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Approximation Properties of Certain Linear Positive Operators in Exponential Weighted Spaces

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Abstract

We introduce certain linear positive operators in exponential weighted spaces of functions of one variable and we study approximation properties of these operators.

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1 Introduction

1.1 Approximation properties of Szasz–Mirakyan operators

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \quad x \in R_0 := [0,+\infty), \ n \in N := \{1,2,\ldots\}, \tag{1}$$

in exponential weighted spaces C_q were examined in [1]. The space C_q, $q > 0$, considered in [1] is associated with the weighted function

$$v_q(x) := e^{-qx}, \quad x \in R_0, \tag{2}$$

and consists of all real-valued functions f continuous on R_0 for which $v_q f$ is uniformly continuous and bounded on R_0. The norm on C_q is defined by

$$\|f\|_q \equiv \|f(\cdot)\|_q := \sup_{x\in R_0} v_q(x)|f(x)|. \tag{3}$$

In [1] was proved that S_n is a positive linear operator from the space C_q into C_p provided that $p > q > 0$ and $n > n_0 > q/\ln(p/q)$. For $f \in C_q$ was proved that

$$v_p(x)|S_n(f;x) - f(x)| \le M_1(q)\omega_2\left(f; C_q; \sqrt{\frac{x}{n}}\right), \quad x \in R_0,\ n > n_0,$$

where $M_1(q) = \text{const.} > 0$ and $\omega_2(f; C_q; \cdot)$ is the modulus of smoothness of the order 2 defined by the formula

$$\omega_2(f; C_q; t) := \sup_{0\le h\le t} \|\Delta_h^2 f(\cdot)\|_q, \quad t \in R_0,$$

where $\Delta_h^2 f(x) := f(x) - 2f(x+h) + f(x+2h)$ for $x, h \in R_0$.

In this paper by $M_k(\alpha, \beta)$ we shall denote suitable positive constants depending only on indicated parameters α, β.

1.2 In this paper we modify the formula (1), i.e. we introduce operators $A_n(f; q, r; \cdot)$ in the space C_q by the following definition.

Definition 1 Let $r \in N$ and $q > 0$ be fixed numbers. For $f \in C_q$ we introduce operators $A_n(f;\cdot) \equiv A_n(f; q, r; \cdot)$ by the formula

$$A_n(f;q,r;x) := \frac{1}{g(nx+1;r)} \sum_{k=0}^{\infty} \frac{(nx+1)^k}{(k+r)!} f\left(\frac{k+r}{n+q}\right), \quad x \in R_0,\ n \in N, \tag{4}$$

where

$$g(t;r) := \sum_{k=0}^{\infty} \frac{t^k}{(k+r)!}, \quad t \in R_0, \tag{5}$$

i.e.

$$g(0;r) = \frac{1}{r!}, \qquad g(t,r) = \frac{1}{t^r}\left(e^t - \sum_{j=0}^{r-1} \frac{t^j}{j!}\right) \quad \text{if } t > 0.$$

In Section 2 we shall prove that $A_n(f; q, r)$, $n \in N$, is a positive linear operator from the space C_q into C_q. Moreover we shall give approximation theorems for these operators.

We shall apply the modulus of continuity of $f \in C_q$ defined by

$$\omega_1(f; C_q; t) := \sup_{0\le h\le t} \|\Delta_h f(\cdot)\|_q, \quad t \in R_0, \tag{6}$$

where $\Delta_h f(x) := f(x+h) - f(x)$ for $x, h \in R_0$. From (6) it follows that

$$\lim_{t\to 0+} \omega_1(f; C_q; t) = 0 \tag{7}$$

for every $f \in C_q$, $q > 0$. Moreover if $f \in C_q^1 = \{f \in C_q : f' \in C_q\}$, then

$$\omega_1(f; C_q; t) \le M_2 t, \quad t \in R_0\ (M_2 = \text{const.} > 0). \tag{8}$$

