Kenneth K. Nwabueze
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Certain Embedding of the Burnside ring into its Ghost ring

K. K. NWABUEZE

Abstract.

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1 Introduction

Let G be a finite group. Consider $\mathcal{U}$, a set of subgroups of G which is closed with respect to intersection and jonjugation and such that $G \in \mathcal{U}$. We define a $(G, \mathcal{U})$-set as a finite left G-set S with, $G_S =: \{g \in G \,|\, gs = s\} \in \mathcal{U}$, for all s in S. Our condition on $\mathcal{U}$ imply that for any $U \in \mathcal{U}$ the set, $G/U = \{gU \,|\, g \in G\}$, of left cosets of U in G is a $(G, \mathcal{U})$*-set*, the G-action on G/U is defined by left multiplication, $G \times G/U \to G/U : (h, gU) \to ghU$, and that for any two G-sets, S_1 and S_2, the G-sets $S_1 \times S_2$ and $S_1 \cup S_2$ are $(G, \mathcal{U})$-sets. We observe that the isomorphism classes of $(G, \mathcal{U})$-sets form a commutative halfring $\Omega^+(G, \mathcal{U})$, because one has the obvious natural isomorphisms;

$$
\begin{gathered}
S_1 \cup S_2 \cong S_2 \cup S_1 \\
(S_1 \cup S_2) \cup S_3 \cong S_1 \cup (S_2 \cup S_3) \\
S_1 \times S_2 \cong S_2 \times S_1 \\
(S_1 \times S_2) \times S_3 \cong S_1 \times (S_2 \times S_3) \\
S_1 \times (S_2 \cup S_3) \cong (S_1 \times S_2) \cup (S_1 \times S_3).
\end{gathered}
$$

Furthermore $1 \in \Omega^+(G, \mathcal{U})$ exists, namely G/G.

Thus a map from the set $\Omega^+(G, \mathcal{U})$ of isomorphism classes of $(G, \mathcal{U})$-sets into a ring R which commutes with sums and products and sends $1 \in \Omega^+(G, \mathcal{U})$ onto 1_R, is nothing else than a homomorphism from the halfring $\Omega^+(G, \mathcal{U})$ into R and this factors uniquely through the universal ring associated to $\Omega^+(G, \mathcal{U})$, *the Burnside rings* $\Omega(G, \mathcal{U})$ *of* G *with respect to* $\mathcal{U}$. We note that if we assume $\mathcal{U}$ to be the set $\mathcal{S}(G)$ of all subgroups of G, then $\Omega(G, \mathcal{U})$ coincides with the usual Burnside ring $\Omega(G)$ of G, constructed from the halfring $\Omega^+(G)$ of isomorphism classes of all finite G-sets.

The following facts are more or less obvious(see [1]):

Theorem 1.1.

1. *$\Omega(G, \mathcal{U})$ is generated freely as an additive group by the isomorphism classes of transitive $(G, \mathcal{U})$-sets, i. e. of G-sets of the form G/U with $U \in \mathcal{U}$: so its rank equals the number $k = k_{\mathcal{U}}$ of G-conjugacy classes of subgroups in $\mathcal{U}$.*

2. *For any subgroup $V \leq G$ of G, whether in $\mathcal{U}$ or not, the mapping,*
$$\chi_V \;:\; S \mapsto \#\{s \in S \,|\, V \leq G_s\}$$
which associates with any $(G, \mathcal{U})$-set S the number of elements in,
$$S^V := \{s \in S \,|\, V \leq G_s\},$$
the set of V-invariant elements in S, induces a homomorphism — also denoted by χ_V or, more precisely $\chi_V^{\mathcal{U}}$ — from $\Omega(G, \mathcal{U})$ into Z.

