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# The Quasi – Real Extension of the Real Numbers

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**Abstract.** Let $\mathbb{R}$ be the set of real numbers ordered by the usual ordering, $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\Xi = \{-, 0, +\}$ with $- < 0 < +$. The set $S = \hat{\mathbb{R}} \times \Xi - \{(-\infty, -), (+\infty, +)\}$, ordered lexicographically and endowed with some partial operations and the order topology, is said to be the quasi–real line and its elements the quasi–real numbers. As the usual operations are partially extended, $S$ fails to be a field, but itself and $S^n$ too, may be considered as order completions of $\mathbb{R}$ and $\hat{\mathbb{R}}$, respectively.

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## Introduction

Let $\mathbb{R}$ be the set of real numbers ordered by the usual ordering, $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\Xi = \{-, 0, +\}$ ordered by $- < 0 < +$. The set $S = \hat{\mathbb{R}} \times \Xi - \{(-\infty, -), (+\infty, +)\}$, ordered lexicographically, endowed with the order topology and on which a partial addition and multiplication are defined (given in def. 2,3, §1), is said to be the *quasi – real line* and its elements the *quasi-real numbers.*

The set $S$, although could have some similarities with some non standard analysis models (see [6], §9.4, p.246), differs substantially from them at least because of not being a field. On the other hand, the quasi–real numbers can play a role in valuation theory (see [5]) or can characterize explicitly the sup, inf or $\overline{\lim}$, $\underline{\lim}$ of a subset, or of a sequence of real numbers (as one can see through §1). In addition, continuous quasi–real functions of a quasi–real variable have the property that their real traces may be discontinuous functions of first kind (see §4). Such a property permits one to classify the discontinuous of first kind functions into classes, as one classifies into Baire classes the real functions of one real variable (see [2]).

The present paper revolves around these subjects. The difficulties that we have to overcome are due to the complexity of forming the elements of $S^n$. In fact, around any element $(\rho_1, \rho_2, \dots, \rho_n) \in \mathbb{R}^n$, there are all the elements $(\rho_1^{\xi_1}, \rho_2^{\xi_2}, \dots, \rho_n^{\xi_n})$, with $\xi_i \in \Xi, i \in \{1, 2, \dots, n\}$.

## §1. The algebraic structure of $S^n$.

Let $S$ be the set of quasi–real numbers defined as above. If $r = (\rho, \xi)$ is a quasi–real number, $\rho$ is called the *real part* and $\xi$ the *kind of* $r$ (+ and − are called opposite). Extend $\hat{I\!R}$ in $S$ identifying each real number $\rho$ with the quasi−real number $(\rho, 0)$.

This extension preserves the order of $\hat{I\!R}$, but not the order topology. In fact, while each real number $\rho$ is, on the real line, a limit point, the corresponding quasi−real number $(\rho, 0)$ is always an isolated point on the quasi−real line. On the other hand, the real projection $\rho : (\rho, \xi) \mapsto \rho$ of the quasi−real numbers is continuous function according to the order topology of the quasi−real line and to arbitrary topology of the real line.

There exist, in general, three quasi−real numbers $(\rho, -)$, $(\rho, 0)$, $(\rho, +)$ with real value $\rho$ (besides, if $\rho = \pm\infty$ there exist two points $(-\infty, 0)$, $(-\infty, +)$ for $-\infty$ and two points $(+\infty, -), (+\infty, 0)$ for $+\infty$). If there is not any ambiguity we use the symbols $\rho^-, \rho^0, \rho^+$, respectively. The corresponding sets are symbolized by $S^-, S^0, S^+$, respectively.

**1.Remark.** Let $J$ be the set of all monotone sequences of $\hat{I\!R}$.
Define into $J$ an equivalence relation:

$$(\rho_n) \sim (\rho'_n) \text{ iff } (\rho_n), (\rho'_n) \text{ have the same limit point}$$

$$\text{and the same monotony ;}$$

then the set $S$ can be defined as the quotient $J/_\sim$ and the common limit $\rho$ of the equivalent sequences is the real value of a quasi−real number $r = (\rho, \xi)$. The kind $\xi \in \{-, 0, +\}$ of $r$ characterizes the corresponding sequences as *increasing, (asymptotically) constant* and *decreasing*, respectively.