2 Main results

2.1 In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems.

By elementary calculations we obtain

Lemma 1 *Let $r \in N$ and $q > 0$ be fixed numbers. Then for all $x \in R_0$ and $n \in N$ we have*

$$A_n(1; q, r; x) = 1, \tag{9}$$

$$A_n(t; q, r; x) = \frac{nx+1}{n+q} + \frac{1}{(n+q)(r-1)!\, g(nx+1; r)},$$

$$A_n(t^2; q, r; x) = \left(\frac{nx+1}{n+q}\right)^2 + \frac{nx+1}{(n+1)^2} + \frac{nx+1+r}{(n+q)^2(r-1)!\, g(nx+1; r)},$$

$$A_n(e^{qt}; q, r; x) = \frac{g\left((nx+1)e^{q/(n+q)}; r\right)}{g(nx+1; r)} e^{qr/(n+q)}, \tag{10}$$

$$A_n(te^{qt}; q, r; x) =$$
$$= \frac{nx+1}{n+q} e^{q/(n+q)} A_n(e^{qt}; q, r; x) + \frac{1}{(n+q)(r-1)!\, g(nx+1; r)} e^{qr/(n+q)},$$

$$A_n(t^2 e^{qt}; q, r; x) = \left\{\left(\frac{nx+1}{n+q} e^{q/(n+q)}\right)^2 + \frac{nx+1}{(n+q)^2} e^{q/(n+q)}\right\} A_n(e^{qt}; q, r; x)$$
$$+ \frac{(nx+1)e^{q/(n+q)} + r}{(n+q)^2(r-1)!\, g(nx+1; r)} e^{qr/(n+q)}.$$

Moreover

$$A_n(t-x; q, r; x) = \frac{1-qx}{n+q} + \frac{1}{(n+q)(r-1)!\, g(nx+1; r)},$$

$$A_n((t-x)^2; q, r; x) =$$
$$= \left(\frac{1-qx}{n+q}\right)^2 + \frac{nx+1}{(n+1)^2} + \frac{1-nx-2qx+r}{(n+q)^2(r-1)!\, g(nx+1; r)}, \tag{11}$$

$$A_n((t-x)^2 e^{qt}; q, r; x) =$$
$$= \left\{\left(\frac{nx+1}{n+q} e^{q/(n+q)} - x\right)^2 + \frac{nx+1}{(n+q)^2} e^{q/(n+q)}\right\} A_n(e^{qt}; q, r; x)$$
$$+ \frac{(nx+1)e^{q/(n+q)} - 2x(n+q) + r}{(n+q)^2(r-1)!\, g(nx+1; r)} e^{qr/(n+q)}, \tag{12}$$

for $x \in R_0$ and $n \in N$.

Now we shall prove two fundamental lemmas.

Lemma 2 *For every fixed $q > 0$ and $r \in N$ there exists a positive constant $M_3(q,r)$, depending only on the parameters q and r, such that*

$$\|A_n(1/v_q(t);q,r;\cdot)\|_q \leq M_3(q,r), \quad n \in N. \tag{13}$$

Moreover for every function $f \in C_q$ we have

$$\|A_n(f;q,r;\cdot)\|_q \leq M_3(q,r)\|f\|_q, \qquad n \in N. \tag{14}$$

The formulas (4)–(5) and the inequality (14) show that $A_n(f;q,r;\cdot)$, $n \in N$, is a positive linear operator on C_q.