3. *Any homomorphism from $\Omega(G, \mathcal{U})$ into Z takes the form as in (2).*

4. *For V, $W \leq G$ one has, $\chi_V^{\mathcal{U}} = \chi_W^{\mathcal{U}}$ if and only if, $\overline{V} := \cap_{V \leq U \in \mathcal{U}} U$, is conjugate to, $\overline{W} := \cap_{W \leq U \in \mathcal{U}} U$. So, in view of $\overline{V} \in \mathcal{U}$ for all $V \in \mathcal{S}(G)$ and $V \in \mathcal{U}$ if and only if $V = \overline{V}$, one has $k = k_{\mathcal{U}}$ different homomorphisms from $\Omega(G, \mathcal{U})$ into Z which — after choosing a system, $\mathcal{U}' = \{U_1, U_2, \ldots, U_k\}$, of representatives of conjugacy classes of subgroups of $\mathcal{U}$ with, $|U_1| \geq |U_2| \geq \ldots \geq |U_k|$, — may be denoted by, $\chi_1 = \chi_{U_1}$, $\chi_2 = \chi_{U_2}$, $\ldots$, $\chi_k = \chi_{U_k}$.*

Definition and Theorem 1.2. *The product map,*
$$\chi := \prod_{i=1}^{k} \chi_i \;:\; \Omega(G, \mathcal{U}) \to \prod_{i=1}^{k} Z,$$
of k different homomorphisms from $\Omega(G, \mathcal{U})$ into Z is injective and maps $\Omega(G, \mathcal{U})$ onto a subring of finite index, $\prod_{i=1}^{k}(N_G(U_i) \,:\, U_i)$ of $\prod_{i=1}^{k} Z$, — this way identifying $\prod_{i=1}^{k} Z$ with the integral closure $\tilde{\Omega}(G, \mathcal{U})$ of $\Omega(G, \mathcal{U})$ in its total quotient ring, $\tilde{\Omega}(G, \mathcal{U}) \cong Q \otimes_Z \Omega(G, \mathcal{U}) \cong \prod_{i=1}^{k} Q$. The product $\prod_{i=1}^{k} Z$ is called the Ghost ring of G (see [1]).

We note from above that for, $\Omega(G, \mathcal{U}) = \Omega(G)$, and for every subgroup U of G there exists a canonical homomorphism, $\chi_{\mathcal{U}}(X) := \#X^U$, of its subset, $X^U = \{x \in X \,|\, ux = x \text{ for all } u \in U\}$, of U invariant elements, — in particular, $\chi_1(X) = \#X$ if $\mathbf{1} = \{\mathbf{1}_G\}$, denotes the trivial subgroup of G.

One also has that, $\chi_U = \chi_V$, if and only if, $U \overset{G}{\sim} V$, for $U, V \leq G$, where $U \overset{G}{\sim} V$ denotes that U and V are G-conjugate and, $\chi_U(X) = \chi_U(X')$ for all $U \leq G$, if and only if, $X = X'$ for X, $X' \in \Omega(G)$. So identifying each $X \in \Omega(G)$ with the associated map, $U \to \chi_U(X)$, from the set $\mathcal{S}(G)$ of all subgroups of G into Z, also denoted by X, we can consider $\Omega(G)$ in a canonical way as a subring of the Ghost ring, $\tilde{\Omega}(G) = \prod'_{U \leq G} Z$, of G, consisting of all maps from $\mathcal{S}(G)$ into Z which are constant on each conjugacy class of subgroups.

Now consider the isomorphism classes of the transitive G-sets of the form, $G/U := \{gU \mid g \in G\}$. These isomorphism classes form a Z basis of $\Omega(G)$ and for $U, V \leq G$, we have, $G/U \cong G/V$, if and only if, $U \overset{G}{\sim} V$.

This then implies that every $X \in \Omega(G)$, can be expressed uniquely in the form,

$$X = \sum_{U \leq G}{}' \mu_U(X) \cdot G/U,$$

where "$\sum'$" indicates that the sum extends over just one subgroup out of every G-conjugacy class of subgroups. That is, every $X \in \Omega(G)$ can be expressed as a linear combination of the isomorphism classes of transitive G-sets of type G/U with uniquely determined integral coefficients $\mu_U(X) \in Z$, subject to the relation, $\mu_U(X) = \mu_V(X)$ if $U \overset{G}{\sim} V$.

