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On the control net of certain multivariate spline functions

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Abstract. For univariate spline functions, control polygons are well-known. This paper presents a similar concept for the multivariate setting in case that the spline functions are linear combinations of the multivariate normalized B-splines due to Dahmen, Micchelli, and Seidel [5]. In addition, we show that the sequence of control nets converges uniformly to the function it represents as the underlying triangulation is uniformly refined.

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1 Introduction and Notations

It has been a quite old desire of numerical mathematicians to generalize the concept of splines and especially of the basis functions called B-splines to the multivariate setting. In 1992, Dahmen, Micchelli, and Seidel [5] presented a multivariate B-spline basis with all the desirable properties of the univariate one preserved, such as affine invariance, convex hull property, locality, partition of unity, and positivity (cf. [5, 10]) while it does not suffer from the regularity (and therefore inflexibility) of tensor product B-splines.

For the rest of the present section we will shortly introduce simplex splines, the fathers of the new B-splines, and the new B-splines themselves to the reader. In the next section, we define the multivariate analogue to univariate control polygons, the so-called control nets. In Section 3, we will show that if we successively refine the underlying triangulation in a uniform way, the respective control nets to a given multivariate spline function converge uniformly to the function they represent.

Now, first of all we give an exact definition of the term "triangulation":

Definition 1. (cf. [7]) A finite family T of sets $\tau \subset \mathbb{R}^s$, $\#\tau = s+1$, is called a *triangulation* of a set $D \subseteq \mathbb{R}^s$ if

(i) $\mathrm{vol}_s[\tau] > 0,\ \tau \in T,$

(ii) $\bigcup_{\tau\in T}[\tau] = D,$

(iii) $[\tau]\cap[\tau'] = [\tau\cap\tau']\ ,\quad \tau,\ \tau' \in T,$

with $[\tau]$ denoting the convex hull of the elements of $\tau \in T$.

For the rest of the paper we will assume D to be a compact, simply connected subset of $\mathbb{R}^s$, and that $n+1 := \#T < \infty$ holds. Moreover, we postulate that for D

there exists at least one triangulation, otherwise we approximate D by a domain for which a triangulation exists.

If we denote by ∂M the boundary and by $\mathrm{int}M$ the interior of a set $M \subset \mathbb{R}^s$, we can give the following

Definition 2. Let T be a triangulation of $D \subset \mathbb{R}^s$, $0 < \mathrm{vol}_s D < \infty$, with

$$T = \{\tau_{\mathbf{i}} = \{\boldsymbol{t}_{i_0}, \ldots, \boldsymbol{t}_{i_s}\} \mid \mathbf{i} \in J \subset \mathbb{N}_0^{s+1}\} .$$

To every vertex $\boldsymbol{t}_i$ of the triangulation we assign knots $\boldsymbol{t}_{i,0}, \ldots, \boldsymbol{t}_{i,k-1}$ (whereas k corresponds to the order of the polynomial spline functions to be represented) in the following way:

$$\begin{array}{lll} \boldsymbol{t}_{i,0} = \ldots = \boldsymbol{t}_{i,k-1} = \boldsymbol{t}_i & , & \text{if } \boldsymbol{t}_i \in \partial D \ , \\ \boldsymbol{t}_{i,0} = \boldsymbol{t}_i \text{ und } \boldsymbol{t}_{i,\nu} \in D \ , \quad \nu = 1, \ldots, k-1 & , & \text{if } \boldsymbol{t}_i \in \mathrm{int}(D) \ . \end{array}$$

In addition, with

$$\begin{aligned} \Delta_{\mathbf{i},\mathbf{b}} &:= [\boldsymbol{t}_{i_0,b_0}, \ldots, \boldsymbol{t}_{i_s,b_s}] \ , \quad \mathbf{i} \in J \ , \quad \mathbf{b} \in \{0, \ldots, k-1\}^{s+1} \ , \\ \Omega_{\mathbf{i},k} &:= \mathrm{int}\left(\bigcap_{0 \le |\mathbf{b}| \le k-1} \Delta_{\mathbf{i},\mathbf{b}} \right) \ , \end{aligned}$$

we require

$$\mathrm{vol}_s(\Omega_{\mathbf{i},k}) > 0 \ , \quad \mathbf{i} \in J \ . \tag{1}$$

(For vectors $\mathbf{b} \in \{0, \ldots, k-1\}^{s+1}$ we denote $|\mathbf{b}| := \|\mathbf{b}\|_1 := \sum_{i=0}^{s} |b_i|$.) Then, the set

$$K(T) := \{\boldsymbol{t}_{i,\nu} | \boldsymbol{t}_i \text{ is a vertex of } T, \nu = 0, \ldots, k-1\}$$

is called a *knot set* of order k of the triangulation T.

