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On Hall Planar Ternary Rings with Ordered Carrier Sets *

MAREK JUKL

*Department of Algebra and Geometry, Faculty of Science,
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: jukl@risc.upol.cz*

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Abstract

This article deals with Hall planar ternary ring $(\mathbf{M}, t)$ such that the ordering on $\mathbf{M}$ is given by a suitable way. Especially, the compatibility of the ordering on the carrier set $\mathbf{M}$ with the addition and multiplication induced on $\mathbf{M}$ by the ternary operation t (in the usual sense) is shown.

Key words: Ternary operation, Hall planar ternary ring, ordered set, loop.

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The starting point is the notion of *Hall planar ternary ring.*[1] According to [1] let us define

Definition 1 An ordered pair (M,t) where $\operatorname{card}\mathbf{M} \geq 2$ and t is a ternary operation on $\mathbf{M}$ fulfilling the following axioms

(1) $\forall x, m, y \in \mathbf{M}\ \exists! b \in \mathbf{M}:\ y = t(x, m, b)$,
(2) $\forall m, b, u, v \in \mathbf{M},\ m \neq u,\ \exists! x \in \mathbf{M}:\ t(x, m, b) = t(x, u, v)$,
(3) $\forall x, y, \bar{x}, \bar{y},\ x \neq \bar{x},\ \exists!(m, b,) \in \mathbf{M}^2:\ t(x, m, b) = y \wedge\ t(\bar{x}, m, b) = \bar{y}$,
(4) $\exists! 0 \in \mathbf{M}\ \forall a, b \in \mathbf{M}:\ t(0, a, b) = b \wedge\ t(a, 0, b) = b$,
(5) $\exists! e \in \mathbf{M}\ \forall a \in \mathbf{M}:\ t(e, a, 0) = a = t(a, e, 0)$,

is called *Hall planar ternary ring* (abb. $HPTR$).[2]

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[1]This algebraic structure was introduced by M. Hall in [1] to coordinatize projective planes. The special types of Hall planar ternary rings are introduced and investigated e.g. in [2]–[4].

[2]The notion *Hall planar ternary field* is used in the equivalent meaning.

It is well known that it may be proved that the unicity of the couple (m, b) in (3) as well as the unicity of elements 0 and e in (4), (5) follows from the conditions above.

Notation 2 Let $(\mathbf{M}, t)$ be a HPTR.

2.1 For every $(\bar{u}, \bar{v}, u) \in \mathbf{M}^3$, $\bar{u} \neq u$, we will denote by $\Phi[\bar{u}, \bar{v}, u]$ the transformation on $\mathbf{M}$ defined for every $\xi \in \mathbf{M}$ by

$$\Phi[\bar{u}, \bar{v}, u](\xi) = x \Leftrightarrow t(x, \bar{u}, \bar{v}) = t(x, u, \xi).$$

2.2 For every $[u, v] \in \mathbf{M}^2$ we will denote by $\varphi[u, v]$ the transformation on $\mathbf{M}$ defined for every $v \in \mathbf{M}$ by

$$\varphi[u, v](x) = t(u, v, x).$$

Now, let us consider the case when the set $\mathbf{M}$ is ordered by certain suitable way.[3]

Definition 3 Let $(\mathbf{M}, t)$ be a HPTR with $\operatorname{card} \mathbf{M} \geq 3$ and let $(\mathbf{M}, <)$ be a linearly ordered set. Then the HPTR $(\mathbf{M}, t)$ is said *admissible* (abb. *APTR*) if

(1) $\forall \bar{u}, \bar{v}, u \in \mathbf{M},\ \bar{u} \neq u:\ \Phi[u, v, u]$ is a monotone mapping,
(2) $\forall u, v \in \mathbf{M}:\ \varphi[u, v]$ is a monotone mapping,
(3) $0 < e$.

The admissible HPTR will be denote by $(\mathbf{M}, t, <)$.

Remark 4 It follows from the Definition 1 and from 2 that the transformations $\Phi[u, v, u]$ and $\varphi[u, v]$, are permutations on $\mathbf{M}$. Therefore 3 implies that corresponding inverse mappings are monotone too.

Notation 5 On HPTR $(\mathbf{M}, t)$ two binary operation may be defined by the following usual way

(1) $\forall a, b \in \mathbf{M}:\ a + b = t(a, e, b)$,
(2) $\forall a, b \in \mathbf{M}:\ a\ .\ b = t(a, b, 0)$.