Now recall that for $U, V \leq G$ one has, $\chi_V(G/U) \neq 0$ if and only if $V \preceq^G U$ (that is U is subconjugate to V in G) in which case one has, $\chi_V(G/U) = \#\{gU \in G/U \mid VgU = gU\} = \frac{1}{|U|} \cdot \#\{g \in G \mid VgU = gU\} = \frac{1}{|U|} \cdot \#\{g \in G \mid V \leq gUg^{-1}\} = (N_G(U) : U) \cdot \#\{U' \leq G \mid V \leq U' \overset{G}{\sim} U\}$, where, as usual, $N_G(U)$ denotes the normalizer of U in G. We also note that given any $X \in \Omega(G)$, a subgroup $U \leq G$ is a maximal subgroup relative to "$\preceq$" with $\mu_U(X) \neq 0$ if and only if it is a maximal subgroup with $\chi_U(X) \neq 0$ because if $U \leq G$ is maximal with $\mu_V(X) \neq 0$ then, $\chi_U(X) = \sum'_{U \preceq V \subseteq G} \mu_V(X)\chi_U(G/V) = \mu_U(X)\chi_U(G/U)$. By assumption $\mu_U(X) \neq 0$ and we know that $\chi_U(G/U) \neq 0$. Therefore $\chi_U(X) \neq 0$. In addition, for any $U' \in \mathcal{S}(G)$ with $U \preceq U'$, but $U \not\sim U'$, we have $\chi_{U'}(X) = 0$ because $\mu_V(X) \cdot \chi_{U'}(G/V) = 0$ for all $V \leq G$ in view of the fact that $\chi_{U'}(G/U) \neq 0$ implies $U' \leq V$ and therefore $\mu_V(X) = 0$. The converse is proved by reversing the argument.

Note that in the foregone case one has: $\chi_U(X) = \mu_U(X) \cdot \chi_U(G/U) = \mu_U(X) \cdot (N_G(U) : U)$. Because, as observed earlier, every $X \in \Omega(G)$ can be expressed uniquely in the form

$$X = \sum_{U \leq G}{}' \mu_U(X) \cdot (G/U),$$

it follows that in the case where G is a p-group one has

$$\chi_1(X) = \sum_{U \leq G} \mu_U(X) \cdot (G : U) \equiv \mu_G(X) = \chi_G(X)(p).$$

Hence, if V is a p-subgroup of an arbitrary finite group G and if U is a subgroup of G with an index $(G : U)$ which is prime to p, then: $\chi_V(G/U) \equiv \chi_1(G/U) = (G : U) \not\equiv 0(p)$, and therefore $V \preceq_G U$. In particular, it is clear that, if Sylow p-subgroups exist in G, they all must be conjugate in G and every p-subgroup must be subconjugate in G to each of them.

Theorem 1.3 (Thevenaz and Kratzer). *Let G be a finite group and consider a subgroup U of G and an integer n. Let $\chi_{n,U} \in \tilde{\Omega}(G)$, defined by:*

$$\chi_{n,U}(V) := \begin{cases} n & \text{if } V \overset{G}{\sim} U, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\chi_{n,U} \in \Omega(G)$ if and only if $(N_G(U) : U) \cdot n_0(U)$ divides n, where $n_0(U)$ denotes the product of all primes p which divide the order $(U : [U, U])$ of the commutator factor group $U/[U, U]$ of $U \leq G$ (see [2]).