Remark. A simple knot set of a triangulation T is obtained by

$$\boldsymbol{t}_{i,0} = \ldots = \boldsymbol{t}_{i,k-1} = \boldsymbol{t}_i$$

for all vertices $\boldsymbol{t}_i$ of T. In this case, $\Delta_{\boldsymbol{i},\boldsymbol{b}} = [\tau_{\mathbf{i}}]$, $\mathbf{i} \in J$, $0 \le |\mathbf{b}| < k$, is true which implies $\Omega_{\boldsymbol{i},k} = [\tau_{\mathbf{i}}]$, $\mathbf{i} \in J$, and because of Definition 1 (i), equation (1) holds.

Over such a knot set we define the simplex spline via

Definition 3. *(cf. [1, 8])* The s-variate *simplex spline* $M(\mathbf{x}|W)$ *of order* k over the knot set $W = \{\boldsymbol{w}_0, \ldots, \boldsymbol{w}_{s+k-1}\} \subset \mathbb{R}^s$ is the particular continuous, nonnegative function which satisfies

$$\int_{\mathbb{R}^s} f(\mathbf{x}) M(\mathbf{x}|W)\, d\mathbf{x} = \\ (k-1)! \int_{S_{s+k-1}} f(\phi_0 \boldsymbol{w}_0 + \cdots + \phi_{s+k-1} \boldsymbol{w}_{s+k-1})\, d\phi_0 \cdots d\phi_{s+k-1} \tag{2}$$

for all $f \in L^1_{\text{loc}}(\mathbb{R}^s)$, the space of all locally Lebesgue integrable functions on $\mathbb{R}^s$. Here,

$$S_{s+k-1} = \{(\phi_0, \cdots, \phi_{s+k-1}) \mid \phi_\nu \geq 0,\ \nu = 0, \ldots, s+k-1,\ \sum_{\nu=0}^{s+k-1} \phi_\nu = 1\}$$

is the standard $(s+k-1)$ simplex. In case $\text{vol}_s(W) = 0$, the simplex spline is to be understood in the distributional sense. (In this paper, all knot sets W satisfy $\text{vol}_s(W) > 0$ because of (1)).

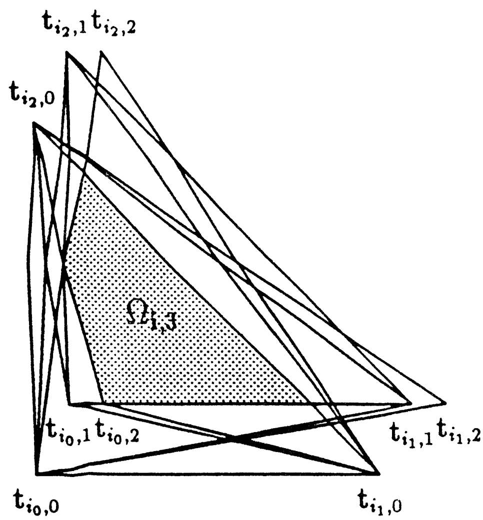

Fig. 1. $\Omega_{\mathbf{i},3}$ in $\mathbb{R}^2$

Remarks. a) The simplex spline $M(\mathbf{x}|W)$ is well-defined and unique.
b) For the classical univariate B-splines, a similar relation as (2) holds (cf. Curry and Schoenberg [4]).

Now, for an arbitrary set $W = \{\boldsymbol{w}_0, \ldots, \boldsymbol{w}_s\}$ of $s+1$ affinely independent points of $\mathbb{R}^s$ we denote

$$d(W) := \det \begin{pmatrix} 1 & \cdots & 1 \\ \boldsymbol{w}_0 & \cdots & \boldsymbol{w}_s \end{pmatrix},$$

and with this notation, we can finally introduce multivariate, normalized B-splines:

Definition 4. (cf. [5]) Let $\tau_{\mathbf{i}} = \{\boldsymbol{t}_{i_0}, \ldots, \boldsymbol{t}_{i_s}\}$ be an element of the triangulation T and let $K(T)$ be a knot set associated with T. We define

$$\begin{aligned} X_{\mathbf{i},\mathbf{b}} &:= \{\boldsymbol{t}_{i_j,b_j} | j = 0, \ldots, s\}, \quad |\boldsymbol{b}| = k-1, \\ V_{\mathbf{i},\mathbf{b}} &:= \{\boldsymbol{t}_{i_j,\nu} | \nu = 0, \ldots, b_j,\ \ j = 0, \ldots, s\}, \quad |\boldsymbol{b}| = k-1, \end{aligned}$$