It is well known that $(\mathbf{M} - \{0\}, .)$, resp. $(M, +)$, forms a loop with the neutral element e, resp. 0.

There exists a natural question—are binary operations + and . compatible with ordering on $\mathbf{M}$?

Let us investigate this by the properties of the ternary operation t.

Proposition 6 *The transformation* $\alpha[u, v]$ *of* $\mathbf{M}$ *defined by*

$$\alpha[u, v](x) = t(x, u, v)$$

is monotone for every $u, v \in \mathbf{M}$, $u \neq 0$.

[3]The demanded properties of ordering on $\mathbf{M}$ and of transformations $\Phi[\ldots]$ and $\varphi[\ldots]$ seem to be natural with respect to the geometric interpretation of mentioned transformations in the projective plane which is coordinated by the considered HPTR.

Proof With respect to definitions above we may for $u \neq 0$ write

$$\alpha[u,v](x) = y \Leftrightarrow y = t(x,u,v) \Leftrightarrow t(x,u,v) = t(x,0,y) \Leftrightarrow x = \Phi[u,v,0](y)$$

which means that $\alpha[u,v]$ is the inverse of $\Phi[u,v,0]$. Now the propositions follows from the Remark 4. □

Using the previous proposition and (2) of Definition 3 we have:

Corollary 7 *The following mappings are monotone:*

$$\begin{array}{ll} (1) & \forall a \in \mathbf{M}: \ x \mapsto x + a, \\ (2) & \forall a \in \mathbf{M}, \ a \neq 0: \ x \mapsto x \,.\, a, \\ (3) & \forall a \in \mathbf{M}: \ x \mapsto a + x. \end{array}$$

Notation 8 Let $(\mathbf{M}, t, <)$ be an APTR. Let us define a mapping $f[u] : \mathbf{M} \to \mathbf{M}$ for every $u \in \mathbf{M}$, $u \neq 0$, by

$$f[u](x) = \xi \Leftrightarrow t(x,u,\xi) = 0.$$

Since $u \neq 0$ the f[u] is a permutation on $\mathbf{M}$. Using $f[u] = (\Phi[0,0,u])^{-1}$ and the Remark 4 we have:

Proposition 9 *The transformation $f[u]$ is monotone for every $u \in \mathbf{M}$, $u \neq 0$.*

Lemma 10 *Let c be no maximal and at the same time no minimal element of* $\mathbf{M}$. *Then the transformation $\varphi[c,v]$ is increasing for every $v \in \mathbf{M}$.*

Proof If $v = 0$ or $c = 0$ then $\varphi[c,v](x) = x$ and the lemma is evident.

Let $v \neq 0 \wedge c \neq 0$, now.

I. Let $f[v]$ be increasing.

Let us suppose $c > 0$ and let b be an arbitrary element with $0 < c < b$. It implies that $0 = f[v](0) < f[v](c) < f[v](b)$.

Using the Proposition 6 and the unequality $0 < c < b$ we have moreover:

$$t(0,v,f[v](b)) < t(c,v,f[v](b)) < t(b,v,f[v](b))$$

or

$$t(0,v,f[v](b)) > t(c,v,f[v](b)) > t(b,v,f[v](b))$$

which means that

$$f[v](b) < t(c,v,f[v](b)) < 0 \quad \text{or} \quad f[v](b) > t(c,v,f[v](b)) > 0.$$

Since $f[v](b) > 0$ we obtain that $t(c,v,f[v](b)) > 0$.

Further we may write

$$\varphi[c,v](f[v](c)) = t(c,v,f[v](c)) = 0, \varphi[c,v](f[v](b)) = t(c,v,f[v](b)) > 0.$$

With respect to the fact $\varphi[u,v]$ is monotone the last relations imply that $\varphi[u,v]$ is increasing.

The case $c < 0$ may be solved analogously.

II. Let f[v] be decreasing. The proof will be analogical to the previous part. □

Lemma 11 *The APTR* $(\mathbf{M}, t, <)$ *has no maximal element.*

Proof Let c be the maximal element in (**M**,t). Since $0 < e$ we have $0 < c$.

Let us consider the monotone transformation $\varphi[c, e]$. If it is increasing the we obtain the following implications

$$0 < c \Rightarrow \varphi[c, e](0) < \varphi[c, e](c) \Rightarrow t(c, e, 0) < t(c, e, c) \Rightarrow c < c + c,$$

which contradics to the maximality of c.