We define two partial operations on $S$. Firstly we refer to a partial operation on $\Xi$.

**2. Definition.** For every quasi−real number $r = (\rho, \xi)$, we say $-\xi$ the *opposite* of $\xi$, hence, $+$ is opposite of $-$, $-$ is opposite of $+$, $0$ is opposite of $0$; we define $-r = (-\rho, -\xi)$. If $r \neq (0, 0)$, then $r^{-1} = (\rho^{-1}, -\xi)$ and we can define the absolute value of $r$ by

$$|r| = r \text{ or } -r, \text{ if } r > (0,0) \text{ or } r < (0,0) \text{ respectively },\ |(0,0)| = (0,0).$$

**3. Definition.** If $\xi_1, \xi_2$ are not opposite, put

$$i(\xi_1, \xi_2) = \begin{cases} + & \text{if at least one of } \xi_1, \xi_2 \text{ is } + \\ 0 & \text{if both of } \xi_1, \xi_2 \text{ are zero} \\ - & \text{if at least one of } \xi_1, \xi_2 \text{ is } - \end{cases}.$$

Consider $r_1 = (\rho_1, \xi_1)$, $r_2 = (\rho_2, \xi_2)$ and $\xi_1, \xi_2$ are not opposite.

(a) We put

$$r_1 + r_2 = (\rho_1 + \rho_2, i(\xi_1, \xi_2)) \tag{1}$$

(b) If $r_1, r_2 > (0, 0)$, then

$$r_1 . r_2 = (\rho_1 . \rho_2, i(\xi_1, \xi_2)) \tag{2}$$

If $r_1, r_2 < (0,0)$, then

$$r_1.r_2 = (\rho_1.\rho_2, \text{ opposite } of i(\xi_1, \xi_2)). \tag{3}$$

If at least one of $r_1, r_2$ is $(0,0)$, then $r_1.r_2 = (0,0)$.
If the one, say $r_1$, is $> (0,0)$ and the other, say $r_2$, is $< (0,0)$, then $r_1.r_2 = -(-\rho_1\rho_2, i(\xi_1, -\xi_2))$, provided $\xi_1$ and $-\xi_2$ are not opposite.

**4. Remark.** If $r_1$ and $r_2$ are of kind, say $-$, then they can be considered as limits of two increasing sequences of real numbers $(\rho_n)$ and $(\rho_{n'})$, respectively. The sequence $(\rho_n + \rho_{n'})$ converges to a quasi–real number of kind $-$. Analogously, if $(\rho_n)$, $(\rho_{n'})$ are two decreasing sequences of real numbers converging to two negative real numbers from the right, then their product $(\rho_n.\rho_{n'})$ is an increasing sequence converging from the left to a positive number. Just these results inspired the partial operations we have referred to.

Consider now the cartesian product $S^n = S \times \ldots \times S$ (n times). We can partially define on $S^n$ the operations $\oplus$ and $\odot$ provided that the $+$ and ., respectively, are defined componentwise as above.

For $x = (\rho_1^{\xi_1}, \ldots, \rho_n^{\xi_n}), y = (q_1^{\xi'_1}, \ldots, q_n^{\xi'_n})$ put

$x \oplus y = (\rho_1^{\xi_1} + q_1^{\xi'_1}, \ldots, \rho_n^{\xi_n} + q_n^{\xi'_n})$ and

$x \odot y = (\rho_1^{\xi_1}.q_1^{\xi'_1}, \ldots, \rho_n^{\xi_n}.q_n^{\xi'_n})$.

The n–ple $(\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ is said to be the *real projection* of $x$.

## § 2. Two order completions

Let $(E, \leq)$ be an order structure ($<$ is the strict relation).