(remember that (1) implies $\mathrm{vol}_s[X_{\mathbf{i},\mathbf{b}}] > 0$ for all $\mathbf{b}$ with $|\boldsymbol{b}| = k-1$). Then, the *normalized B-spline of order k* is defined by

$$N_{\boldsymbol{i},\boldsymbol{b},k}(\mathbf{x}) := |d(X_{\mathbf{i},\mathbf{b}})| \cdot M(\mathbf{x}|V_{\mathbf{i},\mathbf{b}}) \ , \quad |\boldsymbol{b}| = k-1 \ .$$

2 Definition of the control net

In this section, we present an anologue of the control polygon of univariate spline functions in B-spline representation for the multivariate setting in case that the spline functions are linear combinations of normalized B-splines as introduced in Definition 4.

Assume we are given some spline function f on a compact, simply connected domain D as well as a triangulation T of D and a knot set $K(T)$. Assume further that f has the representation

$$f = \sum_{\boldsymbol{i} \in J} \sum_{|\boldsymbol{b}|=k-1} d_{\boldsymbol{i},\boldsymbol{b}} N_{\boldsymbol{i},\boldsymbol{b},k} \ . \tag{3}$$

As in [9], we define the *Greville abscissae* $\xi_{\boldsymbol{i},\boldsymbol{b}}$ by

$$\xi_{\boldsymbol{i},\boldsymbol{b}} := \frac{1}{k-1} \sum_{j=0}^{s} \sum_{\nu=0}^{b_j-1} \boldsymbol{t}_{i_j,\nu} \ .$$

Now, as in the univariate case, the control net is a piecewise linear interpolant of the control points $\binom{\xi_{\boldsymbol{i},\boldsymbol{b}}}{d_{\boldsymbol{i},\boldsymbol{b}}}$. But, in contrast to the univariate case, such an interpolant is not unique. It depends on the triangulation of the abscissae (which are the Greville abscissae in the present situation).

Here, we give a triangulation of the Greville abscissae which is similar to the canonical triangulation of the abscissae of the Bézier points in the case of Bernstein-Bézier segments (cf. [2]). To this end, we define the set

$$\bar{T} := \left\{ \{\xi_{\boldsymbol{i},\boldsymbol{b}-\boldsymbol{e}_j+\boldsymbol{e}_0}, \ldots, \xi_{\boldsymbol{i},\boldsymbol{b}-\boldsymbol{e}_j+\boldsymbol{e}_s}\} \ \mid \ b_j > 0, \ \boldsymbol{i} \in J, \ |\boldsymbol{b}| = k-1 \right\}$$

and denote by $\tilde{T}$ the set of *different* sets in $\bar{T}$. As it is easily verified, $\tilde{T}$ is a triangulation of D whenever (1) is true. In the special case

$$\boldsymbol{t}_{i,j} = \boldsymbol{t}_i \ , \quad j = 0, \ldots, k-1 \ , \tag{4}$$

for all vertices $\boldsymbol{t}_i$ of T (then, the normalized B-splines are exactly the Bernstein polynomials over the respective simplices), $\tilde{T}$ is exactly the canonical triangulation we mentioned above (cf. Fig. 3).

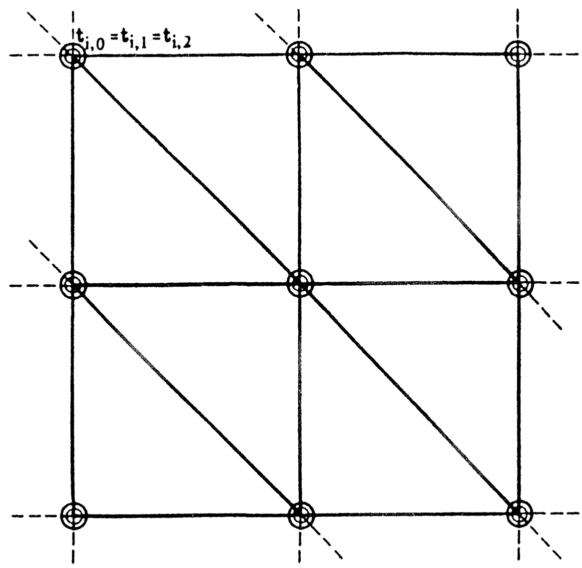

Fig. 2. Part of a triangulation T of D.