Now, we have $\varphi[c, e]$ is decreasing, which means

$$0 < c \Rightarrow \varphi[c, e](0) > \varphi[c, e](c)$$

or equivalently

$$0 < c \Rightarrow c > c + c.$$

Let us suppose that there exists $y \in \mathbf{M}$ with $y < c+c$. Then we have exactly one $p \in \mathbf{M}$ s.t. $c + p = y$, which means $\varphi[c, e](p) = y$.

We may write $\varphi[c, e](p) = y < c + c = \varphi[c, e](c)$ and with the respect to the fact $\varphi[c, e]$ is decreasing we give from this $p > c$ which is not possible.

It follows from this that $c + c$ is the minimal element of **M**.

By the analogical way we may derive that $c + (c + c)$ is a maximal element of **M**, which yields $c + (c + c) = c$ and $c + c = 0$, consequently.

We have proved that 0 is the minimal element of **M**.

In **M** there exists at least one element b s.t. $0 \neq b \neq c$, which implies $0 < b < c$. It follows from the Lemma 10 that $\varphi[b, e]$ is increasing.

Let $\varphi[b, e](c) < c$. There exists $y \in \mathbf{M}$ s.t. $b + y = c$ or $\varphi[b, e](y) = c$, equivalently.

Respecting this fact we get

$$\varphi[b, e](c) < \varphi[b, e](y) \Rightarrow \ c < y,$$

which is a contradiction—therefore the maximality of c gives $\varphi[b, e](c) = c$.

It may be expressed by $t(b, e, c) = t(b, 0, c)$. Considering the evident relation $t(0, e, c) = t(0, 0, c)$, we (by (2) of 1.) have $b = 0$—a contradiction.

Therefore c is not the maximal element of **M**. □

Lemma 12 *The APTR* $(\mathbf{M}, t, <)$ *has no minimal element.*

Proof Let c be the minimal element in $(\mathbf{M}, t)$. Therefore either $c < 0$ or $c = 0$.

I. $c < 0$

In the case $\varphi[c, e]$ is increasing we may write

$$\varphi[c, e](c) < \varphi[c, e](0) \Rightarrow t(c, e, c) < t(c, e, 0) \Rightarrow c + c < c,$$

which contradics to the minimality of c—i.e. $\varphi[c, e]$ is decreasing.

Let us suppose the existence of $y \in \mathbf{M}$ with $y > c+c$. There exists (just one) $z \in \mathbf{M}$ s.t. $c + z = y$. Using the expressions $c + c = \varphi[c, e](c)$ and $y = \varphi[c, e](z)$

and respecting $\varphi[c,e]$ is decreasing we obtain $z < c$—a contradiction to the minimality of c. This implies that $c+c$ is the maximal element.

By an analogical way we may show the minimality of $c+(c+c)$. It follows from this that $c+c=0$ which means that 0 is the maximal element of $\mathbf{M}$. The maximality of 0 contradics to the (3) of 3.

Therefore c is not the minimal element of $\mathbf{M}$.

II. $c=0$

Let us choose some $b > 0$. Denoting by x the solution[4] of the equation $t(x,e,b)=0$ and respecting the fact $b \neq 0$ we get $0=\varphi[x,e](b)$, $x \neq 0$.

Since c is not maximal the transformation $\varphi[x,e]$ is increasing (according to Lemma 10) which means $\varphi[x,e](0) < \varphi[x,e](b)$. The last relation gives $x < 0$—a contradiction. □

Using Lemmas 10, 11 and 12 we have the two following propositions.

Theorem 13 *The APTR* $(\mathbf{M},t,<)$ *has no maximal and no minimal element.*

Proposition 14 *The transformation* $\varphi[u,v]$ *is increasing for every* $u,v \in \mathbf{M}$.

Now the compatibility of addition with the ordering on $\mathbf{M}$ may be shown.

Theorem 15 $\forall a,x,y \in \mathbf{M}:\ x<y \Rightarrow a+x<a+y \wedge x+a<y+a.$

Proof The Proposition 14 says

$$\forall u,a,x,y \in \mathbf{M}:\ x<y \Rightarrow t(u,a,x)<t(u,a,y),$$

which implies for $u=e$ especially

$$x<y \Rightarrow a+x<a+y. \qquad (*)$$

The transformation $x \mapsto x+a$ is monotone for every $a \in \mathbf{M}$ (see 7).