Proof From (2), (4), (5) and (10) we have for all $x \in R_0$, $n \in N$

$$v_q(x)A_n(1/v_q(t);q,r;x) = \frac{g\left((nx+1)e^{q/(n+q)};r\right)}{g(nx+1;r)}\, e^{qr/(n+q)-qx}$$

and

$$\frac{g\left((nx+1)e^{q/(n+q)};r\right)}{g(nx+1;r)} =$$

$$= \left[\frac{e^{(nx+1)(e^{q/(n+q)}-1)+nx+1}}{e^{nx+1}-\sum\limits_{j=0}^{r-1}\frac{(nx+1)^j}{j!}} + \frac{\sum\limits_{j=0}^{r-1}\frac{\left((nx+1)e^{q/(n+q)}\right)^j}{j!}}{e^{nx+1}-\sum\limits_{j=0}^{r-1}\frac{(nx+1)^j}{j!}}\right] e^{-qr/(n+q)}.$$

Using the inequality

$$e^{q/(n+q)} - 1 = \sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{q}{n+q}\right)^k < \sum_{k=1}^{\infty}\left(\frac{q}{n+q}\right)^k = \frac{q}{n},$$

we get by (5)

$$v_q(x)A_n(1/v_q(t);q,r;x) \leq \frac{e^{nx+1+q/n} - e^{-qx}\sum\limits_{j=0}^{r-1}\frac{(nx+1)^j}{j!}}{e^{nx+1}-\sum\limits_{j=0}^{r-1}\frac{(nx+1)^j}{j!}}$$

$$= \frac{e^{q/n}\left(e^{nx+1}-\sum\limits_{j=0}^{r-1}\frac{(nx+1)^j}{j!}\right) + \left(e^{q/n}-e^{-qx}\right)\sum\limits_{j=0}^{r-1}\frac{(nx+1)^j}{j!}}{e^{nx+1}-\sum\limits_{j=0}^{r-1}\frac{(nx+1)^j}{j!}}$$

$$\leq e^q\left(1 + \frac{\sum\limits_{j=0}^{r-1}\frac{(nx+1)^j}{j!}}{e^{nx+1}-\sum\limits_{j=0}^{r-1}\frac{(nx+1)^j}{j!}}\right) = e^q\left(1 + \frac{\sum\limits_{j=0}^{r-1}\frac{(nx+1)^j}{j!}}{g\left(nx+1;r\right)(nx+1)^r}\right)$$

for $x \in R_0$, $n \in N$. From (5) we also get

$$\frac{1}{g(t;r)} \leq r! \quad \text{for } t \in R_0. \tag{15}$$

Hence we can write

$$v_q(x) A_n(1/v_q(t); q, r; x) \leq M_3(q, r),$$

which implies (13).

The formula (4) and (3) yield

$$\|A_n(f(t); q, r; \cdot)\|_q \leq \|f\|_q \|A_n(1/v_q(t); q, r; \cdot)\|_q, \quad n \in N, \; r \in N,$$

for every $f \in C_q$. Applying (13), we obtain (14). This completes the proof of Lemma 2. □

Lemma 3 *For fixed $q > 0$ and $r \in N$ there exists a positive constant $M_4(q,r)$ such that*

$$v_q(x) A_n\left((t-x)^2/v_q(t); q, r; x\right) \leq M_4(q,r) \left[\left(\frac{x+1}{n+q}\right)^2 + \frac{x+1}{n+q}\right] \tag{16}$$

for all $x \in R_0$ and $n \in N$.

Proof From (2) and (12) it follows that

$$v_q(x) A_n\left((t-x)^2/v_q(t); q, r; x\right) =$$

$$= v_q(x) A_n\left(1/v_q(t); q, r; x\right) \left[\left(\frac{nx+1}{n+q} e^{q/(n+q)} - x\right)^2 + \frac{nx+1}{(n+q)^2} e^{q/(n+q)}\right]$$

$$+ \frac{(nx+1)e^{q/(n+q)} - 2x(n+q) + r}{(n+q)^2 (r-1)! g(nx+1; r)} e^{qr/(n+q) - qx}$$

for $x \in R_0$, $n, r \in N$. Observe that

$$\left(\frac{nx+1}{n+q} e^{q/(n+q)} - x\right)^2 \leq 2\left(\frac{nx+1}{n+q}\left(e^{q/(n+q)} - 1\right)\right)^2 + 2\left(\frac{nx+1}{n+q} - x\right)^2$$

for $x \in R_0$, $n \in N$. By the inequality $e^t - 1 \leq te^t$ for $t \in R_0$, we get

$$\left(\frac{nx+1}{n+q} e^{q/(n+q)} - x\right)^2 \leq 2e^2 q^2 \left(\frac{x+1}{n+q}\right)^2 + 2\left(\frac{1-qx}{n+q}\right)^2 \leq M_5(q) \left(\frac{x+1}{n+q}\right)^2,$$