**1. Definition.** (see [7], def. 1). We say a couple $(A, B)$ of two non void subsets of $E$ $f^*$*–cut*, if it fulfils the next:

(1) $A$ (resp $B$) is up–directed (resp. low - directed).

(2) $(\forall x \in A)(\forall y \in B)[x < y]$.

(3) There exist two totally ordered subsets $\alpha_1$ of $A$ and $\beta_1$ of $B$ such that $\alpha_1$ is cofinal with $A$ and $\beta_1$ is coinitial with $B$.

(4) $A$ and $B$ are maximal according to the inclusion relation among the subsets which fulfil (1), (2) and (3).

The subsets $A$ and $B$ are called *lower* and *upper class* of the $f^*$ – cut $(A, B)$, respectively. If there exists the maximum of $A$ and the minimum of $B$, then the $f^*$ – cut is called $f^*$ – *jump*. If there do not exist extreme points, then it is called $f^*$ – *gap*. The set of the $f^*$– gaps is symbolized by $L^*(E)$.

In the set $E^* = E \cup L^*(E)$ we define a relation $R^*$ as following ($x, y$ are elements of $E$ and $(A_1, B_1)$, $(A_2, B_2)$ elements of $L^*(E)$):

$$xR^*y \Leftrightarrow x \leq y$$

$$xR^*(A_1,B_1) \Leftrightarrow x \in A_1$$
$$(A_1,B_1)R^*x \Leftrightarrow x \in B_1$$
$$(A_1,B_1)R^*(A_2,B_2) \Leftrightarrow (A_1,B_1) = (A_2,B_2) or A_2 \cap B_1 \neq \emptyset.$$

**2. Proposition.** *(see [7], prop.1). The relation $R^*$ is an ordering and its restriction on $E$ coincides with $\leq$.*

**3. Proposition.** *(see [7], prop. 2). There do not exist $f^*$ – gaps in $E^*$ and $(E^*)^* = E^*$.*

**4. Remark.** The above completion would coincide with the system of linear subsets which define the completions of partially ordered sets (see [4],§1).

**5.** We consider now the well–known Mac Neille's completion $\bar{E}$ of $E$, which consists of the Mac Neille's cuts (see [1], p. 128). Such a cut is a couple $(A,B)$ of non void subsets of $E$ which fulfils the next:

(1) $(\forall x \in A)(\forall y \in B)[x < y]$
(2) Each $y \in E$, properly greater than all the elements of $A$, belongs to $B$.
(3) Each $x \in E$ , properly smaller than all the elements of $B$, belongs to $A$.

The sets $A$ and $B$ are called *lower* and *upper class*, respectively. Every $x \in E$ would be presented as a Mac Neille's cut $(A_x, B_x)$, where $x$ will be end of $A_x$ or of $B_x$.

The notions of jumps and gaps are as in 1.Definition.

**6. Definition.** (see [4], p. 2505). Let $\Lambda(E)$ be the set of the classes of the Mac Neille's cuts which have no ends.

In the set $E_{ku} = E \cup \Lambda(E)$ we define a relation $R$ as follows ($x,y,z$ in $E$, $(A,B),(A',B')$ Mac Neille's cuts):

$$xRy \Leftrightarrow x \leq y, xRA \Leftrightarrow x \in A, ARx \Leftrightarrow (\forall z \in A)[z < x],$$
$$xRB \Leftrightarrow (\forall z \in B)[x < z], BRx \Leftrightarrow x \in B,$$
$$ARA' \Leftrightarrow A = A' or (\exists x \in A')(\forall y \in A)[y < x],$$
$$BRB' \Leftrightarrow B = B' or (\exists x \in B)(\forall y \in B')[x < y],$$
$$ARB \Leftrightarrow (A,B) \text{ is a Mac Neille' s cut}$$
$$\text{or}(\exists x \in E)(\forall y \in A)(\forall z \in B)[y < x < z]$$
$$BRA \Leftrightarrow B \cap A \neq \emptyset.$$

The relation $R$ is an ordering which extends $\leq$ on $E_{ku}$. We also symbolize the relation $R$ by $\leq$. The structure $(E_{ku}, \leq)$ – or simply the set $E_{ku} = E \cup \Lambda(E)$– is said to be the *Kurepa's completion* (see [3], p. 2506).

