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Archimedean GMV-chains

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Abstract

In this paper we examine *GMV*-algebras among naturally ordered monoids. We show that some classical results of Clifford, Fuchs and Hölder and some known properties of linearly ordered semigroups can be applied to prove that any archimedean *GMV*-chain is commutative and, moreover, the corresponding linearly ordered semigroup can be embedded into the usually ordered real interval $[0,1]$ equipped with the addition defined via $x \oplus y = \min(1, x+y)$.

Key words: *GMV*-algebra, archimedean *GMV*-chain, naturally ordered semigroup, archimedean linearly ordered monoid, archimedean ℓ-group.

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A *GMV-algebra* is an algebra $A = (A, \oplus, \neg, \sim, 0, 1)$ of type $\langle 2,1,1,0,0\rangle$ satisfying the following identities:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(A2) $x \oplus 0 = 0 \oplus x = x$,

(A3) $x \oplus 1 = 1 \oplus x = 1$,

(A4) $\neg 1 = \sim 1 = 0$,

(A5) $\neg(\sim x \oplus \sim y) = \sim(\neg x \oplus \neg y)$,

(A6) $x \oplus (y \odot \sim x) = y \oplus (x \odot \sim y) = (\neg y \odot x) \oplus y = (\neg x \odot y) \oplus x$,

(A7) $(\neg x \oplus y) \odot x = y \odot (x \oplus \sim y)$,

(A8) $\sim \neg x = x$,

where $x \odot y := \; \sim (\neg x \oplus \neg y)$.

GMV-algebras were introduced by J. Rachůnek in [10] and by G. Georgescu and A. Iorgulescu in [7] and [8] (called here *pseudo MV-algebras*; the operation $\odot$ was defined via $x \odot y = \; \sim (\neg y \oplus \neg x)$) as a noncommutative generalization of *MV*-algebras.

If we set $x \leq y$ if and only if $\neg x \oplus y = 1$ then $(A, \leq)$ is a distributive lattice with the join $x \vee y = x \oplus (y \odot \sim x)$ and the meet $x \wedge y = (\neg x \oplus y) \odot x$ (see e.g. [10]).

A *GMV*-algebra A is said to be a *GMV-chain* (or a *linearly ordered GMV-algebra*) if $(A, \leq)$ is a chain.

Let $G = (G, +, 0, -, \vee, \wedge)$ be an ℓ-group, $0 \leq u \in G$ and the operations on $\Gamma(G, u) = [0, u] = \{x \in G \mid 0 \leq x \leq u\}$ as follows:

$$\begin{aligned} x \oplus y &:= (x + y) \wedge u, \\ \neg x &:= u - x, \\ \sim x &:= -x + u, \\ 1 &:= u. \end{aligned}$$

Then $\Gamma(G, u) = ([0, u], \oplus, \neg, \sim, 0, 1)$ is a *GMV*-algebra. *GMV*-algebras in the form $\Gamma(G, u)$ are universal because by [3], Theorem 3.9, for any *GMV*-algebra A there exists an ℓ-group G and a strong order unit $0 \leq u \in G$ such that $A \cong \Gamma(G, u)$.

Let us recall some notions (see e.g. [6] or [9]). A *partially ordered semigroup* is a system $S = (S, +, \leq)$ such that $(S, +)$ is a semigroup, $(S, \leq)$ is a partially ordered set and the partial order relation $\leq$ is compatible with the addition $+$, i.e., for all $x, y, z \in S$, $x \leq y$ implies $x + z \leq y + z$ and $z + x \leq z + y$.

A *positive cone* of a partially ordered semigroup S is the set

$$S^+ = \{a \in S \mid a + x \geq x \leq x + a \text{ for all } x \in S\}.$$

Similarly, a *negative cone* of S is the set

$$S^- = \{a \in S \mid a + x \leq x \geq x + a \text{ for all } x \in S\}.$$

Obviously, if $S = (S, +, 0, \leq)$ is a partially ordered monoid then $S^+ = \{a \in S \mid a \geq 0\}$ and $S^- = \{a \in S \mid a \leq 0\}$.

We say that S is a *naturally ordered semigroup* if $S = S^+$ and for any $x, y \in S$, $x < y$ iff $y = x + s = t + x$ for some $s, t \in S$.

Summarizing some basic properties of *GMV*-algebras we obtain the following lemma.

Lemma 1 *Let A be a GMV-algebra. Then $(A, \oplus, 0, \leq)$ and $(A, \odot, 1, \leq)$ are partially ordered monoids. Moreover, $(A, \oplus, 0, \leq)$ is naturally ordered.*

Proof It is well-known that $(A, \oplus, 0)$ and $(A, \odot, 1)$ are monoids and $\leq$ is compatible with $\oplus$ and $\odot$. Hence, $(A, \oplus, 0, \leq)$ and $(A, \odot, 1, \leq)$ are partially ordered monoids. Since $x \vee y = x \oplus (y \odot \sim x) = (\neg x \odot y) \oplus x$, it is clear that $(A, \oplus, 0, \leq)$ is naturally ordered. □

A linearly ordered monoid $S = (S, +, 0, \leq)$ satisfying the conditions (i) and (ii) is called an *archimedean linearly ordered monoid*:

(i) for all $x, y \in S^+$, if $nx < y$ for all $n \in \mathbb{N}$ then $x = 0$,

(ii) for all $x, y \in S^-$, if $nx > y$ for all $n \in \mathbb{N}$ then $x = 0$.