Considering $a \in \mathbf{M}, 0 < a$, and using $(*)$ we have $a+0 < a+a$. Since $a+0=0+a$ we get that $0+a<a+a$ from this. It implies that the considered transformation is increasing for every $a<0$.

The cases $a>0, a=0$ gives the same result.

Thus we have $\forall a,x,y \in \mathbf{M}: x<y \Rightarrow x+a<y+a.$ □

Notation 16 Let $k,l,m \in \mathbf{M}$. In what follows we will by the symbol μ denote the ternary relation on $\mathbf{M}$ defined by

$$(k,l,m) \in \mu \Leftrightarrow (k<l<m \vee\ k>l>m).$$

Lemma 17 *If* $u,v,\bar{u},\bar{v},a,b$ *are elements of* $\mathbf{M}$ *such that*

$$t(a,\bar{u},\bar{v})<t(a,u,v) \wedge\ t(b,\bar{u},\bar{v})>t(b,u,v),$$

then the following hold

(1) $u \neq \bar{u}$,
(2) $\exists! c \in \mathbf{M}: (a,c,b) \in \mu \wedge t(c,\bar{u},\bar{v})=t(c,u,v).$

[4]See the axiom (2) of Definition 1.

Proof Using the fact $\varphi[a,u]$ is increasing for every $a,u \in \mathbf{M}$ and the definition of mappings $\varphi[a,u]$ we have that $u = \bar{u}$ gives the following implication

$$t(a,\bar{u},\bar{v},) < t(a,u,v) \Rightarrow \varphi[a,u](\bar{v}) < \varphi[a,u](v) \Rightarrow \bar{v} < v.$$

It follows from this that $\varphi[b,u](\bar{v}) < \varphi[b,u](v)$, analogously. It yields

$$t(b,\bar{u},\bar{v}) < t(b,u,v),$$

a contradiction.

Thus $u \neq \bar{u}$ and there exists exactly one $c \in \mathbf{M}$ with $t(c,\bar{u},\bar{v},) = t(c,u,v)$, consequently (see Definition 1).

Let x,y be elements of $\mathbf{M}$ such that

$$t(a,\bar{u},\bar{v}) = t(a,u,x) \wedge t(b,\bar{u},\bar{v}) = t(b,u,y)$$

the existence of which is guarantied by the definition of HPTR.

Therefore we may write $\varphi[a,u](x) = t(a,u,x) < t(a,u,v) = \varphi[a,u](v)$ which implies that $x < v$. By the same way we will obtain that $v < y$. It means that $x < v < y$, summa summarum. Simultaneously, we have the following relations

$$a = \Phi[\bar{u},\bar{v},u](x), \quad c = \Phi[\bar{u},\bar{v},u](v), \quad b = \Phi[\bar{u},\bar{v},u](y),$$

which (according to 3.(1)) yield that $(a,c,b) \in \mu$. □

Lemma 18 *If $u,v,\bar{u},\bar{v},a,b$ are elements of $\mathbf{M}$ such that*

(1) $t(a,\bar{u},\bar{v},) < t(a,u,v)$,
(2) $u \neq \bar{u}$,
(3) $\exists c \in \mathbf{M} : (a,c,b) \in \mu \wedge t(c,\bar{u},\bar{v}) = t(c,u,v)$,

then $t(b,\bar{u},\bar{v}) > t(b,u,v)$.

Proof With respect to the definition of HPTR we get the elements x,y of $\mathbf{M}$ such that

$$t(a,\bar{u},\bar{v}) = t(a,u,x), \qquad (*)$$

$$t(b,\bar{u},\bar{v}) = t(b,u,y). \qquad (**)$$

Using the supposition (1) of this Lemma, the relation $(*)$ and the fact $\varphi[a,u]$ is increasing we obtain the following implication

$$\varphi[a,u](x) = t(a,u,x) < t(a,u,v) = \varphi[a,u](v) \Rightarrow x < v.$$

By the same way as in the previous proof we have that

$$a = \Phi[\bar{u},\bar{v},u](x), \quad c = \Phi[\bar{u},\bar{v},u](v), \quad b = \Phi[\bar{u},\bar{v},u](y).$$