$n \in N$. Applying (15) and the inequality $te^{-at} \leq a^{-1}$ for $a > 0$ and $t \in R_0$, we obtain

$$\frac{(nx+1)e^{q/(n+q)} - 2x(n+q) + r}{(n+q)^2 (r-1)! g(nx+1; r)} e^{qr/(n+q) - qx} \leq$$

$$\leq \frac{(n/q+1)e^{q/(n+q)} + 2\,(n+q)\,/q + r}{(n+q)^2} r e^{qr/(n+q)} \leq \frac{M_6(q,r)}{n+q}$$

for $x \in R_0$, $n \in N$. Using the above inequalities and (13), we get

$$v_q(x)A_n\left((t-x)^2/v_q(t);q,r;x\right) \le M_4(q,r)\left[\left(\frac{x+1}{n+q}\right)^2+\frac{x+1}{n+q}\right].$$

This ends the proof of (16). □

2.2 Now we shall give approximation theorems for A_n.

Theorem 1 *For every fixed $q > 0$ and $r \in N$ there exists a positive constant $M_7(q,r)$ such that for every $f \in C_q^1$ we have*

$$v_q(x)|A_n(f;q,r;x)-f(x)| \le M_7(q,r)\|f'\|_q\left[\left(\frac{x+1}{n+q}\right)^2+\frac{x+1}{n+q}\right]^{1/2}, \quad (17)$$

$x \in R_0$, $n \in N$.

Proof Let $x \in R_0$ be a fixed point. For $f \in C_q^1$ we have

$$f(t)-f(x)=\int_x^t f'(u)\,du, \quad t \in R_0.$$

From this and by (4) and (9) we get

$$A_n\left(f(t);q,r;x\right)-f(x)=A_n\left(\int_x^t f'(u)\,du;q,r;x\right), \quad n \in N.$$

But by (2) and (3) we have

$$\left|\int_x^t f'(u)\,du\right| \le \|f'\|_q\left(\frac{1}{v_q(t)}+\frac{1}{v_q(x)}\right)|t-x|, \quad t \in R_0.$$

This implies that

$$v_q(x)|A_n(f;q,r;x)-f(x)| \le$$
$$\le \|f'\|_q\left\{A_n\left(|t-x|;q,r;x\right)+v_q(x)A_n\left(|t-x|/v_q(t);q,r;x\right)\right\} \quad (18)$$

for $n \in N$. By the Hölder inequality, (9) and Lemmas 1–3, we obtain

$$A_n\left(|t-x|;q,r;x\right) \le \left\{A_n\left((t-x)^2;q,r;x\right)A_n\left(1;q,r;x\right)\right\}^{1/2}$$
$$\le M_8(q,r)\left[\left(\frac{x+1}{n+q}\right)^2+\frac{x+1}{n+q}\right]^{1/2}$$

and

$$v_q(x)A_n\left(|t-x|/v_q(t);q,r;x\right) \le$$
$$\le v_q(x)\left\{A_n\left((t-x)^2/v_q(t);q,r;x\right)\right\}^{1/2}\left\{A_n\left(1/v_q(t);q,r;x\right)\right\}^{1/2}$$
$$\le M_9(q,r)\left[\left(\frac{x+1}{n+q}\right)^2+\frac{x+1}{n+q}\right]^{1/2}, \quad n \in N.$$

From this and by (18) we immediately obtain (17). □

Theorem 2 *Suppose that $q > 0$, $r \in N$ are fixed numbers and $f \in C_q$. Then there exists a positive constant $M_{10}(q,r)$ such that*

$$v_q(x)|A_n(f;q,r;x) - f(x)| \le M_{10}(q,r)\omega_1\left(f;C_q;\left[\left(\frac{x+1}{n+q}\right)^2 + \frac{x+1}{n+q}\right]^{1/2}\right) \tag{19}$$

for all $x \in R_0$ and $n \in N$.

