) - v_t(s) + \hat{D}(v(t)) - \hat{D}(v(s)) + \\ + \int_s^t \int_\Omega f(u(x, \tau))\,\mathrm{d}x\,\mathrm{d}\tau = \int_s^t \int_\Omega q(x, \tau)\,\mathrm{d}x\,\mathrm{d}\tau. \end{gathered}$$

In accordance with (A_8), (4.1) everything is bounded and, consequently

$$\int_J \int_\Omega f(u(x, \tau))\,\mathrm{d}x\,\mathrm{d}\tau \leq c_{12}R + c_4 = c_{13}. \tag{4.5}$$

If j is large enough, there is a subinterval $I \subset J$, length $(I) = \ell$ (cf. (A_9)) such that

$$\int_I \int_\Omega f(u(x, \tau))\,\mathrm{d}x\,\mathrm{d}\tau \leq c_{13}\frac{\ell}{j}. \tag{4.6}$$

As a consequence of (1.4), (4.1) we have

$$\max_{x,\,\tau \geq T} |u(x, t)|, \quad \max_{t \geq T} |v(t)| \geq c_{14}. \tag{4.7}$$

As $u \geq 0$, we may use (A_6) to obtain

$$\begin{gathered} \int_I \int_\Omega f(u(x, \tau))\,\mathrm{d}x\,\mathrm{d}\tau \geq c_{15}|u(x, \tau)|\,\mathrm{d}x\,\mathrm{d}\tau \geq \\ \geq c_{15}/c_{14}|u(x, \tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau \geq c_{16}\int_I v^2(\tau)\,\mathrm{d}\tau. \end{gathered}$$

Thus we infer that

$$\int_I v^2(\tau)\,\mathrm{d}\tau \leq c_{17}\frac{\ell}{j}. \tag{4.8}$$

We recall that

$$|\frac{\mathrm{d}}{\mathrm{d}t}v^2(t)| = 2|v(t)v_t(t)| \leq c_{18} \text{ on } I. \tag{4.9}$$

Thanks to (4.9) we may estimate

$$\int_I v^2(\tau)\,\mathrm{d}\tau \geq \frac{1}{4c_{18}}(\max_I v^2)^2$$

and, consequently,

$$\max_{t\in J}|v(t)| \leq c_{19}(\frac{\ell}{j})^{\frac{1}{4}}. \tag{4.10}$$

Next, we use (4.3) to obtain

$$|v_{tt}| \leq c_{20} \text{ on } I. \tag{4.11}$$

By means of the Taylor expansion formula, we get

$$v(t) - v(s) = v_t(s)(t-s) + \frac{1}{2}(t-s)^2 v_{tt}(\xi), \quad \xi \in [s,\, t].$$

Thus the choice $(t-s) = (\frac{\ell}{j})^{\frac{1}{8}}$ along with (4.10) lead to

$$\max_{t\in I}|v_t(t)| \leq c_{21}(\frac{\ell}{j})^{\frac{1}{8}}. \tag{4.12}$$

To conclude with, we combine (4.3) with (4.10), (4.12) to obtain the relation

$$\max_{t\in I}|v_{tt}(t) - \int_\Omega q(x,\, t)\,\mathrm{d}x| \leq c_{22}(\frac{\ell}{j})^{\frac{1}{8}}. \tag{4.13}$$

To follows from (A_9) that

$$v_t(t_2^+) - v_t(t_1^+) = \int_{t_1^+}^{t_2^+} v_{tt}(\tau)\,\mathrm{d} \geq$$
$$-\delta c_{22}(\tfrac{\ell}{j})^{\frac{1}{8}} + \int_{t_1 nn+}^{t_2^+}\int_\Omega q(x,\, \tau)\,\mathrm{d}x\,\mathrm{d}\tau \geq K\delta - c_{22}(\tfrac{\ell}{j})^{\frac{1}{8}}.$$

Thus there is a point $s \in I$ such that

$$|v_t(s)| \geq \frac{1}{2}K\delta - \delta c_{22}(\frac{\ell}{j})^{\frac{1}{8}}. \tag{4.14}$$

Comparing (4.12), (4.14) we get

$$\frac{1}{2}K\delta \leq c_{23}(\frac{\ell}{j})^{\frac{1}{8}} \tag{4.15}$$

which yields a bound for the length j.

The case $u \leq 0$ in (4.2) may be treated in a similar fashion.

Theorem 2 has been proved.

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