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Regular Linear Model with the Nuisance Parameters with Constraints of the Type I *

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Abstract

The regular linear model in which the vector of the first order parameters is divided into two parts: to the vector of the useful parameters and to the vector of the nuisance parameters is considered. We study the situation when constraints of the type I are given on the useful parameters.

Key words: Regular linear regression model, useful and nuisance parameters, BLUE , constraints of the type I on the first order parameters.

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1 Introduction, notations

There are situations in practice, that in the linear model with useful and nuisance parameters some constraints are given on the useful parameters. For example (author prof. Kubáček) by measuring a gravimetrical closed traverse, possibly complexes of such closed traverses creating a net, the gravimeters with following insufficiency are used: their time drift is not insignificant. The registration of the measured quantity (which does not change) changes in time. These

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changes are modelled by polynomials of the 3rd and 4th degree with unknown parameters. These parameters are estimated from the results of measurement.

The values of the differences of gravitation which are the object of measurement have to fulfil the obvious condition on the closed traverse: the sum of the differences is equal to zero. In case of measuring on a net several of these conditions come into existence.

The conditions are obviously relevant to the useful parameters (the differences of accelerations of gravitation), no conditions are assigned to the nuisance parameters (the coefficients of the drift polynomials).

The following notation will be used throughout the paper:

R^n	the space of all n-dimensional real vectors;
$\mathbf{u}_p$, $\mathbf{A}_{m,n}$	the real column p-dimensional vector, the real $m \times n$ matrix;
$\mathbf{A}', r(\mathbf{A})$	the transpose, the rank of the matrix $\mathbf{A}$;
$\mathscr{M}(\mathbf{A}), Ker(\mathbf{A})$	the range, the null space of the matrix $\mathbf{A}$;
$\mathbf{A}^-$	a generalized inverse of a matrix $\mathbf{A}$ (satisfying $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$);
$\mathbf{A}^+$	the Moore-Penrose generalized inverse of a matrix $\mathbf{A}$ (satisfying $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}, \mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+, (\mathbf{A}\mathbf{A}^+)' = \mathbf{A}\mathbf{A}^+, (\mathbf{A}^+\mathbf{A})' = \mathbf{A}^+\mathbf{A}$);
$\mathbf{P}_A$	the orthogonal projector onto $\mathscr{M}(\mathbf{A})$;
$\mathbf{M}_A = \mathbf{I} - \mathbf{P}_A$	the orthogonal projector onto $\mathscr{M}^{\perp}(\mathbf{A}) = Ker(\mathbf{A}')$;
$\mathbf{I}_k$	the $k \times k$ identity matrix;
$\mathbf{0}_{m,n}$	the $m \times n$ null matrix.

If $\mathscr{M}(\mathbf{A}) \subset \mathscr{M}(\mathbf{S})$, $\mathbf{S}$ p.s.d., then the symbol $\mathbf{P}_A^{S^-}$ denotes the projector projecting vectors in $\mathscr{M}(\mathbf{S})$ onto $\mathscr{M}(\mathbf{A})$ along $\mathscr{M}(\mathbf{S}\mathbf{A}^{\perp})$. A general representation of all such projectors $\mathbf{P}_A^{S^-}$ is given by $\mathbf{A}(\mathbf{A}'\mathbf{S}^-\mathbf{A})^-\mathbf{A}'\mathbf{S}^- + \mathbf{B}(\mathbf{I} - \mathbf{S}\mathbf{S}^-)$, where $\mathbf{B}$ is arbitrary, (see [5, (2.14)]). $\mathbf{M}_A^{S^-} = \mathbf{I} - \mathbf{P}_A^{S^-}$.

Let us consider the following linear model

$$\mathbf{Y} = (\mathbf{W}, \mathbf{Z}) \begin{pmatrix} \beta \\ \kappa \end{pmatrix} + \varepsilon, \tag{1}$$

where $\mathbf{Y} = (\mathbf{Y}_1, \ldots, \mathbf{Y}_n)'$ is a random observation vector, $\beta \in R^r$ is a vector of the useful parameters, $\kappa \in R^s$ is a vector of the nuisance parameters, $\mathbf{W}_{n,r}$ is a design matrix belonging to the vector β, $\mathbf{Z}_{n,s}$ is a design matrix belonging to the vector κ.

