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Periodicity and Stability Results for Solutions of Some Fifth Order Non-Linear Differential Equations

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Abstract

This paper investigates the existence of a globally exponentially stable solution which is bounded and periodic (or almost periodic) for a class of fifth-order non-linear differential equations of the forms (1.1) and (1.2).

Key words: Periodic solutions, almost periodic solutions, bounded solutions, exponential stability and frequency-domain.

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1 Introduction

We shall be concerned here with the differential equations of the form:

$$x^{(v)} + ax^{(iv)} + bx''' + f(x'') + g(x') + ex = p(t) \tag{1.1}$$

and

$$x^{(v)} + ax^{(iv)} + bx''' + f(x'') + g_1(x)x' + ex = p(t) \tag{1.2}$$

where a, b, e are positive constants and f, g, g_1, and p are continuous functions which depend only on the arguments displayed explicitly.

The problem of interest here is to determine conditions on these functions under which all solutions of (1.1) and (1.2) are bounded, globally exponentially stable and periodic (or almost periodic). Many authors have worked on these properties of solutions for various kind of fifth order non-linear differential equations using Lyapunov's direct method (see, e.g., [1, 10, 15] and the references therein). However, our purpose here is to use frequency-domain method (see, eg., [2–9, 16–19] and [14, pp. 84–88]) to study the above mentioned properties for the solutions of (1.1) and (1.2).

In [6], Afuwape and Adesina recently initiated the use of the frequency-domain method to investigate these properties of solutions for (1.1) and (1.2) when $f(x'') = cx''$. The results obtained in this work generalize the results in [6] and also generalize to fifth order non-linear differential equations the results of Afuwape [4] and Barbalat [9]. The frequency-domain conditions obtained for equations (1.1) and (1.2) are necessary conditions for the existence of a positive definite Lyapunov function of the Luré–Postnikov form with a negative sign derivative. Our work shall utilize substantially, the generalized theorem of Yacubovich [7] represented by the following :

Theorem A *Consider the system*

$$X' = AX - B\varphi(\sigma) + P(t), \quad \sigma = C^{\star}X, \tag{1.3}$$

where A is an $n \times n$ real matrix, B and C are $n \times m$ real matrices with $C^{\star}$ as the transpose of C, $\varphi(\sigma) = col\varphi_j(\sigma_j)$, $(j = 1, 2, \dots, m)$ and $P(t)$ is an n-vector.

Suppose that in (1.3), the following assumptions are true:

(i) A is a stable matrix;

(ii) $P(t)$ is bounded for all t in $\mathbb{R}$;

(iii) for some constants $\hat{\mu}_j \geq 0$, $(j = 1, 2, \dots, m)$

$$0 \leq \frac{\varphi_j(\sigma_j) - \varphi_j(\hat{\sigma}_j)}{\sigma_j - \hat{\sigma}_j} \leq \hat{\mu}_j, \quad (\sigma_j \neq \hat{\sigma}_j), \tag{1.4}$$

(iv) there exists a diagonal matrix $D > 0$, such that the frequency-domain inequality

$$\pi(\omega) = MD + ReDG(i\omega) > 0 \tag{1.5}$$

holds for all ω in $\mathbb{R}$, where $G(i\omega) = C^{\star}(i\omega I - A)^{-1}B$ is the transfer function and $M = diag(\frac{1}{\hat{\mu}_j})$, $(j = 1, 2, \dots, m)$. Then, system (1.3) has the following properties.

(I) existence of a bounded solution which is globally exponentially stable;

(II) existence of a solution which is periodic (almost periodic).

Definition Following [7], we shall say that system

$$X' = A_1X - B_1\varphi(\sigma_1) + P_1(t), \quad \sigma_1 = C_1{}^{\star}X \tag{1.6}$$

is a dual to system (1.3) if $A_1 = A$, $B_1 = C$, $C_1 = B$ and $P_1(t) = TP(t)$, where T is a non-singular matrix transformation.