Respecting $(a,c,b) \in \mu$ we obtain $(x,v,y) \in \mu$ from this. As $x < v$ we get the following chain of implications (by $(**)$)

$$(x,v,y) \in \mu \Rightarrow v < y \Rightarrow \varphi[b,u](v) < \varphi[b,u](y) \Rightarrow t(b,u,v) < t(b,u,y) = t(b,\bar{u},\bar{v}).$$

□

Notation 19 Let $(\mathbf{M}, t, <)$ be an APTR. Let us define a transformation $g[a, v]$ of $\mathbf{M}$ for every $a, v \in \mathbf{M}$, $a \neq 0$ by

$$g[a, v](u) = t(a, u, v).$$

Remark 20 With respect to the fact

$$g[a, v](u) = y \Leftrightarrow t(a, u, v) = y \Leftrightarrow t(a, u, v) = y \wedge t(0, u, v) = v$$

and to the (3) of the definition of HPTR we have, that $g[a, v]$ is for every $a \neq 0$ a permutation of $\mathbf{M}$.

Proposition 21 *Let $v \in \mathbf{M}$. If $\exists b \in \mathbf{M}$, $b > 0$, s.t. $g[b, v]$ is increasing, then*

$$\begin{array}{ll} (1) & a > 0 \Rightarrow g[a, v] \text{ is increasing,} \\ (2) & a < 0 \Rightarrow g[a, v] \text{ is decreasing.} \end{array}$$

Proof I. Let us suppose $a > 0$.

Considering some elements $x, y \in \mathbf{M}$, $x < y$, with $g[a, v](x) > g[a, v](y)$ we have that $t(a, x, v) > t(a, y, v)$ and $t(b, x, v) < t(b, y, v)$, simultaneously. According to the Lemma 17 there exists $c \in \mathbf{M}$ s.t. $(a, c, b) \in \mu \wedge t(c, x, v) = t(c, y, v)$. Putting $c = 0$ we get the unique solution of the equation $t(c, x, v) = t(c, y, v)$, evidently. But it means that $(a, 0, b) \in \mu$, which contradics to the suposition of the positivity of elements a, b.

II. Let us suppose $a < 0$.

In this case $(a, 0, b) \in \mu$. Let again x, y be elements of $\mathbf{M}$ satisfying $x < y$. Therefore we have that $t(b, x, v) > t(b, y, v)$ and $t(0, x, v) = t(0, y, v)$, indeed. According to the Lemma 18 we obtain that $t(a, x, v) < t(a, y, v)$ which means that $g[a, v](x) > g[a, v](y)$. □

Theorem 22 *Let $a \in \mathbf{M}$, $a \neq 0$. Then*

$$\begin{array}{ll} (1) & a > 0 \Rightarrow \text{the transformation } x \mapsto a \,.\, x \text{ is increasing,} \\ (2) & a < 0 \Rightarrow \text{the transformation } x \mapsto a \,.\, x \text{ is decreasing.} \end{array}$$

Proof Firstly, $ax = t(a, x, 0) = g[a, 0](x)$. Using the clear fact $g[e, 0](x) = x$ for every $x \in \mathbf{M}$ we get that the transformation $g[e, 0]$ is increasing. Now, this theorem follows from 21.

Now, let us prove the compatibility of multiplication with the ordering on $\mathbf{M}$. □

Theorem 23 $\forall a, x, y \in \mathbf{M}:$

$$\begin{array}{ll} (1) & 0 < a,\ x < y \Rightarrow x \,.\, a < y \,.\, a \wedge a \,.\, x < a \,.\, y, \\ (2) & 0 > a,\ x < y \Rightarrow x \,.\, a > y \,.\, a \wedge a \,.\, x > a \,.\, y. \end{array}$$

Proof It follows from 22 that

$$0 < a,\ x < y \Rightarrow\ ax < ay \tag{$*$}$$

$$0 > a,\ x < y \Rightarrow\ ax > ay \tag{$**$}$$

The transformation $x \mapsto x\ .\ a$ is monotone (due to 7).

If $0 < a$ then $(*)$ gives $a\ .\ 0 < a\ .\ a$. Since $a\ .\ 0 = 0 = 0\ .\ a$, we get that $0\ .\ a < a\ .\ a$. It means that the considered transformation is increasing for $a > 0$.

Using the relation $(**)$ we may the case $a < 0$ investigate analogously. □

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