**7. Definition.** We consider the set $\Lambda^*(E)$ of the classes of the $f^*$ – cuts which have no end and on $E^*_{ku} = E \cup \Lambda^*(E)$ we define a relation $R^*$, just as in defin. 6

it has been defined the relation $R$ in the set $E_{ku}$ with one exception: the relation "$ARB \Leftrightarrow (A,B)$ is a Mac Neille's cut " becomes "$AR^*B \Leftrightarrow (A,B)$ is an $f^*$ – cut".

The relation $R^*$ is an ordering on the set $E^*_{ku}$, which we also symbolize by $\leq$.

The structure $(E^*_{ku}, \leq)$ – or simply the set $E^*_{ku}$ – is called *Kurepa's $f^*$ –completion* (see [7], p. 82 and [4], p. 199).

**8. Remark.** Consider a Mac Neille's cut $(A,B)$ of $E$ and decompose the classes $A$ and $B$ into maximal up-directed and low-directed sets, respectively, each of which is cofinal and coinitial, respectively, with a totally ordered subset of itself (c.f. [7], p. 83). Let $A = (A_i)_i$ and $B = (B_j)_j$ these decompositions. If a directed subset, say $A_i$, does not have a maximum element, then there corresponds to it an element of the Kurepa's $f^*$ – completion and conversely, to each new element of the Kurepa's $f^*$ – completion, corresponds a directed set, say $A_i$, without end and hence it corresponds to a Mac Neille's cut $(A,B)$, in which the class $A$ does not have an end, too. So, to each new element $A_i$ of Kurepa's $f^*$ – completion corresponds a new element $A$ of Kurepa's completion. Likewise, if an element of $E$ will be considered as an element of $E^*_{ku}$, then it corresponds generally to two elements in $E_{ku}$. These correspondences motivate to the following definition.

**9. Definition.** The element $x^*$ of Kurepa's completion which assigns to an element $x$ of Kurepa's $f^*$ – completion according to the above correspondence is said to be the *Kurepa's value* of $x$ for the Mac Neille's cut $(A,B)$. (For further details see [3], p. 2506 and [7] p.p. 81, 82).

## § 3. The set $S^n$ as an algebraic ordered system

Firstly we define on $R^n$ a family of orderings as follows:

**1. Definition.** ($\alpha$) If $\Im = \Xi^n, \Xi = \{-,0,+\}$, $n$ a natural number, then for every element $j = (\xi_1, \xi_2, \dots, \xi_n)$ in $\Im$, define on $\mathbb{R}^n$ the relation $\leq_j$: For $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n)$

$$x \leq_j y \Leftrightarrow x = y \text{ or } \begin{cases} x_i = y_i, if \xi_i = 0 \\ x_i < y_i, if \xi_i = + \\ x_i > y_i, if \xi_i = - \end{cases},$$

Evidently, the relation $\leq_j$, for every $j \in \Im$, is an ordering. Put $G = (\leq_j)_{j \in \Im}$.

($\beta$) By $B^j$ we symbolize the Kurepa's completion of the structure $(\mathbb{R}^n, \leq_j)$ and put

$$\mathbb{R}^n_{ku} = \bigcup_{j \in \Im} B^j.$$

($\gamma$) If $x \in B^j \setminus \mathbb{R}^n$, then $x = ]\rho_x \to [_j = \{y \in \mathbb{R}^n : \rho_x <_j y\}$

or $\quad x = ] \leftarrow \rho_x[_j = \{y \in \mathbb{R}^n : y <_j \rho_x\}$, where $\rho_x \in \mathbb{R}^n$.

The element $\rho_x$ is called *real projection* of $x$. If $x \in \mathbb{R}^n$, the real projection is $x$ itself.