Theorem B ([7]) *The frequency-domain inequalities for dual systems are equivalent.*

2 Formulation of Results

We introduce the following: The Routh–Hurwitz conditions for stability of solutions of the linear homogeneous equation of (1.1) and (1.2) are:

$$\begin{gathered} a > 0, \quad (ab - c) > 0, \quad (ab - c)c - (ad - e)a > 0, \\ (ab - c)(cd - be) - (ad - e)^2 > 0, \quad e > 0 \end{gathered} \tag{2.1}$$

and the consequences of these conditions are:

$$b > 0, \quad c > 0, \quad cd - be > 0, \quad ad - e > 0. \tag{2.2}$$

The following notations shall be basic throughout this work. Equations $v^2a - vc + e = 0$ and $v^2 - vb + d = 0$ have two real positive roots given by v_1, v_2 and v_3, v_4 respectively, where

$$v_1 = \frac{1}{2a}\left[c - (c^2 - 4ae)^{\frac{1}{2}}\right] \tag{2.3}$$

$$v_2 = \frac{1}{2a}[c + (c^2 - 4ae)^{\frac{1}{2}}] \tag{2.4}$$

$$v_3 = \frac{1}{2}[b - (b^2 - 4d)^{\frac{1}{2}}] \tag{2.5}$$

$$v_4 = \frac{1}{2}[b + (b^2 - 4d)^{\frac{1}{2}}] \tag{2.6}$$

such that $b^2 - 4d > 0$, $c^2 - 4ae > 0$ and $0 < v_1 < v_3 < v_2 < v_4$.

The main objective of this paper is to prove the following:

Theorem 2.1 *Consider (1.1) where the functions f, g, and p are continuous with $f(0) = g(0) = 0$ and $p(t)$ bounded in $\mathbb{R}$. Suppose that there exist positive parameters c, d, μ_1 and μ_2 such that inequality*

$$(\mu_1\mu_2)^2 \leq 16(d\mu_2 - e\mu_1)(c\mu_1 - b\mu_2) \tag{2.7}$$

is satisfied and the functions f and g satisfy respectively the following inequalities

$$c \leq \frac{f(z) - f(\bar{z})}{z - \bar{z}} \leq c + \mu_1, \quad (z \neq \bar{z}) \tag{2.8}$$

$$d \leq \frac{g(z) - g(\bar{z})}{z - \bar{z}} \leq d + \mu_2, \quad (z \neq \bar{z}) \tag{2.9}$$

Then equation (1.1) has property (I) and if in addition $p(t)$ is periodic (or almost periodic), then it has property (II).

Theorem 2.2 *Let the functions f, g_1, and p be continuous in (1.2) with $f(0) = g_1(0) = 0$. Suppose that there exist positive parameters c, d, μ_1 and μ_2 such that inequalities (2.7), (2.8) and*

$$d \leq \frac{1}{x}\int_0^x g_1(s)ds \leq d + \mu_2, \quad (x \neq 0) \tag{2.10}$$

are satisfied, then equation (1.2) has property (I) and if in addition $p(t)$ is periodic (or almost periodic), then it has property (II).

3 Preliminary Results

The main tool in the proof of our theorems is the function $\pi(\omega)$ defined by inequality (1.5). For us to determine the function $\pi(\omega)$, we shall, by setting $x_1 = x$, reduce (1.1) to system (1.3) with

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}; \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -e & -d & -c & -b & -a \end{pmatrix};$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}; \qquad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix};$$

$$P(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ p(t) \end{pmatrix}; \qquad \varphi(\sigma) = \begin{pmatrix} \hat{g}(x_2) \\ \hat{f}(x_1) \end{pmatrix} \tag{3.1}$$