The foregone are basic reviews of some well known facts about the Burnside rings. The purpose of this paper is to use all these facts and of course some results in group theory, to give a simple condition under which an element of the Ghost ring is contained in the Burnside ring for noncyclic p-groups ($p \neq 2$). So we shall be interested in considering the case where G is a p-group ($p \neq 2$) of order p^α, say. More precisely ve shall prove the following:

Main Theorem. *(see section 3); Let G be a noncyclic p-Group ($p \neq 2$) of order p^α. We consider $\tilde{\Omega}(G) := \prod' Z := Z^{S(G)/\sim}$, and define (for $2 \leq \beta \leq \alpha$)*

$$\chi_\beta = \chi_\beta^G \in \tilde{\Omega}(G) := \prod_{U \leq G}{}' Z$$

by

$$\chi_\beta := \begin{cases} p^{\alpha-\beta} & \text{if } |U| < p^\beta, \\ p^{\alpha-\beta} & \text{if } |U| = p^\beta \text{ and } U \text{ is noncyclic,} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We claim that χ_β is contained in $\Omega(G)$.

2 Some basic results

Theorem 2.1(Frobenius). *The number of solutions of $x^r = c$ where c belongs to a fixed class C of h elements conjugate in a finite group G of order n, is divisible by the greatest common divisor of hn and r.*

PROOF: See any book on group theory. □

Proposition 2.2. *Let G be a p-group, ($p \neq 2$). Let $x^{p^2} = 1$, $x^p \neq 1$, $y^p = 1$ and $[x, y] \in C(G)$, where $C(G)$ denotes the commutators of G, then,*

$$(x, y^i)^p = [x, y]^{\frac{p+1}{2} \cdot p} y^{ip} x^p = [x, y^{ip}]^{\frac{p+1}{2}} \cdot y^{ip} x^p = x^p; \quad i = 0, 1, \ldots, p-1.$$

Moreover,

$$\langle xy^i \rangle = \langle xy^j \rangle \Leftrightarrow i = j, \text{ for } i, j \in \{0, 1, \ldots, p-1\} \Leftrightarrow \langle x^p \rangle \neq \langle y \rangle.$$

PROOF: easy. □

Corollary 2.3. *In a noncyclic group G, of order p^2 there are two different subgroups of order p.*

PROOF: easy. □

Theorem 2.4(Kulakoff). *In a noncyclic p-group of order p^n, $p \neq 2$ the number of solutions of $x^{p^m} = e$ (the neutral element) is divisible by p^{m+1} where $(0 < m < n)$.*

PROOF: See any book on group theory. □

Lemma 2.5. *If $g \in G$, and $U \leq G$ then $\langle gU \rangle$ is cyclic and $|\langle gU \rangle| = p^\beta$ (p a prime) if and only if $|\langle g \rangle| = p^\beta$ and $U \subseteq \langle g \rangle$.*

PROOF: easy. □

Lemma 2.6. *Let G be a noncyclic p-group, $p \neq 2$. If G is noncyclic then,*

$$\#\{x \in G \mid x^p = 1\} \equiv 0(p^2).$$

PROOF: easy. □

Theorem 2.7. *Let G be a noncyclic p-group ($p \neq 2$) and $U \trianglelefteq G$, $|U| = p$. Then there exists a normal subgroup $V \trianglelefteq G$ of G of order p^2, containing U and isomorphic to $Z_p \times Z_p$.*

PROOF: If G is abelian, then $G \cong C_{p^{n_1}} \times C_{p^{n_2}} \times \ldots \times C_{p^{n_k}} \trianglelefteq Z_p \times Z_p$, $k > 1$, where $C_{p^{n_i}}$ denotes the cyclic subgroup of order p^{n_i}.

If $Z(G)$ is abelian and noncyclic then we are home. So we assume from now on that,

$$1 \neq Z(G) = \langle g \rangle, \quad g^{p^\alpha} = 1 \neq h_0, \quad h_0^p = 1 \neq h_0.$$

Consider, $\{x \in G \mid x^p = 1\} \supseteq \{1, h_0, \ldots, h_0^{p-1}\}$. We claim that there exists $x \in G$ with $x^p = 1$, but $x \notin \{1, h_0, h_0^2, \ldots, h_0^{p-1}\} = \langle h_0 \rangle$.