There are three typical examples of archimedean naturally linearly ordered semigroups ($\leq$ denotes the usual ordering of $\mathbb{R}$) (see [6]):

Example 1 $S_1 = (\mathbb{R}_{\geq 0}, +, \leq)$, i.e. the set of all real numbers $x \geq 0$ with respect to the usual addition $+$.

Example 2 $S_2 = ([0,1], \oplus, \leq)$, where $x \oplus y := \min(1, x + y)$.

Example 3 $S_3 = ([0,1] \cup \{u\}, \boxplus, \leq)$, where $u \in \mathbb{R}$ is such that $u > x$ for all $x \in [0,1]$ and the addition is defined via

$$x \boxplus y := \begin{cases} x + y & \text{for } x + y \leq 1, \\ u & \text{for } x + y > 1. \end{cases}$$

Some properties of archimedean naturally linearly ordered semigroups were studied in [6], where it is proved that any archimedean naturally linearly ordered semigroup S is commutative (see Hölder–Fuchs theorem). Moreover, by Hölder–Clifford theorem, S can be embedded into S_1 (if and only if S is a semigroup with the cancellation property), S_2 or S_3.

Lemma 2 *Let A be a GMV-chain. Then $(A, \oplus, 0, \leq)$ is an archimedean linearly ordered monoid if and only if $(A, \odot, 1, \leq)$ is an archimedean linearly ordered monoid.*

Proof Let us denote $n \odot x = x \oplus \ldots \oplus x$ (n times) and $x^n = x \odot \ldots \odot x$ (n times) for $x \in A$, $n \in \mathbb{N}$. For any $x \in A$ and $n \in \mathbb{N}$ we have

$$x^n = \neg(n \odot \sim x) = \sim (n \odot \neg x) \text{ and } n \odot x = \neg(\sim x)^n = \sim (\neg x)^n.$$

(1) Let $(A, \oplus, 0, \leq)$ be archimedean. By (i), for each $0 \neq \neg x \in A$, i.e. $x \neq 1$, and $\neg y \in A$ there is $k \in \mathbb{N}$ such that $k \odot \neg x \geq \neg y$. Therefore $x^k = \sim (k \odot \neg x) \leq y$ and $(A, \odot, 1, \leq)$ is an archimedean linearly ordered monoid.

(2) Conversely, suppose that $(A, \odot, 1, \leq)$ is archimedean. Let $1 \neq \neg x \in A$, i.e. $x \neq 0$, $\neg y \in A$. By (ii), $(\neg x)^k \leq \neg y$ for some $k \in \mathbb{N}$. Hence $k \odot x = \sim (\neg x)^k \geq y$. □

In [3], [4] and [5], archimedean *GMV*-algebras are introduced by means of the *partial addition* $+$ in *GMV*-algebras. We define $x + y$ if and only if $x \leq \neg y$, and in this case $x + y := x \oplus y$.

A *GMV*-algebra A is called *archimedean* if the existence of $nx = x + \ldots + x$ (n times) for any $n \in \mathbb{N}$ entails $x = 0$.

Now, we will show a connection between archimedean linearly ordered semigroups and archimedean *GMV*-chains.

Lemma 3 *A GMV-chain A is an archimedean GMV-algebra if an only if* $(A, \oplus, 0, \leq)$, *and also* $(A, \odot, 1, \leq)$, *is an archimedean linearly ordered monoid.*

Proof (1) Let A be an archimedean *GMV*-chain and $x, y \in A$. Suppose that $n \odot x < y$ for each $n \in \mathbb{N}$. Then by [5], Proposition 6.4.21, there exists nx for any $n \in \mathbb{N}$. Hence $x = 0$.

(2) Let $(A, \oplus, 0, \leq)$ be an archimedean linearly ordered monoid. If nx exists in A for all $n \in \mathbb{N}$ then $n \odot x \leq \neg x$ for all $n \in \mathbb{N}$. Hence $x = 0$. Indeed, since $x > 0$ implies $\neg x < 1$, we obtain $n \odot x < 1$ for each $n \geq 1$, which yields $x = 0$. □

Archimedean *GMV*-chains were investigated in [4] and [5] (as *archimedean linear pseudo MV-algebras*). There it is proved that every archimedean *GMV*-chain is commutative. This result was generalized in [3], Theorem 4.2, for archimedean (not necessarily linearly ordered) *GMV*-algebras.