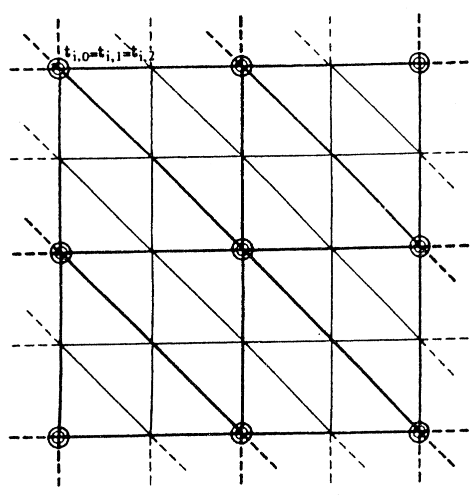

Fig. 3. The same part of D as in Fig. 2, together with the triangulation for the third order Bézier control net (thinner lines)

In addition, the Bernstein polynomial $b_{i,b} = N_{i,b,k}$ over a particular simplex $[\tau_i]$

of T reaches its maximum in $\xi_{\boldsymbol{i},\boldsymbol{b}}$.

In general, the topology of $\tilde{T}$ for all knot sets satisfying (1) is the same as in the special case of (4). An example for this is shown in Figures 3 and 4.

Over $\tilde{T}$ we now define hat functions which are exactly the normalized second order B-splines $\tilde{N}_{\boldsymbol{i},\boldsymbol{b},2}$. For these B-splines

$$\tilde{N}_{\boldsymbol{i},\boldsymbol{b},2}(\xi_{\mathbf{j},\mathbf{g}}) = \delta_{\mathbf{ij}} \cdot \delta_{\mathbf{bg}} ,$$

is true. Here, the generalized Kronecker symbol $\delta_{\mathbf{ij}}$ for integer vectors $\mathbf{i}$, $\mathbf{j} \in \mathbb{N}_0^{s+1}$, $\mathbf{i} = (i_0, \ldots, i_s)^T$, $\mathbf{j} = (j_0, \ldots, j_s)^T$, is given by

$$\delta_{\mathbf{ij}} := \prod_{\nu=0}^{s} \delta_{i_\nu j_\nu} .$$

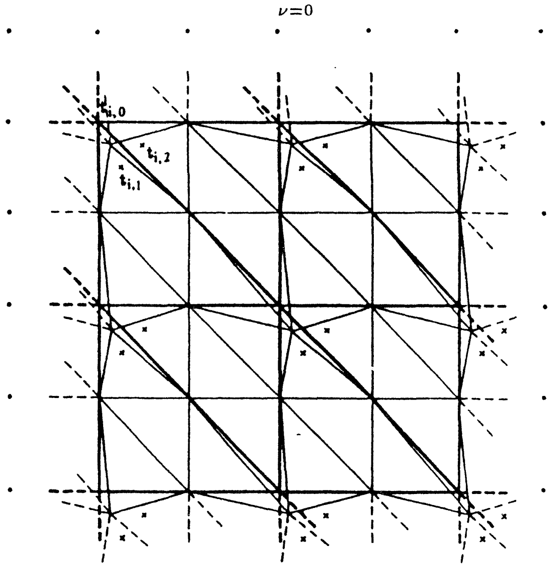

Fig. 4. The same part of D as in Fig. 2, together with the triangulation $\tilde{T}$ for the control net of a space of third order spline functions (thinner lines)

Now, if some spline function f over T has the representation (3), then the operator L,

$$Lf := \sum_{\mathbf{i}\in J} \sum_{|\mathbf{b}|=k-1} d_{\mathbf{i},\mathbf{b}} \tilde{N}_{\boldsymbol{i},\boldsymbol{b},2}$$

maps f onto its control net with respect to the knot set $K(T)$.

Remark. Inspecting Figures 5 and 6 in Fong and Seidel [6] indicates that there, a similar operator to L must have been used by the authors in order to visualize a control net for multivariate spline functions. But neither the operator nor the control net have been explicitely described there.