We suppose that

1. $E(\mathbf{Y}) = \mathbf{W}\beta + \mathbf{Z}\kappa$, $\forall \beta \in R^r$, $\forall \kappa \in R^s$,

2. $var(\mathbf{Y}) = \Sigma_{\vartheta} = \sum_{i=1}^{p} \vartheta_i V_i$, $\forall \vartheta = (\vartheta_1, \ldots, \vartheta_p)' \in \underline{\vartheta} \subset R^p$, $V_1, \ldots, V_p$ given symmetric matrices,

3. $\underline{\vartheta} \subset R^p$ contains an open sphere in R^p,

4. if $\vartheta \in \underline{\vartheta}$, the matrix Σ_{ϑ} is positive semidefinite,

5. the matrix Σ_{ϑ} is not a function of the vector $(\beta', \kappa')'$.

If the matrix Σ_{ϑ} is positive definite for any $\vartheta \in \underline{\vartheta}$ and $r(\mathbf{W}, \mathbf{Z}) = r + s < n$, the model is said to be *regular*, (see [2, p.13]).

Theorem 1 *In the regular model (1) the ϑ-LBLUE of the parameter $(\beta', \kappa')'$ is given by*

$$\begin{pmatrix} \hat{\beta} \\ \hat{\kappa} \end{pmatrix} = \begin{pmatrix} (\mathbf{W}'\Sigma_\vartheta^{-1}\mathbf{M}_Z^{\Sigma_\vartheta^{-1}}\mathbf{W})^{-1}\mathbf{W}'\Sigma_\vartheta^{-1}\mathbf{M}_Z^{\Sigma_\vartheta^{-1}} \\ (\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{M}_W^{\Sigma_\vartheta^{-1} M_Z^{\Sigma_\vartheta^{-1}}} \end{pmatrix} \mathbf{Y}$$

$$= \begin{pmatrix} (\mathbf{W}'[\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z]^{+}\mathbf{W})^{-1}\mathbf{W}'[\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z]^{+} \\ (\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\{\mathbf{I} - \mathbf{W}[\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^{+}\mathbf{W}]^{-1}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^{+}\} \end{pmatrix} \mathbf{Y}.$$

Proof see [4, Theorem 1].

2 Regular model with constraints on the useful parameters

In the introduction the situation was described when the vector of the useful parameters has to fulfil some conditions.

Definition 1 (see [2, p. 57]) If the parametric space R^r of the parameter β in the linear model (1) is reduced into the linear manifold

$$\mathscr{B} = \{\beta : \beta \in R^r, \mathbf{b} + \mathbf{B}\beta = 0\}, \tag{2}$$

where $\mathbf{B}$ is a given $q \times r$ matrix and $b \in \mathscr{M}(\mathbf{B})$ is a given q-dimensional vector, then the model is called *linear model with constraints of the type I on the useful parameters.*

Consider the regular model $[\mathbf{Y}, (\mathbf{W}, \mathbf{Z}) \begin{pmatrix} \beta \\ \kappa \end{pmatrix}, \Sigma_\vartheta]$ with a system of constrains (2) on the useful parameters.

Let us suppose that

$$r(\mathbf{B}) = q < r.$$

Theorem 2 *In the regular model (1) with constrains (2) the ϑ-LBLUEs of the parameters β and κ are given by*

$$\hat{\hat{\beta}} = [\mathbf{I} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}]\hat{\beta} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B})^{-1}\mathbf{b},$$

$$\hat{\hat{\kappa}} = \hat{\kappa} + (\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}(\mathbf{b} + \mathbf{B}\hat{\beta}),$$

where $\hat{\beta}$, $\hat{\kappa}$ are the estimators in the regular model without constrains (see Theorem 1) and

$$\mathbf{C} = \mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^{+}\mathbf{W}.$$

The variance matrix of the estimator $\hat{\hat{\beta}}$ is given by

$$var(\hat{\hat{\beta}}) = [\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'}]^{+}.$$