We give a different proof of the commutativity of archimedean *GMV*-chains by means of classical results from [6], the theorems of Clifford, Fuchs and Hölder.

Theorem 4 ([4], Theorem 4.3, [5], Theorem 6.4.23) *Every archimedean GMV-chain is commutative, i.e. an MV-algebra.*

Proof By Lemmas 1 and 3, archimedean *GMV*-chains can be considered as archimedean naturally linearly ordered semigroups. But by [6], Hölder–Fuchs theorem, every such a linearly ordered semigroup is commutative. □

As a consequence of the theorem of Clifford and Hölder we get the next theorem.

Theorem 5 *If A is an archimedean GMV-chain then* $(A, \oplus, 0, \leq)$ *is isomorphic to some submonoid of* S_2.

Proof Suppose that A is a nontrivial archimedean *GMV*-chain. Clearly, $(A, \oplus, 0, \leq)$ can be embedded into S_2 or into S_3. The archimedeanicity entails that $0 \mapsto 0$ and $1 \mapsto 1$ or $1 \mapsto u$, respectively. If A is finite, $|A| = n$, then by Hölder–Fuchs theorem and its proof it follows that $(A, \oplus, 0, \leq)$ is isomorphic with $S(n-1) = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ with respect to $\oplus$ and $\leq$. If A is infinite then, again by Hölder–Fuchs theorem and its proof, it does not contain any atom and, consequently, any dual atom. Since every submonoid of S_3 without the element 1 can be embedded into S_2, we obtain that $(A, \oplus, 0, \leq)$ is isomorphic with some submonoid of S_2. □

Remark 1 An embedding of the linearly ordered monoid $(A, \oplus, 0, \leq)$ into S_2 is not necessarily an embedding of the MV-algebra A into the MV-algebra $\Gamma(\mathbb{R}, 1)$. For example, consider the following MV-chain $(M, \oplus, \neg, 0, 1)$:

$\oplus$	0	$\frac{2}{3}$	1
0	0	$\frac{2}{3}$	1
$\frac{2}{3}$	$\frac{2}{3}$	1	1
1	1	1	1

Then the linearly ordered monoid $(M, \oplus, 0, \leq)$ is a submonoid of S_2, but M is not a subalgebra in $\Gamma(\mathbb{R}, 1)$ since $\neg\frac{2}{3} = \frac{2}{3} \neq 1 - \frac{2}{3}$.

Recall that an *archimedean ℓ-group* is an ℓ-group G satisfying the following condition, for each $x, y \in G$:

$$\text{If } nx < y \text{ for all } n \in \mathbb{Z} \text{ then } x = 0.$$

It is known that a linearly ordered group is archimedean if and only if it is isomorphic to some subgroup in $(\mathbb{R}, +, 0, -, \leq)$ (see e.g. [6] or [9]).

Lemma 6 *Let G be an abelian linearly ordered group with a strong order unit $u \in G$. Then $\Gamma(G, u)$ is an archimedean MV-chain if and only if G is an archimedean linearly ordered group.*

Proof (1) Suppose that $\Gamma(G, u)$ is an archimedean MV-chain. We shall verify that for each $x, y \in G, x \neq 0$, there is $n \in \mathbb{Z}$ such that $nx \geq y$. Since u is a strong order unit in G, for each $y \in G$ there exists $k \in \mathbb{N}$ such that $ku \geq y$.

Case 1: If $x \geq u$ then $kx \geq ku \geq y$, i.e. $n = k$.

Case 2: For $0 < x < u$ there is $r \in \mathbb{N}$ such that $r \odot x = \min(rx, u) = u$, i.e. $rx \geq u$. Hence $(kr)x = k(rx) \geq ku \geq y$, so that $n = kr$.

Case 3: If $x < 0$ then $-x > 0$ and by the cases 1 and 2 $m(-x) \geq y$ for some $m \in \mathbb{N}$. It follows that $(-m)x \geq y$, i.e. $n = -m$.

(2) The converse is obvious. □

Remark 2 The previous assertion can be extended for arbitrary GMV-algebras. By [3], Proposition 4.1, a GMV-algebra $\Gamma(G, u)$ is archimedean if and only if G is an archimedean ℓ-group. The commutativity of archimedean GMV-algebras follows from the commutativity of archimedean ℓ-groups.

Corollary 7 *Any nontrivial archimedean GMV-chain is isomorphic with some subalgebra in $\Gamma(\mathbb{R}, 1)$.*

Proof By Theorem 4, any archimedean GMV-chain A is an MV-chain and, by [2], Lemma 6, any MV-chain can be viewed as an interval $\Gamma(G, u)$ in an abelian linearly ordered group G with a strong order unit u. Therefore, by Lemma 6, we can suppose that $A = \Gamma(G, u)$ for an archimedean linearly ordered group G and a strong order unit $u \in G$. However, every such an ordered group is (up to isomorphism) a subgroup of $\mathbb{R}$. □

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