3 Convergence of the control net

In order to inquire the convergence properties of the multivariate control nets introduced in the previous section we start with a sequence of triangulations $\{T_l\}_{l=0}^{\infty}$,

$$T_l = \{\tau_{\mathbf{i}}^l = \{t_{i_0}^l, \ldots, t_{i_s}^l\} \mid \mathbf{i} \in J_l \subset \mathbb{N}_0^{s+1}\} ,$$

of the compact, simply connected subset D of $\mathbb{R}^s$, where each triangulation is a refinement of its predecessor. Moreover, with the number $n_l + 1$ of vertices in T_l as well as

$$h_l := \max_{\mathbf{i}\in J_l} \operatorname{diam}[\tau_{\mathbf{i}}^l] \quad \text{and} \quad \phi_l := \min_{\substack{0\le i,j\le n_l \\ i\neq j}} \|t_i^l - t_j^l\|_2 , \quad l \in \mathbb{N}_0 ,$$

we postulate the following:

$$\lim_{l\to\infty} h_l = 0 \quad \text{and} \quad \frac{\phi_l}{h_l} \geq q \quad \text{with} \quad 0 < q \leq 1 . \tag{5}$$

(5) is sort of a equilibrium postulation: With this condition, it cannot happen that the refinement of the triangulations proceeds at different speeds in different areas of D.

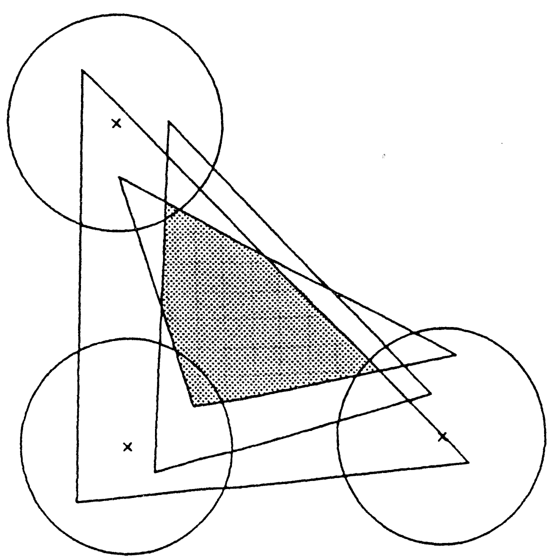

Fig. 5. Illustration of equation (6).

To every triangulation T_l we assign some knot set $K(T_l)$. The choice for the knot sets $K(T_l)$ is arbitrary as far as Definition 2 is fulfilled and f is in the respective spline space. In addition, we postulate

$$||\boldsymbol{t}_{i,j}^l - \boldsymbol{t}_i^l||_2 \leq \frac{1}{3} \cdot \phi_l \ , \quad j = 0, \ldots, k-1 \ , \quad i = 0, \ldots, n_l \ , \tag{6}$$

with ϕ_l as in (5). (NB: (6) does *not* imply (1) although Fig. 5 seems to propose that).

In preparation of Theorem 5 we define the operators L_l for T_l and $K(T_l)$ by

$$L_l f := \sum_{\mathbf{i} \in J_l} \sum_{|\mathbf{b}|=k-1} d_{\mathbf{i},\mathbf{b}}^l \tilde{N}_{\boldsymbol{i},\boldsymbol{b},2}^l \tag{7}$$

for f with the B-spline representation

$$f = \sum_{\boldsymbol{i} \in J_l} \sum_{|\boldsymbol{b}|=k-1} d_{\boldsymbol{i},\boldsymbol{b}}^l N_{\boldsymbol{i},\boldsymbol{b},k}^l \tag{8}$$

(cp. Section 2) as well as another class of (obviously linear) operators Q_l, $l \in \mathbb{N}_0$, by

$$Q_l f := \sum_{\mathbf{i} \in J_l} \sum_{|\mathbf{b}|=k-1} f(\xi_{\boldsymbol{i},\boldsymbol{b}}^l) N_{\boldsymbol{i},\boldsymbol{b},k}^l \ .$$

The latter are the analogues of the classical variation diminishing quasiinterpolants due to Schoenberg.

The composition $L_l Q_l$ maps the spline function f onto the piecewise linear function $L_l Q_l f$ over $\tilde{T}_l$ which interpolates f at the Greville abscissae $\xi_{\boldsymbol{i},\boldsymbol{b}}^l$, $\mathbf{i} \in J_l$, $|\boldsymbol{b}| = k-1$.

With these operators we can now state and prove our central theorem on the convergence of control nets. It is similar to a result of Cohen and Schumaker [3] for the control polygons of univariate spline functions.