Proof We will follow the proof of the [3, Theorem IV.3.1.]. Let $\mathbf{K}_B$ be the $r \times (r-q)$ matrix with $r(\mathbf{K}_B) = r - q$ and $\mathscr{M}(\mathbf{K}_B) = Ker(\mathbf{B})$. Let β_0 be an arbitrary but fixed particular solution of equation $\mathbf{b} + \mathbf{B}\beta = 0$. Then each vector $\beta \in \mathscr{B}$ can be expressed as

$$\beta = \beta_0 + \mathbf{K}_B\gamma, \quad \gamma \in R^{r-q}.$$

Hence the model (1) with constraints (2) is equivalent to the regular linear model (without constraints)

$$\mathbf{Y} - \mathbf{W}\beta_0 = (\mathbf{W}\mathbf{K}_B, \mathbf{Z}) \begin{pmatrix} \gamma \\ \kappa \end{pmatrix} + \varepsilon, \ \gamma \in R^{r-q}, \ \kappa \in R^s,$$

where

$$var(\varepsilon) = \Sigma_\vartheta, \quad r(\mathbf{W}\mathbf{K}_B, \mathbf{Z}) = r - q + s < n.$$

According to [2, Theorem 1.1.1.] the ϑ-LBLUE of the parameters in this model

$$\begin{pmatrix} \hat{\hat{\gamma}} \\ \hat{\hat{\kappa}} \end{pmatrix} = [(\mathbf{W}\mathbf{K}_B, \mathbf{Z})'\Sigma_\vartheta^{-1}(\mathbf{W}\mathbf{K}_B, \mathbf{Z})]^{-1}(\mathbf{W}\mathbf{K}_B, \mathbf{Z})'\Sigma_\vartheta^{-1}(\mathbf{Y} - \mathbf{W}\beta_0),$$

i.e.

$$(\mathbf{W}\mathbf{K}_B, \mathbf{Z}) \begin{pmatrix} \hat{\hat{\gamma}} \\ \hat{\hat{\kappa}} \end{pmatrix} = \mathbf{P}_{(WK_B, Z)}^{\Sigma_\vartheta^{-1}}[\mathbf{Y} - \mathbf{W}\beta_0] = \mathbf{P}_{(WM_{B'}, Z)}^{\Sigma_\vartheta^{-1}}[\mathbf{Y} - \mathbf{W}\beta_0].$$

$$\mathbf{P}_{(WM_{B'}, Z)}^{\Sigma_\vartheta^{-1}} = (\mathbf{W}\mathbf{M}_{B'}, \mathbf{Z})[(\mathbf{W}\mathbf{M}_{B'}, \mathbf{Z})'\Sigma_\vartheta^{-1}(\mathbf{W}\mathbf{M}_{B'}, \mathbf{Z})]^{-}(\mathbf{W}\mathbf{M}_{B'}, \mathbf{Z})'\Sigma_\vartheta^{-1}$$

$$= (\mathbf{W}\mathbf{M}_{B'}, \mathbf{Z}) \begin{bmatrix} \mathbf{M}_{B'}\mathbf{W}'\Sigma_\vartheta^{-1}\mathbf{W}\mathbf{M}_{B'}; & \mathbf{M}_{B'}\mathbf{W}'\Sigma_\vartheta^{-1}\mathbf{Z} \\ \mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}\mathbf{M}_{B'}; & \mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z} \end{bmatrix}^{-} \begin{pmatrix} \mathbf{M}_{B'}\mathbf{W}' \\ \mathbf{Z}' \end{pmatrix} \Sigma_\vartheta^{-1},$$

$$= (\mathbf{W}\mathbf{M}_{B'}, \mathbf{Z}) \begin{bmatrix} \mathbf{A}_{11}; & \mathbf{A}_{12} \\ \mathbf{A}_{21}; & \mathbf{A}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{M}_{B'}\mathbf{W}' \\ \mathbf{Z}' \end{pmatrix} \Sigma_\vartheta^{-1},$$

where (we have used Rohde's formula for inverse of partitioned p.d. matrix, see [1, Lemma 13])