Proof of claim: We proceed by induction on $|G|$, and take that all proper subgroups are cyclic. Consider $\Phi(G)$, the Frattini subgroup of G. One knows that $G/\Phi(G) \cong Z_p^e$ with e the minimal number of generators og $G > 1$.

If $e \geq 3$, pick $g_1, g_2, \ldots, g_e \in G$ such that $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_e \in G/\Phi(G)$ generate $G/\Phi(G)$. Then $\langle g_1, \ldots, g_e \rangle = G$ and $\langle g_1, g_2 \rangle < G$, $\langle g_1, g_2 \rangle \neq G$ is noncyclic. This implies that $\langle g_1, g_2 \rangle$ contains more than p elements of order p or 1. More precisely, there would be noncyclic proper subgroups. Here again we are home.

If $e = 2$, $G = \langle g_1, g_2 \rangle$ where $g_1^p = g_0 \neq g_2^p = g_0^{-1}$. Define a map

$$G \times G \to \Phi(G) : (g, h) \to [g, h].$$

If $\{x \in G \mid x^p = 1\} = \langle h_0 \rangle$, then there exists some power h_0^i with $[g_1, g_2] = h_0^i \neq 1$ (because G is not abelian). Hence without loss of generality,

$$[g_1^2, g_2] = h_0^2, \quad [g_1, g_2] = h_0, \quad g_1 g_2 = h_0 g_2 g_1.$$
$$(g_1 g_2)^p = h_0^{1+2+3+\ldots+p} g_2^p \cdot g_1^p = 1 \Rightarrow (g_1 g_2)^p = \mathbf{1}.$$

But $(g_1 g_2) \in \Phi(G)$ hence $x = g_1 g_2$ is of order p, but not in $\langle h_0 \rangle$.

Now consider all $x \in G$ with $x^p = 1$, but $x \notin \{1, h_0, \ldots, h_0^{p-1}\} = \langle h_0 \rangle$. Obviously $\langle x, h_0 \rangle \cong Z_p \times Z_p$. It follows that the order of existing subgroups is equal to

$$\frac{\#\{x \in G \mid x^p = 1, x \notin \{1, h_0, \ldots, h_0^{p-1}\}\}}{p^2 - p} = \frac{\#\{x \in G \mid x^p = 1\} - p}{p^2 - p}.$$

So our claim now follows from (2.6) because if,

$$\#\{x \in G \mid x^p = 1\} = p^\beta \cdot q,$$

with $\beta \geq 2$, $(p, q) = 1$, then

$$\frac{\#\{x \in G \mid x^p = 1\} - p}{p^2 - p} = \frac{p^\alpha q - p}{p^2 - p} = \frac{p^{\alpha - 1} q - 1}{p - 1},$$

is prime to p. Hence under conjugation of G on

$$\{V \subseteq G \mid h_0 \in V \cong Z_p \times Z_p\},$$

there must be an invariant one. □

Theorem 2.8. *Let G be a finite cyclic group of order n. Then we have:*

1. *for each divisor d of n there is precisely one subgroup of order d.*
2. *given d_1, d_2 that divide n then the following are equivalent:*
   (a) *d_1 divides d_2.*
   (b) *If $U(d_1)$ is the subgroup of order d_1 and $U(d_2)$ is the subgroup of order d_2 then $U(d_1) \subseteq U(d_2)$.*
3. *if $n = p^\alpha$ we have for any two subgroups $U_1, U_2 \leq G : U_1 \subseteq U_2 \Leftrightarrow |U_1| \leq |U_2|$.*
4. *if $U \trianglelefteq G$, U cyclic, $U_0 \leq U \to U_0 \trianglelefteq G$.*

PROOF: easy. □

3 Characterizations

Here we now solve our main problem. Before this we need

Lemma 3.1. *Let G be a p-group and define $y_\beta := y_\beta^G \in \tilde{\Omega}(G)$ by*

$$y_\beta(U) := \begin{cases} p^{\alpha-\beta} & \text{if } |U| \leq p^\beta, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $p^\alpha = |G|$. then $y_\beta \in \Omega(G)$.