Theorem 5. *For $f \in C^2(D)$ with the representations*

$$f = \sum_{\mathbf{i} \in J_l} \sum_{|\mathbf{b}|=k-1} d_{\mathbf{i},\mathbf{b}}^l N_{\boldsymbol{i},\boldsymbol{b},k}^l$$

over the triangulations T_l with respect to the knot sets $K(T_l)$ the following is true:

$$||f - L_l f||_{\infty,D} \leq C \cdot h_l^2 \cdot \max_{|\mathbf{a}|=2} ||D^{\mathbf{a}} f||_{\infty,D} \ . \tag{9}$$

Here, we have $C = const. > 0$, $h_l := \max_{\tau \in T_l} \operatorname{diam}[\tau]$ and $D^{\mathbf{a}} := \dfrac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \dfrac{\partial^{a_s}}{\partial x_s^{a_s}}$, $\mathbf{a} \in \mathbb{N}_0^s$, the usual multiindex notation for partial derivatives in s variables.

Remark. Since we take T_l to be a refined triangulation with respect to T_{l-1}, $l = 1, 2, \ldots$, (9) is true for all functions f that can be represented in the following form:

$$f = \sum_{\mathbf{i} \in J_0} \sum_{|\mathbf{b}|=k-1} d_{\mathbf{i},\mathbf{b}}^0 N_{\boldsymbol{i},\boldsymbol{b},k}^0 \ ,$$

i.e., (9) is true for all f that are linear combinations of normalized B-splines over the original triangulation, together with the original knot set $K(T_0)$.

PROOF: We start with the same standard trick as in [3], here in the multivariate setting:

$$||f - L_l f||_{\infty,D} \le ||f - L_l Q_l f||_{\infty,D} + ||L_l Q_l f - L_l f||_{\infty,D} \ . \tag{10}$$

Now, from Lemma 6 below (all Lemmata that we use here will be given below), we know that the operators L_l are continuous and bounded by $\frac{1}{\gamma_{s,k}}$ where $\gamma_{s,k}$ is a constant that depends only on the number s of independent variables and the order k of the spline spaces we use. So, equation (10) becomes

$$||f - L_l f||_{\infty,D} \le ||f - L_l Q_l f||_{\infty,D} + \frac{1}{\gamma_{s,k}} ||Q_l f - f||_{\infty,D} \ . \tag{11}$$

Since in [5], Section 4, the relation

$$||f - Q_l f||_{\infty,D} \le C_0 \cdot h_l^2 \cdot \max_{|\mathbf{a}|=2} ||D^{\mathbf{a}} f||_{\infty,D} \tag{12}$$

is shown (with $h_l := \max_{\tau \in T_l} \operatorname{diam}[\tau]$), we only have to prove sort of an $O(h_l^2)$ asymptotics for

$$||f - L_l Q_l f||_{\infty,D} \ .$$

In order to do so, for an arbitrary $\mathbf{x} \in D$ with $\mathbf{x} \in [\tilde{\tau}]$, $\tilde{\tau} \in \tilde{T}_l$, we choose some fixed $\tilde{\boldsymbol{x}}$ within the same simplex $\tilde{\tau}$. By Taylor's Theorem (cf. [11]) for the same representation as here) we get

$$f(\mathbf{x}) = f(\tilde{\boldsymbol{x}}) + f'(\tilde{\boldsymbol{x}}) \cdot (\mathbf{x} - \tilde{\boldsymbol{x}}) + \sum_{|\mathbf{a}|=2} \frac{1}{\mathbf{a}!} D^{\mathbf{a}} f\left(\tilde{\boldsymbol{x}} + \theta_{\mathbf{x},\tilde{\boldsymbol{x}}}(\mathbf{x} - \tilde{\boldsymbol{x}})\right) \cdot (\mathbf{x} - \tilde{\boldsymbol{x}})^{\mathbf{a}} \tag{13}$$

where $f'(\tilde{\boldsymbol{x}}) = \operatorname{grad} f(\tilde{\boldsymbol{x}})$ and $\theta_{\mathbf{x},\tilde{\boldsymbol{x}}} \in (0,1)$.

Since linear functions get reproduced by the linear operators $L_l Q_l$ (cf. Lemma 7), (13) furnishes

$$f(\mathbf{x}) - L_l Q_l f(\mathbf{x}) = (I - L_l Q_l) \left(\sum_{|\mathbf{a}|=2} \frac{1}{\mathbf{a}!} D^{\mathbf{a}} f\left(\tilde{\boldsymbol{x}} + \theta_{\bullet,\tilde{\boldsymbol{x}}}(\bullet - \tilde{\boldsymbol{x}})\right) \cdot (\bullet - \tilde{\boldsymbol{x}})^{\mathbf{a}} \right) (\mathbf{x}) \ ,$$