$$\mathbf{A}_{11} = \left[\mathbf{M}_{B'}\mathbf{W}'(\Sigma_\vartheta^{-1} - \Sigma_\vartheta^{-1}\mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1})\mathbf{W}\mathbf{M}_{B'}\right]^{-}$$

$$= \left[\mathbf{M}_{B'}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^{+}\mathbf{W}\mathbf{M}_{B'}\right]^{-},$$

$$\mathbf{A}_{12} = (\mathbf{A}_{21})' = -\left[\mathbf{M}_{B'}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^{+}\mathbf{W}\mathbf{M}_{B'}\right]^{-}\mathbf{M}_{B'}\mathbf{W}'\Sigma_\vartheta^{-1}\mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1},$$

$$\mathbf{A}_{22} = (\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1} + (\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}\mathbf{M}_{B'}$$

$$\times \left[\mathbf{M}_{B'}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^{+}\mathbf{W}\mathbf{M}_{B'}\right]^{-}\mathbf{M}_{B'}\mathbf{W}'\Sigma_\vartheta^{-1}\mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}.$$

Thus

$$(\mathbf{W},\mathbf{Z})\widehat{\widehat{\begin{pmatrix}\mathbf{K}_B\gamma\\ \kappa\end{pmatrix}}} = \widehat{\widehat{\mathbf{W}\mathbf{K}_B\gamma}} + \widehat{\widehat{\mathbf{Z}\kappa}} = \mathbf{P}^{\Sigma_\vartheta^{-1}}_{(WM_{B'},Z)}(\mathbf{Y}-\mathbf{W}\beta_0)$$

$$= [\mathbf{W}\mathbf{M}_{B'}\mathbf{A}_{11}\mathbf{M}_{B'}\mathbf{W}' + \mathbf{W}\mathbf{M}_{B'}\mathbf{A}_{12}\mathbf{Z}' + \mathbf{Z}\mathbf{A}_{21}\mathbf{M}_{B'}\mathbf{W}' + \mathbf{Z}\mathbf{A}_{22}\mathbf{Z}'] \times \Sigma_\vartheta^{-1}(\mathbf{Y}-\mathbf{W}\beta_0). \tag{3}$$

Let us denote

$$\mathbf{C} = \mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^+\mathbf{W}.$$

In the following we use that

a) in the expression $\mathbf{W}\mathbf{M}_{B'}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^-\mathbf{M}_{B'}\mathbf{W}'$ an arbitrary generalized inverse can be used,

b) $\mathbf{M}_{B'}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^+\mathbf{M}_{B'} = (\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^+$,

c) $(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^+ = \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}$.

After some calculations we get

$$\widehat{\widehat{\mathbf{W}\mathbf{K}_B\gamma}} = \mathbf{W}\mathbf{M}_{B'}\left[\mathbf{M}_{B'}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^+\mathbf{W}\mathbf{M}_{B'}\right]^+$$

$$\times \mathbf{M}_{B'}\mathbf{W}'\{\Sigma_\vartheta^{-1} - \Sigma_\vartheta^{-1}\mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\}(\mathbf{Y}-\mathbf{W}\beta_0)$$

$$= \mathbf{W}[\mathbf{M}_{B'}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^+\mathbf{W}\mathbf{M}_{B'}]^+\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^+(\mathbf{Y}-\mathbf{W}\beta_0).$$

Thus

$$\widehat{\widehat{\mathbf{W}\mathbf{K}_B\gamma}} = \mathbf{W}[\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'}]^+\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^+(\mathbf{Y}-\mathbf{W}\beta_0)$$

$$= \mathbf{W}[\mathbf{I} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}]\mathbf{C}^{-1}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^+(\mathbf{Y}-\mathbf{W}\beta_0),$$

and

$$\widehat{\widehat{\mathbf{W}\beta}} = \mathbf{W}\beta_0 + \widehat{\widehat{\mathbf{W}\mathbf{K}_B\gamma}}$$

$$= \mathbf{W}\beta_0 + \mathbf{W}[\mathbf{I} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}]\mathbf{C}^{-1}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^+\mathbf{Y}$$