**2. Remarks.** (1) For every $x \in I\!R^n$, three elements of $B^j$ have the same real projection; these are the $x$ itself, the $]x \to [_j$ and the $] \leftarrow x[_j$. However to each $x \in I\!R^n$ correspond $3^n$ elements of $I\!R^n_{ku}$.
(2) If $x \in I\!R^n_{ku} \setminus I\!R^n$, then $x$ is a class, say, a lower class $A$ and, say, $B$ the corresponding upper class in Mac Neille's cut $(A, B)$. Because of the completeness of $I\!R^n$, just one of these two classes will have an end.
(3) If $j = (\xi_1, \ldots, \xi_n)$, $\bar{j} = (-\xi_1, \ldots, -\xi_n)$, then $B^j = B^{\bar{j}}$. In fact, if $x \in I\!R^n_{ku} \setminus I\!R^n$, then $] \leftarrow \rho_x[_j = ]\rho_x \to [_{\bar{j}}$, where $\rho_x$ is the real projection and the two completions finally coincide.

We now define an ordering on $I\!R^n_{ku}$ and $S^n$.

**3. Definition.** $\alpha$) Firstly we define two functions: Let $j = (\xi_1, \ldots, \xi_n) \in \Im$,

$\varepsilon : \{\xi_1, \ldots, \xi_n\} \to \{-1, 0, +1\}$ with value

$$\varepsilon(\xi_i) = \begin{cases} -1, & \text{for } \xi_i = - \\ 0, & \text{for } \xi_i = 0 \text{ , } i \in \{1, 2, \ldots, n\} \\ +1, & \text{for } \xi_i = + \end{cases}$$

and

$$h : \Im \mapsto \{0, -1, 1\}^n, \quad \Im = \Xi^n, \text{ such that}$$

$$h(\xi_1, \ldots, \xi_n) = (\varepsilon(\xi_1), \ldots, \varepsilon(\xi_n)).$$

$\beta$) Define on the set $\Im$ a relation $\preceq$ as follows: if $i = (\xi_1, \ldots, \xi_n)$ and $j = (\xi'_1, \ldots, \xi'_n)$ in $\Im$, there holds

$i \preceq j \Leftrightarrow h(\xi_1, \ldots, \xi_n) \leq h(\xi'_1, \ldots, \xi'_n)$ ($\leq$ the natural ordering on $I\!R^n$).
Evidently the relation $\preceq$ is an ordering.

**4. Definition.** We define on the $S^n$ an ordering $\leq$, as following: let be $x = (\rho_1^{\xi_1}, \ldots, \rho_n^{\xi_n})$, $y = (q_1^{\xi'_1}, \ldots, q_n^{\xi'_n})$ in $S^n$.
($\alpha$) If $(\rho_1, \ldots, \rho_n) \leq (q_1, \ldots, q_n)$ for the componentwise ordering on $I\!R^n$, then $x \leq y$.
($\beta$) If $(\rho_1, \ldots, \rho_n) = (q_1, \ldots, q_n)$ and $(\xi_1, \ldots, \xi_n) \preceq (\xi'_1, \ldots, \xi'_n)$ (as it was defined in $3\beta$ def.), then $x \leq y$, again.

The relation $\leq$ on $S^n$ is called *componentwise ordering* of it.

**5. Definition.** Let $x$ and $y$ be elements of $I\!R^n_{ku}$ and $\rho_x, \rho_y$ their real projections, respectively. (1) For $\rho_x \neq \rho_y$ define:

$$x \tilde{\leq} y \Leftrightarrow \rho_x \leq \rho_y.$$

(2) For $\rho_x = \rho_y$, we distinguish the cases:

$\alpha$) If $x, y$ in $I\!R^n_{ku} \setminus I\!R^n$ and $i, j$ in $\Im$, such that $x = ]\rho_x \to [_i, y = ]\rho_y \to [_j$, then

$$x \tilde{\leq} y \Leftrightarrow i \preceq j \text{ (see def } 3\beta).$$

$\beta$) If the one of $x$ and $y$ belongs in $\mathbb{R}^n$ and the other equals to $]\rho \to [_j$ or to $]\rho \to [_{\bar{j}}$, where $(0,0,\ldots,0) \preceq j = (\xi_1,\xi_2,\ldots,\xi_n)$,then

$$]\rho_x \to [_{\bar{j}} \tilde{\leq} \rho_x \tilde{\leq} ]\rho_x \to ]_j, \quad \text{where } \bar{j} = (-\xi_1,-\xi_2,\ldots,-\xi_n).$$

Evidently the relation $\tilde{\leq}$ is an ordering.