Proof: We check by congruences; assume $U \trianglelefteq V \leq G$. It suffices to show that

$$\sum_{vU \in V/U} y_\beta(\langle vU \rangle) \equiv 0(V : U).$$

Note that $(V : U)$ is a power of p, say p^δ. Now

$$\sum_{vU \in V/U} y_\beta(\langle vU \rangle) := p^{\alpha-\beta} \# \{vU \in V/U \,|\, |\langle vU \rangle| \leq p^\beta\} \tag{3}$$

and

$$\mathcal{C}(v, U) := \#\{vU \in V/U \,|\, |\langle vU \rangle| \leq p^\beta\} = 0$$

unless

$$p^\gamma := |U| \leq p^\beta,$$

that is $\gamma \leq \beta$. In this case for any $v \in V$ we have,

$$|\langle vU \rangle| = (\langle vU \rangle : U)|U| \leq p^\beta \Leftrightarrow (\langle vU \rangle : U) \leq p^{\beta-\gamma} \Leftrightarrow (vU)^{p(\beta-\gamma)} = U,$$

in V/U. Hence, by (2.1), the left hand side of equation (2) is divisible by

$$p^{\alpha-\beta} g \cdot c \cdot d \cdot [(V : U), p^{\beta-\gamma}] = g \cdot c \cdot d[p^{\alpha-\beta}(V : U), p^{\alpha-\gamma}].$$

Since $p^{\alpha-\gamma} := (G : U)$, this must be divisible by $(V : U)$. So all congruences we need to conclude that y_β is in $\Omega(G)$ are satisfied. □

Theorem 3.2(Main Result). *Let G be a noncyclic p-group ($p \neq 2$) of order p^α. We consider $\tilde{\Omega}(G) := \prod' Z := Z^{S(G)/\sim}$. For $2 \leq \beta \leq \alpha$ define*

$$\chi_\beta = \chi_\beta^G \in \tilde{\Omega}(G) := \prod_{U \leq G}{}' Z$$

by

$$\chi_\beta := \begin{cases} p^{\alpha-\beta} & \text{if } |U| < p^\beta, \\ p^{\alpha-\beta} & \text{if } |U| = p^\beta \text{ and } U \text{ is noncyclic}, \\ 0 & \text{otherwise}. \end{cases} \tag{4}$$

Then χ_β is contained in $\Omega(G)$.

PROOF: We first observe from (3.2) that the element χ_β is in $\Omega(G)$ if and only if

$$\rho_\beta := \rho_\beta^G := y_\beta^G - \chi_\beta^G$$

is in $\Omega(G)$, where,

$$\rho_\beta(U) := \begin{cases} p^{\alpha-\beta} & \text{if } |U| = p^\beta, \text{ cyclic and } p^{\alpha-\beta} = (G : U), \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

For $W \trianglelefteq V \leq G$ we have

$$\rho_\beta^G(W) := \begin{cases} (G : W) & \text{if } W \text{ is cyclic of order } p^\beta, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (G : W)(V : W) & \text{if } W \text{ is cyclic of order } p^\beta, \\ 0 & \text{otherwise} \end{cases} = (G : V)\rho_\beta^V(W).$$