$$-\mathbf{W}[\mathbf{I} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}]\mathbf{C}^{-1}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^+\mathbf{W}\beta_0$$

$$= \mathbf{W}[\mathbf{I} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}]\mathbf{C}^{-1}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^+\mathbf{Y}$$

$$+ \mathbf{W}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\beta_0$$

$$= \mathbf{W}[\mathbf{I} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}]\hat{\beta} - \mathbf{W}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{b},$$

where

$$\hat{\beta} = [\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^+\mathbf{W}]^{-1}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^+\mathbf{Y} = \mathbf{C}^{-1}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^+\mathbf{Y},$$

is the ϑ-LBLUE of β in the regular linear model without constraints.

$$\widehat{\widehat{\mathbf{Z}\kappa}} = [\mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1} + \mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^+$$

$$\times \mathbf{W}'\Sigma_\vartheta^{-1}\mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}$$

$$- \mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^+\mathbf{W}'\Sigma_\vartheta^{-1}](\mathbf{Y}-\mathbf{W}\beta_0)$$

$$= \mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}[\mathbf{I} - \mathbf{W}(\mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1})\mathbf{W}'\Sigma_\vartheta^{-1}\mathbf{M}_Z^{\Sigma_\vartheta^{-1}}]$$
$$\times (\mathbf{Y} - \mathbf{W}\beta_0) = \mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}(\mathbf{I} - \mathbf{W}\mathbf{C}^{-1}\mathbf{W}'\Sigma_\vartheta^{-1}\mathbf{M}_Z^{\Sigma_\vartheta^{-1}})\mathbf{Y}$$
$$+ \mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{W}'\Sigma_\vartheta^{-1}\mathbf{M}_Z^{\Sigma_\vartheta^{-1}}\mathbf{Y}$$
$$-\mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}[\mathbf{I} - \mathbf{W}(\mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1})\mathbf{W}'\Sigma_\vartheta^{-1}\mathbf{M}_Z^{\Sigma_\vartheta^{-1}}]\mathbf{W}\beta_0$$
$$= \mathbf{Z}\hat{\kappa} + \mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^{+}\mathbf{Y}$$
$$-\mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}\beta_0 + \mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}\mathbf{C}^{-1}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^{+}\mathbf{W}\beta_0$$
$$- \mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{W}'(\mathbf{M}_Z\Sigma_\vartheta\mathbf{M}_Z)^{+}\mathbf{W}\beta_0$$
$$= \mathbf{Z}\hat{\kappa} + \mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\hat{\beta}$$
$$+ \mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{b}$$
$$= \mathbf{Z}\hat{\kappa} + \mathbf{Z}(\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_\vartheta^{-1}\mathbf{W}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}[\mathbf{b} + \mathbf{B}\hat{\beta}].$$

As $var(\hat{\beta}) = \mathbf{C}^{-1}$, is easy to determine $var(\hat{\hat{\beta}})$. □

Theorem 3 *In the regular model (1) with constraints (2) on the useful parameters is the statistic* $\mathbf{g}'\mathbf{Y}$ *the UBLUE of its expectation if and only if*

$$\mathbf{g} \in \mathscr{K} = Ker\left(\sum_{i=1}^{p}\mathbf{V}_i\mathbf{M}_{(W,Z)}\mathbf{V}_i + \sum_{i=1}^{p}\mathbf{V}_i\mathbf{M}_Z\mathbf{P}^{M_Z}_{W(W'M_ZW)^{-1}B'}\mathbf{V}_i\right). \qquad (4)$$

Proof Model (1) with constraints (2) is equivalent to the regular linear model

$$\left[\mathbf{Y} - \mathbf{W}\beta_0, (\mathbf{W}\mathbf{K}_B, \mathbf{Z})\begin{pmatrix}\gamma\\ \kappa\end{pmatrix}, \Sigma_\vartheta\right],$$

where vector β_0 and the matrix $\mathbf{K}_B$ are given in the proof of the Theorem 2. According to [2, Theorem 1.2.1] the statistic $\mathbf{g}'(\mathbf{Y} - \mathbf{W}\beta_0)$ is the UBLUE of its expectation iff

$$\mathbf{g} \in Ker\left[\sum_{i=1}^{p}\mathbf{V}_i\mathbf{M}_{(WK_B,Z)}\mathbf{V}_i\right].$$