**6. Theorem.** *The structure $(\mathbb{R}^n_{ku}, \tilde{\leq})$ is order-isomorphic to the structure $(S^n, \leq)$, where $\leq$ is the componentwise ordering of $S^n$.*

PROOF: Define a function $f$ of $\mathbb{R}^n_{ku}$ in $S^n$ as follows:
If $x = (\rho_1,\ldots,\rho_n) \in \mathbb{R}^n$, then put $f(x) = (\rho_1^0,\ldots,\rho_n^0) \in S^n$.

Let be $x \in \mathbb{R}^n_{ku} \setminus \mathbb{R}^n, j = (\xi_1,\ldots,\xi_n)$ and $\rho_x = (\rho_1,\ldots,\rho_n)$ the real projection of $x$. If $x = ]\rho_x \to [_j$, then put $f(x) = (\rho_1^{\xi_1},\ldots,\rho_n^{\xi_n})$, while if $x = ] \longleftarrow \rho_x [_j$, put $f(x) = (\rho_1^{-\xi_1},\ldots,\rho_n^{-\xi_n})$.

The bijection of $f$ is obvious.

We will prove that $f$ is *onto*. Let be $x'$ in $S^n$.
If $x' = (\rho_1^0,\ldots,\rho_n^0)$, then $x = (\rho_1,\ldots,\rho_n)$ in $\mathbb{R}^n_{ku}$ and $f(x) = x'$. Let $x' = (\rho_1^{\xi_1},\ldots,\rho_n^{\xi_n})$ with $\xi_1,\ldots,\xi_n$ not all zero and $\rho_{x'}$ the real projection of $x'$. If $j = (\xi_1,\ldots,\xi_n)$ and $x = ]\rho_{x'} \to [_j$, then from the definition of $f$ it will be $f(x) = x'$.

Finally we prove that $f$ preserves the ordering.

Let $x,y$ in $\mathbb{R}^n_{ku}, \rho_x = (\rho_1,\ldots,\rho_n), \rho_y = (q_1,\ldots,q_n)$ their real projections respectively, with $x \tilde{\leq} y$. It is evident that the real projections of $f(x), f(y)$ will be $\rho_x, \rho_y$, respectively.

We distinguish the cases:

(i) $\rho_x \leq \rho_y, \rho_x \neq \rho_y$. Then $f(x) \leq f(y)$.

(ii) $\rho_x = \rho_y = (\rho_1,\ldots,\rho_n)$ and $x,y$ in $\mathbb{R}^n_{ku} \setminus \mathbb{R}^n$.
Because $x \tilde{\leq} y$ , it means that there exist $i = (\xi_1,\cdots,\xi_n)$,
$j = (\xi'_1,\ldots,\xi'_n)$, such that $x = ]\rho_x \to [_i$, $y = ]\rho_x \to [_j$ and $i \prec j$. It results $f(x) = (\rho_1^{\xi_1},\ldots,\rho_n^{\xi_n}) \leq f(y) = (\rho_1^{\xi'_1},\ldots,\rho_n^{\xi'_n})$

(iii) $\rho_x = \rho_y = (\rho_1,\ldots,\rho_n), x \in \mathbb{R}^n$ and$y = ]\rho_x \to [_j$,
where $j = (\xi_1,\cdots,\xi_n)$. Because $x \tilde{\leq} y$, it means that
$(0,\ldots,0) \preceq j$. Hence

$$f(x) = (\rho_1^0,\ldots,\rho_n^0) \leq f(y) = (\rho_1^{\xi_1},\ldots,\rho_n^{\xi_n}).$$