Hence, the restriction of V in G, res $\downarrow_V^G (\rho_\beta^G)$ is in $\Omega(V)$ for $V \leq G$, $V \neq G$ in view of the following simple analysis; that is from an observation that from (1.6), that for every $U \leq G$ (G a p-group) the element $e_U \in \tilde{\Omega}(G)$ defined by

$$e_U := \begin{cases} p(G : U) & \text{if } V \sim U, \\ 0 & \text{if } V \not\sim U \end{cases} \tag{6}$$

is always in $\Omega(G)$. Hence,

$$p \cdot \rho_\beta^G := \sum_{U \text{ cyclic, } |U| = p^\beta} e_U,$$

is also in $\Omega(G)$ for every p-group G, cyclic or noncyclic. To show that $\rho_\beta \in \Omega(G)$, we have to prove that

$$\rho_\beta(V, U) := \sum_{vU \in V/U} \rho_\beta(\langle vU \rangle) \equiv 0(V : U)$$

holds for all $U \trianglelefteq V \leq G$. Obviously,

$$\sum_{vU \in V/U} \rho_\beta(\langle vU \rangle) = p^{\alpha-\beta} \cdot \#\{vU \in V/U \mid \langle vU \rangle \text{ cyclic of order } p^\beta\}.$$

Hence $\rho_\beta(V, U) = 0$ unless U is cyclic of order p^γ with $\gamma \leq \beta$. So we assume that to be the case from now on. As we have seen above, for $V \leq G$, $V \neq G$ the restriction map res $\downarrow_V^G$ from $\tilde{\Omega}(G)$ into $\tilde{\Omega}(V)$ maps ρ_β^G onto an element in $\Omega(V)$. So for $V \leq G$, $V \neq G$ the above congruences necessarily hold.

Hence, without loss of generality, we can assume in addition that $V = G$, that is, we can altogether assume that U is cyclic normal subgroup of G, $p^\gamma \leq p^\beta$ and we have to show that $\rho_\beta(G : U) \equiv 0(G : U)$.

If $|U| = 1$, we have;

$$\rho_\beta(G,\,1) = \sum_{g\in G} \rho_\beta(\langle g\rangle) = p^{\alpha-\beta}\cdot \#\{g\in G\,|\,|\langle g\rangle| = p^\beta\}. \tag{7}$$

Now

$$\#\{g\in G\,|\,|\langle g\rangle| = p^\beta\} = \#\{C\leq G\,|\,C \text{ is cyclic of order } p^\beta\}\cdot (p^\beta - p^{\beta-1}.$$

So

$$\begin{aligned} \rho_\beta(G:1) = \textstyle\sum_{g\in G}\rho_\beta(\langle g\rangle) = \\ = p^{\alpha-\beta}\cdot(p^\beta - p^{\beta-1})\#\{C\leq G\,|\,C \text{ is cyclic of order } p^\beta\} = \\ = p^{\alpha-1}\cdot(p-1)\cdot\#\{C\leq G\,|\,C \text{ is cyclic of order } p^\beta\}. \end{aligned}$$

Hence we have to show that if G is a noncyclic p-group for some prime $p\neq 2$ and if $\beta\geq 2$, then

$$\#\{g\in G\,|\,|\langle g\rangle| = p^\beta\}\equiv 0(p^\beta).$$

But according to (2.4), our assumptions imply that

$$F(G,\,p^\beta) := \#\{g\in G\,|\,g^{p^\beta} = 1\}$$

is divisible by $p^{\beta+1}$ and we have:

$$F(G,\,p^\beta) = F(G,\,p^{\beta-1}) + \#\{g\in G\,|\,|\langle g\rangle| = p^\beta\}.$$

So indeed we have

$$\#\{g\in G\,|\,|\langle g\rangle| = p^\beta\} = F(G,\,p^\beta) - F(G,\,p^{\beta-1})\equiv 0(p^\beta).$$

Now assume $|U| = p^\gamma > 1$. As we have to show that

$$p^{\alpha-\beta}\cdot\#\{gU\in G/U\,|\,\langle gU\rangle \text{ is cyclic of order } p^\beta\}$$

is divisible by,

$$(G\,:\,U) = p^{\alpha-\gamma} = p^{\alpha-\beta}\cdot p^{\beta-\gamma},$$

we are left to prove that $p^{\beta-\gamma}$ divides

$$\#\{gU\in G/U\,|\,\langle gU\rangle \text{ cyclic of order } p^\beta\}.$$

This is trivial in case $\beta=\gamma$. Otherwise observe from (2.5) that $\#\{gU\in G/U\,|\,\langle gU\rangle$ cyclic of order $p^\beta\} = \#\{C\leq G\,|\,C$ is cyclic of order p^β, $C\supseteq U\}\cdot(p^{\beta-\gamma}-p^{\beta-\gamma-1})$ is divisible by $p^{\beta-\gamma}$ if and only if,

$$\#\{C\leq G\,|\,U\leq C,\,U\neq C,\,C \text{ cyclic of order } p^\beta\}\equiv 0(p).$$