As

$$\mathbf{M}_{(WK_B,Z)} = \mathbf{I} - \mathbf{P}_{(WK_B,Z)} = \mathbf{I} - \mathbf{P}_{(WM_{B'},Z)},$$

we use

$$\mathbf{P}_{(WM_{B'},Z)} = (\mathbf{W}\mathbf{M}_{B'}, \mathbf{Z})[(\mathbf{W}\mathbf{M}_{B'}, \mathbf{Z})'(\mathbf{W}\mathbf{M}_{B'}, \mathbf{Z})]^{-}(\mathbf{W}\mathbf{M}_{B'}, \mathbf{Z})'$$

$$= (\mathbf{W}\mathbf{M}_{B'}, \mathbf{Z})\begin{bmatrix}\mathbf{M}_{B'}\mathbf{W}'\mathbf{W}\mathbf{M}_{B'}; & \mathbf{M}_{B'}\mathbf{W}'\mathbf{Z}\\ \mathbf{Z}'\mathbf{W}\mathbf{M}_{B'}; & \mathbf{Z}'\mathbf{Z}\end{bmatrix}^{-}\begin{pmatrix}\mathbf{M}_{B'}\mathbf{W}'\\ \mathbf{Z}'\end{pmatrix}.$$

Thus

$$\mathbf{P}_{(WM_{B'},Z)} = (\mathbf{W}\mathbf{M}_{B'}, \mathbf{Z})\begin{bmatrix}\mathbf{B}_{11}, & \mathbf{B}_{12}\\ \mathbf{B}_{21}, & \mathbf{B}_{22}\end{bmatrix}\begin{pmatrix}\mathbf{M}_{B'}\mathbf{W}'\\ \mathbf{Z}'\end{pmatrix},$$

where

$$\mathbf{B}_{11} = \left[\mathbf{M}_{B'}\mathbf{W}'(\mathbf{I}-\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}')\mathbf{W}\mathbf{M}_{B'}\right]^{-} = \left[\mathbf{M}_{B'}\mathbf{W}'\mathbf{M}_Z\mathbf{W}\mathbf{M}_{B'}\right]^{-},$$

$$\mathbf{B}_{12} = (\mathbf{B}_{21})' = -\left[\mathbf{M}_{B'}\mathbf{W}'\mathbf{M}_Z\mathbf{W}\mathbf{M}_{B'}\right]^{-}\mathbf{M}_{B'}\mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1},$$

$$\mathbf{B}_{22} = (\mathbf{Z}'\mathbf{Z})^{-1} + (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{W}\mathbf{M}_{B'}\left[\mathbf{M}_{B'}\mathbf{W}'\mathbf{M}_Z\mathbf{W}\mathbf{M}_{B'}\right]^{-}\mathbf{M}_{B'}\mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}.$$

Let us denote

$$\mathbf{D} = \mathbf{W}'\mathbf{M}_Z\mathbf{W},$$

then

$$\begin{aligned}
&\mathbf{P}_{(WM_{B'},Z)} = \\
&= \mathbf{W}\mathbf{M}_{B'}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{M}_{B'}\mathbf{W}' - \mathbf{W}'\mathbf{M}_{B'}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{M}_{B'}\mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \\
&\quad - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{W}\mathbf{M}_{B'}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{M}_{B'}\mathbf{W}' + \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \\
&\quad + \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{W}\mathbf{M}_{B'}(\mathbf{M}_B'\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{M}_{B'}\mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \\
&= \mathbf{W}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{W}' - \mathbf{W}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \\
&\quad - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{W}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{W}' \\
&\quad + \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' + \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{W}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \\
&= \mathbf{W}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{W}' - \mathbf{W}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{W}'\mathbf{P}_Z - \mathbf{P}_Z\mathbf{W}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{W}' \\
&\quad + \mathbf{P}_Z + \mathbf{P}_Z\mathbf{W}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{W}'\mathbf{P}_Z.
\end{aligned}$$