where I is the identity on $C^2(D)$. Now, using Lemma 7 again, we get

$$\begin{aligned} ||f - L_l Q_l f||_{\infty,[\tilde{\tau}]} &= \left\| (I - L_l Q_l) \left(\sum_{|\mathbf{a}|=2} \frac{1}{\mathbf{a}!} D^{\mathbf{a}} f\left(\tilde{\boldsymbol{x}} + \theta_{\bullet,\tilde{\boldsymbol{x}}}(\bullet - \tilde{\boldsymbol{x}})\right) \cdot (\bullet - \tilde{\boldsymbol{x}})^{\mathbf{a}} \right) \right\|_{\infty,[\tilde{\tau}]} \\ &\le C_1 \cdot ||(I - L_l Q_l)||_{\infty,D} \cdot \tilde{h}_l^2 \cdot \max_{|\mathbf{a}|=2} ||D^{\mathbf{a}} f||_{\infty,D} \\ &\le C_2 \cdot \tilde{h}_l^2 \cdot \max_{|\mathbf{a}|=2} ||D^{\mathbf{a}} f||_{\infty,D} \ . \end{aligned}$$

Here, we set $\tilde{h}_l := \max_{\tilde{\tau}\in\tilde{T}_l} \operatorname{diam}[\tilde{\tau}]$. Note that neither C_1 nor C_2 depend on l !

Since $\tilde{h}$ and h have the same asymptotic behaviour as can be seen from Lemma 8, we obtain

$$\|f - L_l Q_l f\|_{\infty,[\tilde{\tau}]} \leq C_3 \cdot h_l^2 \cdot \max_{|\mathbf{a}|=2} \|D^{\mathbf{a}} f\|_{\infty,D} \ . \tag{14}$$

And since the right hand side of (14) does not depend on our special choice for $\mathbf{x}$ and therefore $\tilde{\tau}$, we can take the supremum norm over D on the left hand side and at the end, we have

$$\|f - L_l Q_l f\|_{\infty,D} \leq C_3 \cdot h_l^2 \cdot \max_{|\mathbf{a}|=2} \|D^{\mathbf{a}} f\|_{\infty,D} \ .$$

Together with (11) and (12), this proves our theorem. □

Lemma 6. *The operators L_l are linear, continuous, and bounded. Moreover, the following relation is true:*

$$\|L_l\|_{\infty,D} \leq \frac{1}{\gamma_{s,k}} \ . \tag{15}$$

Here, the constant $\gamma_{s,k}$ only depends on the number s of independent variables and on the order k of the spline space under scope.

PROOF: Each of the operators L_l is an isomorphism between finite dimensional vector spaces as can be seen from their definition (cf. (7) and (8)). Knowing this, the linearity, continuity, and boundedness are obvious.

In [5], Theorem 4.2, the authors show that for any sequence $\mathbf{c} = \{c_{\boldsymbol{i},\boldsymbol{b}}\}_{\mathbf{i}\in J,|\mathbf{b}|=k-1}$ the inequation

$$\gamma_{s,k}\|\mathbf{c}\|_\infty \leq \|\sum_{\mathbf{i}\in J} \sum_{|\mathbf{b}|=k-1} c_{\mathbf{i},\mathbf{b}} N_{\boldsymbol{i},\boldsymbol{b},k}\|_{\infty,D} \leq \|\mathbf{c}\|_\infty \tag{16}$$

is true. Here, the maximum norm $\|\mathbf{c}\|_\infty$ of a vector $\mathbf{c} = \{c_{\mathbf{i},\mathbf{b}}\}_{\mathbf{i}\in J,|\mathbf{b}|=k-1}$ is given as usual by

$$\|\mathbf{c}\|_\infty := \max_{\mathbf{i}\in J,|\boldsymbol{b}|=k-1} |c_{\boldsymbol{i},\boldsymbol{b}}| \ .$$

By inequality (16) we see the following:

$$\begin{aligned}
\|L_l f\|_{\infty,D} &= \|\sum_{\mathbf{i}\in J_l} \sum_{|\mathbf{b}|=k-1} d^l_{\mathbf{i},\mathbf{b}} \tilde{N}^l_{\boldsymbol{i},\boldsymbol{b},2}\|_{\infty,D} \\
&\overset{(16)}{\leq} \|\{d^l_{\mathbf{i},\mathbf{b}}\}_{\mathbf{i}\in J_l,|\mathbf{b}|=k-1}\|_\infty \\
&\overset{(16)}{\leq} \frac{1}{\gamma_{s,k}} \cdot \|\sum_{\mathbf{i}\in J_l} \sum_{|\mathbf{b}|=k-1} d^l_{\mathbf{i},\mathbf{b}} N^l_{\boldsymbol{i},\boldsymbol{b},k}\|_{\infty,D} \\
&= \frac{1}{\gamma_{s,k}} \cdot \|f\|_{\infty,D} \ .
\end{aligned}$$