Proof We use Steklov function f_h of $f \in C_q$

$$f_h(x) := \frac{1}{h}\int_0^h f(x+t)\,dt, \quad x \in R_0, \; h > 0. \tag{20}$$

From (20) we get

$$f_h(x) - f(x) = \frac{1}{h}\int_0^h \Delta_t f(x)\,dt, \qquad f_h'(x) = \frac{1}{h}\Delta_h f(x), \; x \in R_0, \; h > 0.$$

This implies that $f_h \in C_q^1$ for $f \in C_q$ and $h > 0$. Moreover

$$\|f_h - f\|_q \le \omega_1(f;C_q;h), \tag{21}$$

$$\|f_h'\|_q \le h^{-1}\omega(f;C_q;h), \tag{22}$$

for $h > 0$. Observe that

$$v_q(x)|A_n(f;q,r;x) - f(x)| \le$$
$$\le v_q(x)\left[|A_n(f - f_h;q,r;x)| + |A_n(f_h;q,r;x) - f_h(x)| + |f_h(x) - f(x)|\right]$$
$$:= L_1(x) + L_2(x) + L_3(x)$$

for $x \in R_0$, $n \in N$, $r \in N$ and $h > 0$. From (14) and (21) we obtain

$$L_1(x) \le M_3(q,r)\|f_h - f\|_q \le M_3(q,r)\omega_1(f;C_q;h),$$
$$L_3(x) \le \omega_1(f:C_q;h).$$

Using Theorem 1 and (22), we get

$$L_2(x) \le M_7(q,r)\|f_h'\|_q\left[\left(\frac{x+1}{n+q}\right)^2 + \frac{x+1}{n+q}\right]^{1/2}$$

$$\le \frac{M_7(q,r)}{h}\left[\left(\frac{x+1}{n+q}\right)^2 + \frac{x+1}{n+q}\right]^{1/2}\omega_1(f;C_q;h).$$

Hence

$$v_q(x)|A_n(f;q,r;x) - f(x)| \le$$
$$\le \left(1 + M_3(q,r) + \frac{M_7(q,r)}{h}\left[\left(\frac{x+1}{n+q}\right)^2 + \frac{x+1}{n+q}\right]^{1/2}\right)\omega_1(f;C_q;h)$$

for $x \in R_0$, $n \in N$, $r \in N$ and $h > 0$. Setting

$$h = \left[\left(\frac{x+1}{n+q} \right)^2 + \frac{x+1}{n+q} \right]^{1/2},$$

for fixed $x \in R_0$, $n \in N$ and $q > 0$, we obtain the assertion of Theorem 2.

From Theorem 1 and Theorem 2 and by (5) we obtain

Corollary *If $f \in C_q$ with some $q > 0$ and $r \in N$, then*

$$\lim_{n\to\infty} \{A_n(f; q, r; x) - f(x)\} = 0 \tag{23}$$

for all $x \in R_0$. Moreover (23) *holds uniformly on every interval* $[x_1, x_2]$, $x_2 > x_1 \geq 0$.

Remark It is easily verified that analogous approximation properties hold for the following operators on C_q.

$$B_n(f; q, r; x) := \frac{1}{g(nx+1; r)} \sum_{k=0}^{\infty} \frac{(nx+1)^k}{(k+r)!} (n+q) \int_{(k+r)/(n+q)}^{(k+1+r)/(n+q)} f(t)\, dt \quad (f \in C_q)$$

for fixed $q > 0$, $x \in R_0$, $n \in N$ and $r \in N$.

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