(iv) $\rho_x = \rho_y = (\rho_1,\ldots,\rho_n), x = ]\rho_x \to [_j, y \in \mathbb{R}^n$, where $j = (\xi_1,\ldots,\xi_n)$. Because $x \tilde{\leq} y$, it is $j \preceq (0,\ldots,0)$, hence

$$f(x) = (\rho_1^{\xi_1},\ldots,\rho_n^{\xi_n}) \leq f(y) = (\rho_1^0,\ldots,\rho_n^0).$$

□

**7. Definition.** We consider the family $G$ of the orderings $(\leq_j)_j \in \Im$ (of def. 1.a, § 3). By $A^i$ we symbolize the Kurepa's $f^*$-completion of the structure $(\mathbb{R}^n, \leq_j)$ and put

$$(\mathbb{R}^n)^*_{ku} = \bigcup_{i \in \Im} A^i.$$

**8. Remark.** Every Kurepa's value $x^*$, according to the ordering $\leq_j$, of an element $x \in (\mathbb{R}^n)^*_{ku}$, is a class $]\rho_{x^*} \to [_j$ or
$] \longleftarrow \rho_{x^*}[_j$, where $\rho_{x^*}$ is the real projection of $x^*$. Let us call the $\rho_{x^*}$ *the real value of* $x$ as well.

On the set $(\mathbb{R}^n)^*_{ku}$ we define a relation $\leq^*$, as follows:

**9. Definition.** Let be $x$ and $y$ elements of $(\mathbb{R}^n)^*_{ku}$ and $x^*, y^*$ their Kurepa's values, respectively. Define

$$x \leq^* y \Leftrightarrow \quad x^* \tilde{\leq} y^*.$$

The relation $\leq^*$ is an ordering.

**10. Theorem.** *The completion structures* $(\mathbb{R}^n_{ku}, \tilde{\leq})$ *and* $((\mathbb{R}^n)^*_{ku}, \leq^*)$ *are order – isomorphic.*

PROOF: Define a function $f$ of $\mathbb{R}^n_{ku}$ in $(\mathbb{R}^n)^*_{ku}$ as follows:

If $x \in \mathbb{R}^n$, then put $f(x) = x$.

Let be now $x \in \mathbb{R}^n_{ku} \setminus \mathbb{R}^n$ ; then there exists
$j = (\xi_1, \ldots, \xi_n) \in \Im$ such that $x \in B^j \setminus \mathbb{R}^n$ and $x = ]\rho_x \to [_j$ or
$x = ] \longleftarrow \rho_x[_j$, where $\rho_x$ is the real projection of $x$.

Suppose $x = ]\rho_x \to [_j$. Consider the set:

$$A_j = \{[\rho_1 + \varepsilon(\xi_1)t, \ldots, \rho_n + \varepsilon(\xi_n)t]; t > 0\},$$

where $\varepsilon$ is the function which was defined in def. $3\alpha$, § 3. If $t \to 0$ monotonically, then the set $A_j$ is a decreasing chain for $\leq_j$, coinitial with $]\rho_x \to [_j$.

In fact:

($\alpha$) $A_j \subseteq ]\rho_x \to [_j$, because, if $\xi_i = +$, then $\rho_i + \varepsilon(\xi_i)t > \rho_i$, while if $\xi_i = -$, then $\rho_i + \varepsilon(\xi_i)t < \rho_i$, for every $t > 0$ and $i \in \{1, 2, ..., n\}$.

Hence, every $x\prime \in A_j$ belongs to $]\rho_x \to [_j$.

($\beta$) Let now be $x\prime = (\rho'_1, \ldots, \rho'_n) \in ]\rho_x \to [_j$.

Then there exists a $t_1 > 0$, such that for every $i \in \{1, \ldots, n\}$, there hold: if $\xi_i = +$, then $\rho_i + t_1 \in ]\rho_i, \rho'_i[$, if $\xi_i = -$, then $\rho_i - t_1 \in ]\rho'_1, \rho_i[$, and if $\xi_i = 0$, then $\rho_i + 0.t_1 = \rho_i = \rho'_i$.