Thus using that (see [4, Lemma])

$$\mathbf{M}_{(W,Z)} = \mathbf{M}_Z - \mathbf{M}_Z\mathbf{P}_W^{M_Z},$$

we get

$$\begin{aligned}
\mathbf{M}_{(WK_B,Z)} &= \mathbf{I} - \mathbf{P}_{(WM_{B'},Z)} \\
&= \mathbf{I} - \mathbf{P}_Z - \mathbf{W}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{W}'\mathbf{M}_Z + \mathbf{P}_Z\mathbf{W}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{W}'(\mathbf{I}-\mathbf{P}_Z) \\
&= \mathbf{M}_Z - \mathbf{W}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{W}'\mathbf{M}_Z + \mathbf{P}_Z\mathbf{W}(\mathbf{M}_{B'}\mathbf{D}\mathbf{M}_{B'})^{+}\mathbf{W}'\mathbf{M}_Z \\
&= \mathbf{M}_Z - \mathbf{M}_Z\mathbf{W}[\mathbf{D}^{-1} - \mathbf{D}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{D}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{D}^{-1}]\mathbf{W}'\mathbf{M}_Z \\
&= \mathbf{M}_Z - \mathbf{M}_Z\mathbf{W}(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1}\mathbf{W}'\mathbf{M}_Z \\
&\quad + \mathbf{M}_Z\mathbf{W}(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1}\mathbf{B}')^{-1}\mathbf{B}(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1}\mathbf{W}'\mathbf{M}_Z \\
&= (\mathbf{M}_Z - \mathbf{M}_Z\mathbf{P}_W^{M_Z}) + \mathbf{M}_Z\mathbf{P}_{W(W'M_ZW)^{-1}B'}^{M_Z} \\
&= \mathbf{M}_{(W,Z)} + \mathbf{M}_Z\mathbf{P}_{W(W'M_ZW)^{-1}B'}^{M_Z}.
\end{aligned}$$

Finally

$$\begin{aligned}
&Ker\left[\sum_{i=1}^{p}\mathbf{V}_i\mathbf{M}_{(WK_B,Z)}\mathbf{V}_i\right] \\
&= Ker\left[\sum_{i=1}^{p}\mathbf{V}_i\mathbf{M}_{(W,Z)}\mathbf{V}_i + \sum_{i=1}^{p}\mathbf{V}_i(\mathbf{M}_Z\mathbf{P}_{W(W'M_ZW)^{-1}B'}^{M_Z})\mathbf{V}_i\right].
\end{aligned}$$

□

Notation 1 Let **N** be such matrix that

$$\mathcal{M}(\mathbf{N}) = \mathcal{K} = Ker\left[\sum_{i=1}^{p} \mathbf{V}_i \mathbf{M}_{(W,Z)} \mathbf{V}_i + \sum_{i=1}^{p} \mathbf{V}_i (\mathbf{M}_Z \mathbf{P}^{M_Z}_{W(W'M_Z W)^{-1}B'}) \mathbf{V}_i\right].$$

Theorem 4 *In the regular model (1) with constraints (2) the function*

$$\mathbf{f}'\begin{pmatrix}\beta\\ \kappa\end{pmatrix} = (\mathbf{f}'_1, \mathbf{f}'_2)\begin{pmatrix}\beta\\ \kappa\end{pmatrix}$$

has its UBLUE iff

$$\mathbf{f} = \begin{pmatrix}\mathbf{f}_1\\ \mathbf{f}_2\end{pmatrix} \in \mathcal{M}\begin{pmatrix}\mathbf{W}'\mathbf{N}, & \mathbf{B}'\\ \mathbf{Z}'\mathbf{N}, & \mathbf{0}\end{pmatrix}. \tag{5}$$