So it suffices to show that

$$z(U,\,\beta) := z_G(U,\,\beta) := \#\{C\leq G\,|\,C\supseteq U,\,|C| = p^\beta,\,C \text{ cyclic}\}\equiv 0(p).$$

Now assume $\beta - 1 > \gamma$ and let

$$Z(U,\,\beta) := Z_G(U,\,\beta) := \{C \le G \mid C \supseteq U,\ |C| = p^{\beta-1},\ C \text{ cyclic}\},$$

then there exists a canonical map,

$$\phi\,:\, Z_G(U,\,\beta) \to Z_G(U,\,\beta-1),$$

defined by,

$$\phi(C) = C^p.$$

Observe that given $V \in Z(U,\,\beta-1)$ then $C \in \phi^{-1}(V)$ if and only if $C \in Z(U,\,\beta)$ and $V \le C$ if and only if $C \in Z(V,\,\beta)$ and so

$$z(U,\,\beta) = \sum_{V\in Z(U,\beta-1)} \#\phi^{-1}(V) = \sum_{V\in Z(U,\beta-1)} \#Z(V,\,\beta) = \sum_{V\in Z(U,\beta-1)} z(V,\,\beta)$$

So we may assume without loss of generality, that $\gamma = \beta - 1$. Since for U_0 the unique subgroup of order $p^{\gamma-1} = |U|/p$ one has

$$z_G(U,\,\beta) = z_{G/U_0}(U/U_0,\,\beta-\gamma+1),$$

we may as well assume $|U| = p$ and $\beta = 2$. So to prove our claim it suffices to show that $z_G(C_p,\,2) \equiv 0(p)$.

Now, for $z(C_p,\,2)$, $(C_p \trianglelefteq G)$, let,

$$z_0(C_p,\,2) := \#\{Z \trianglelefteq G \mid Z \in Z(C_p,\,2)\}.$$

Then $z_0(C_p,\,2) \equiv z(C_p,\,2)(p)$. Since G is noncyclic then there exists by (2.3) a further subgroup $\langle\pi\rangle = \Pi \le G$ of order p, different from C_p. Let Π act on

$$Z_0(U,\,2) := \{Z \trianglelefteq G \mid Z \text{ is cyclic of order } p^2,\ Z \supseteq C_p\}$$

as follows:
Assume $C_p = \langle a\rangle$ and pick $x \in Z \in Z_0(U,\,2)$ with $x^p = a$ so that $Z = \langle x\rangle$. Note that x is determined modulo C_p. Choose $\pi_i \in \Pi$ and define,

$$\pi_i * Z = \langle \pi^i x\rangle.$$

We observe that if $x \equiv x' \bmod (C_p)$, then $\langle\pi^i x\rangle = \langle\pi^i x'\rangle$. More precisely, since $(\pi^i x)^p = x^p = a = x'^p = (\pi^i x')^p$ and,

$$\begin{gathered} 1_\pi * Z = Z, \\ \pi^i * (\pi^j * Z) = \pi^{i+j} * Z, \\ \pi^i * Z = Z \Leftrightarrow \pi^i = 1_\pi \ (\text{since } \Pi \ne Z_p). \end{gathered}$$

every orbit has exactly p elements, so the proof follows. □

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Address: K. K. Nwabueze, Dept. of Mathematics, Usman-Danfodio University, Sokoto, Nigeria

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