Hence, $(\rho_1 + \varepsilon(\xi_1)t, \ldots, \rho_n + \varepsilon(\xi_n)t) \leq_j \quad x'$. If $x^*$ is the element of the $f^*$–completion which equals $]\rho_x \to [_j$, then put $f(x) = x^*$.

Analogously we define the $x^*$, if $x = ] \longleftarrow \rho_n[_j$. Evidently the above defined function is a *bijection.*

We prove that $f$ is *onto.*

In fact, let be $x^*$ in $(M^n)\backslash_{u\mathcal{I}}p_x^*$ its reál projection and $Aj$ a totally ordered subset for an order <j, cofinal with the /*— class of $x^*$. So, $Aj$ is coinitial with $]px^*$ $—^*$ $[j$ $^{or}$ cofinal with ] <— $p_x$*[j according to whether $Aj$ is decreasing or increasing, respectively. If $x$ is the Kurepa's value of x*, then it will be $x = ]p_T^*$ $—^*$ $[j$ or $x =]$ <— $p_x^*$ $[j$ according to whether $Aj$ is coinitial with the first or cofinal with the latter. For both of the cases $f(x) = x^*$.

To complete the proof of the theorem we show that / preserves the ordering.

Suppose $x$ and y are elernents of $M\%_u$ satisfying x<y. Since / is a bijection onto, there exist elernents x',y' in (JR$^n$)Jl$_u$, such that $f\sim^l(x^l) = x, f\sim^l(y') =$ y, that is the elernents $x$ and $y$ are the Kurepa's values of $x'$ and y', respectively. Then, by the definition of <*, it is

x<y <& $x'$ <* y', hence

$x<y$ & /(*)<*/(»),

and the proof is over. D

An immediate consequence of theorems 6 and 10, as well as defin. 5, is the following:

**Corollary.** *The structures* $(S^n,<)_y(B\%_u,<)$ *and* $((lR^n)^*_{ku},<^*)$ *are order - isomorphic.*

## §4. An application

In this last section we give an application of theorems 6 and 10. When n = 1 the two completions coincide with the well known set $S$ (Pensemble des nombres semi-reels in [2] and [5]). On the other hand, it is easy to generalize the completions concerning any totally ordered set as the ground set, provided this set does not háve jumps. In such a čase, the set is dense in itself.

Let (£?, X) be a structure of partial ordering endowed with the topology of the open intervals; let also $F$ be a reál function defined on $E$. Given a Mac Neille's cut on 23, it is natural to say that $F$ has limit in (A, $B$) from the left (resp. from the right) the number / € $M$, if and only if

(Ve > 0)(3x$_0$ € $A$)(\fx € $A$)[$x_Q$ X $x$ -* \F(x) -l\<é\

( resp.(V6 > 0)(3y$_0$ € J5)(Vy <E $B$)[$y$ X y$_0$ -> \F(y) - l\ < e})

The function $F$ has limit in (A, $B$) the number /, if it has limit in (A, $B$) the number / from the left and from the right.

It is easy to prove the following theorem:

**Theorem.** *If a reál monotone function is defined on a totally ordered structure without jumps, endowed with the topology of open intervals, and which function has* *limits in every Mac Neille's cut of the ordered structure, then there is an order-completion of the given structure and an extension of the function in such a way that it becomes continuous.*

*In fact, if* $(E, \preceq)$ *is the order structure,* $F$ *is the real function and* $(\tilde{E}, \tilde{\preceq})$ *the Kurepa's completion of the given structure, then we extend* $F$ *in* $\tilde{F}$ *as follows:* $\tilde{F}(x) = F(x)$*, for* $x \in E$ *and if* $x = (A, B)$ *is a new point of the Kurepa's completion and* $l_x$ *the limit of* $F$ *in* $(A, B)$*, then* $\tilde{F}(x) = l_x$.

It is easy to prove the continuity of $\tilde{F}$. It is a natural question whether the last result may be generalized in the case where the ground structure is partially ordered.

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