From this we easily conclude (15). □

Lemma 7. *The operators $L_l Q_l$ are linear, continuous, and bounded. They satisfy the relation*

$$L_l Q_l f = f \tag{17}$$

for every linear function f as well as

$$\|L_l Q_l\|_{\infty,D} = 1 \ . \tag{18}$$

PROOF: As in Lemma 6 above, each of the operators $L_l Q_l$ is an isomorphism between finite dimensional vector spaces and therefore the linearity, continuity, and boundedness are obvious again.

In [5], the relation

$$Q_l f = f$$

for every linear function is shown, and since $L_l f$ is the (unique) piecewise linear interpolant of f over $\tilde{T}_l$, we have

$$L_l Q_l f = L_l f = f$$

for all linear functions f and for all $l \in \mathbb{N}_0$.

Equation (18) can be realized as follows:

$$\begin{aligned} \|L_l Q_l f\|_{\infty,D} &= \|\sum_{\mathbf{i}\in J_l} \sum_{|\mathbf{b}|=k-1} f(\xi^l_{\boldsymbol{i},\boldsymbol{b}}) \tilde{N}^l_{\boldsymbol{i},\boldsymbol{b},2}\|_{\infty,D} \\ &\overset{(16)}{\leq} \|\{f(\xi^l_{\boldsymbol{i},\boldsymbol{b}})\}_{\mathbf{i}\in J_l, |\mathbf{b}|=k-1}\|_\infty \\ &\leq \|f\|_{\infty,D} \ , \end{aligned}$$

i.e. $\|L_l Q_l\|_{\infty,D} \leq 1$ for all $l \in \mathbb{N}_0$. With the test function f, $f \equiv 1$ on D, we get

$$\|L_l Q_l f\|_{\infty,D} = \|f\|_{\infty,D}$$

because of (17), and so $\|L_l Q_l\|_{\infty,D} = 1$, $l \in \mathbb{N}_0$, is true. □

Lemma 8. *For the quantities h_l and $\tilde{h}_l$ as in the proof of Theorem 5 and q from (5) we have the following:*

$$\frac{1}{3} \cdot q \cdot h_l \leq \tilde{h}_l \leq \frac{5}{3} \cdot h_l \ , \quad l \in \mathbb{N}_0 \ . \tag{19}$$

PROOF: For any simplex $\tilde{\tau}_l \in \tilde{T}_l$, $\tilde{\tau}_l = \{\xi^l_{\mathbf{i},\mathbf{b}-e_j+e_0}, \dots, \xi^l_{\mathbf{i},\mathbf{b}-e_j+e_s}\}$, we have

$$\begin{aligned} \operatorname{diam}[\tilde{\tau}_l] &= \max_{0\leq\mu,\nu\leq s} \|\xi^l_{\mathbf{i},\mathbf{b}-e_j+e_\mu} - \xi^l_{\mathbf{i},\mathbf{b}-e_j+e_\nu}\}\|_2 \\ &= \max_{\substack{0\leq\mu,\nu\leq s \\ \mu\neq\nu}} \|\boldsymbol{t}^l_{i_\mu,b_\mu} - \boldsymbol{t}^l_{i_\nu,b_\nu}\|_2 \ , \quad |\boldsymbol{b}| = k-1 \ , \quad l \in \mathbb{N}_0 \ . \end{aligned}$$

Now, it is easily seen by (6) that

$$\min_{\substack{0\le\mu,\nu\le s\\ \mu\ne\nu}} \|t^l_{i_\mu} - t^l_{i_\nu}\|_2 - \frac{2}{3}\phi_l \le \mathrm{diam}[\tilde{\tau}_l] \le \max_{\substack{0\le\mu,\nu\le s\\ \mu\ne\nu}} \|t^l_{i_\mu} - t^l_{i_\nu}\|_2 + \frac{2}{3}\phi_l\ , \quad l \in \mathbb{N}_0\ ,$$

is true. This implies

$$\frac{1}{3}\phi_l \le \max_{\tilde{\tau}_l\in\tilde{T}_l} \mathrm{diam}[\tilde{\tau}_l] = \tilde{h}_l \le h_l + \frac{2}{3}\phi_l\ , \quad l \in \mathbb{N}_0\ ,$$

and finally with (5) and $\phi_l \le h_l$ we obtain (19). □

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