Proof (see [2, Theorem 2.1.4]) The UBLUE of the function $\mathbf{f}'\begin{pmatrix}\beta\\ \kappa\end{pmatrix}$ exists iff there exists a vector
$\mathbf{g} \in \mathcal{K} = \mathcal{M}(\mathbf{N})$ and a number $c \in R^1$ such that

$$\mathbf{f}'\begin{pmatrix}\beta\\ \kappa\end{pmatrix} = \mathbf{g}'(\mathbf{W}, \mathbf{Z})\begin{pmatrix}\beta\\ \kappa\end{pmatrix} + \mathbf{c}, \quad \forall\beta \in \mathcal{B}, \quad \forall\kappa \in R^s,$$

what is equivalent to

$$(\mathbf{f}'_1, \mathbf{f}'_2)\begin{pmatrix}\beta_0 + \mathbf{K}_B\gamma\\ \kappa\end{pmatrix} = \mathbf{g}'(\mathbf{W}, \mathbf{Z})\begin{pmatrix}\beta_0 + \mathbf{K}_B\gamma\\ \kappa\end{pmatrix} + \mathbf{c},$$

i.e. equivalent to

$$\mathbf{f}'_1\beta_0 = \mathbf{g}'\mathbf{W}\beta_0 + c,$$

together with

$$\mathbf{f}_2 = \mathbf{Z}'\mathbf{g} \quad \& \quad \mathbf{K}'_B\mathbf{f}_1 = \mathbf{K}'_B\mathbf{W}'\mathbf{g}. \tag{6}$$

The number **c** always exists.

a) If

$$\begin{pmatrix}\mathbf{f}_1\\ \mathbf{f}_2\end{pmatrix} \in \mathcal{M}\begin{pmatrix}\mathbf{W}'\mathbf{N}, & \mathbf{B}'\\ \mathbf{Z}'\mathbf{N}, & \mathbf{0}\end{pmatrix},$$

then exists a vector $\mathbf{u} = \begin{pmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{pmatrix}$ such that

$$\mathbf{f}_1 = \mathbf{W}'\mathbf{N}\mathbf{u}_1 + \mathbf{B}'\mathbf{u}_2, \quad \& \quad \mathbf{f}_2 = \mathbf{Z}'\mathbf{N}\mathbf{u}_1.$$

Thus the vector $\mathbf{g} = \mathbf{N}\mathbf{u}_1 \in \mathcal{M}(\mathbf{N})$ exists such that

$$\mathbf{K}'_B\mathbf{f}_1 = \mathbf{K}'_B\mathbf{W}'\mathbf{g}, \quad \mathbf{f}_2 = \mathbf{Z}'\mathbf{g},$$

i.e. (6) is fulfilled.

b) Conversely let the vector $\mathbf{g} \in \mathcal{M}(\mathbf{N})$ exist, such that (6) holds:

$$\mathbf{K}'_B\mathbf{f}_1 = \mathbf{K}'_B\mathbf{W}'\mathbf{g}, \quad \& \quad \mathbf{f}_2 = \mathbf{Z}'\mathbf{g}.$$

Then $\mathbf{f}_2 \in \mathscr{M}(\mathbf{Z}'\mathbf{N})$ and

$$\mathbf{f}_1 \in \{\mathbf{v} \in R^r : \mathbf{W}'\mathbf{g} + [\mathbf{I} - (\mathbf{K}'_B)^-(\mathbf{K}'_B)]\mathbf{v}\},$$

since $\mathbf{W}'\mathbf{g}$ is a particular solution of the equation $\mathbf{K}'_B\mathbf{f}_1 = \mathbf{K}'_B\mathbf{W}'\mathbf{g}$. As

$$\mathscr{M}[\mathbf{I} - (\mathbf{K}'_B)^-(\mathbf{K}'_B)] = \mathscr{M}(\mathbf{B}'),$$

the vector $\mathbf{f}_1 = \mathbf{W}'\mathbf{g} + \mathbf{B}'\mathbf{v} = \mathbf{W}'\mathbf{N}\mathbf{u} + \mathbf{B}'\mathbf{v}$ belongs to $\mathscr{M}(\mathbf{W}'\mathbf{N}, \mathbf{B}')$, i.e.

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix} \in \mathscr{M}\begin{pmatrix} \mathbf{W}'\mathbf{N}, & \mathbf{B}' \\ \mathbf{Z}'\mathbf{N}, & \mathbf{0} \end{pmatrix}.$$

□

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