The transfer function $G(i\omega) = C^{\star}(i\omega I - A)^{-1}B$ for system (3.1) becomes

$$G(i\omega) = \frac{1}{\Delta}\begin{pmatrix} i\omega & i\omega \\ -\omega^2 & -\omega^2 \end{pmatrix} \tag{3.2}$$

where $\Delta = (\omega^4 a - \omega^2 c + e) + i\omega(\omega^4 - b\omega^2 + d)$. In order for us to get the function $\pi(\omega)$, we shall make use of the generalised Theorem of Yacubovich as given in the introduction and this requires the existence of strictly positive numbers τ_1 and τ_2 such that $D = \mathrm{diag}(\tau_j)$ and $M = \mathrm{diag}(\frac{1}{\mu_j})$ $(j = 1, 2,)$. After some calculations, we obtain

$$\pi(\omega) = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} > 0 \tag{3.3}$$

where

$$\pi_{11} = \tau_1\left(\mu_1^{-1} - \omega^2\frac{(\omega^4 a - \omega^2 c + e)}{|\Delta|^2}\right) \tag{3.4}$$

$$\pi_{22} = \tau_2\left(\mu_2^{-1} + \omega^2\frac{(\omega^4 - \omega^2 b + d)}{|\Delta|^2}\right) \tag{3.5}$$

$$\begin{aligned} \pi_{12} = \; & \tfrac{1}{2|\Delta|^2}\{\left(\omega^2\tau_2(\omega^4 - \omega^2 b + d) - \tau_1(\omega^4 a - \omega^2 c + e)\right) \\ & + i\omega\left(\tau_2(\omega^4 a - \omega^2 c + e) - \tau_1\omega^2(\omega^4 - \omega^2 b + d)\right)\} = \bar{\pi}_{21} \end{aligned} \tag{3.6}$$

with $\bar{\pi_{21}}$ as the complex conjugate of $\bar{\pi_{12}}$ and $|\Delta|^2 = \Delta\bar{\Delta}$. We shall employ Sylvester's criterion to verify inequality (3.3) and this requires that π_{11}, π_{22} and $\det \pi(\omega)$ be positive for all ω in $\mathbb{R}$. These shall now be proved in a series of lemma.

Lemma 1 *Let*

$$S_1(v) = \frac{(v^2a - vc + e)}{v} + \frac{(v^2 - vb + d)^2}{(v^2a - vc + e)}$$

where $\omega^2 = v$. *Then,* $\pi_{11}(\omega)$ *is positive for all* $v > 0$ *provided that*

$$\mu_1 \leq M_1(c) = S_1(v_0) = \min_{v_1 < v < v_2} S_1(v)$$

and

$$S_1(v_0) = S_1(v_3) = (ab - c) - \frac{2(ad - e)}{(b - (b^2 - 4d)^{\frac{1}{2}}}$$

where v_0 *is the unique real root of* $A_1(v) = 0$ *with* $v_1 < v_0 < v_2$ *and* $M_1(c)$ *is the minimum value of* $S_1(v)$ *and attainable at say* $v = v_0$.

Proof For $\pi_{11}(\omega)$ to be positive, we shall have from equation (3.4);

$$\mu_1 < \frac{(v^2a - vc + e)}{v} + \frac{(v^2 - vb + d)^2}{(v^2a - vc + e)} \tag{3.7}$$

Let

$$\mu_1 < S_1(v) = \frac{(v^2a - vc + e)}{v} + \frac{(v^2 - vb + d)^2}{(v^2a - vc + e)} \tag{3.8}$$

On differentiating the right hand side of inequality (3.8), we get

$$\begin{aligned} A_1(v) &= S_1'(v) \cdot (v^2a - vc - +e)^2 \\ &= v(2av - c)\{(v^2a - vc + e)^2 - v(v^2 - vb + d)^2\} \\ &\quad + (v^2a - vc + e)\{2v^2(2v - b)(v^2 - vb + d) - (v^2a - vc + e)^2\} \end{aligned}$$

Thus, $S_1(v)$ can be zero in the interval (v_1, v_2) if

$$\begin{aligned} A_1(v) = {} & 3av^7 + (a^3 - 2ab - 5c)v^6 + (2a^2c + 8ad + 4ab + 4e)v^5 \\ & + (a^2e + 3ac^2 + 2abd - b^2c - 6be - 2cd)v^4 \\ & + (5ace - 4ad^2 + 2bc - 2bcd + c^3 + 4de)v^3 \\ & + (-ae^2 - 2bde + c^2e - 2c^2 + cd^2)v^2 + ce^2v - e^3 = 0 \end{aligned} \tag{3.9}$$

On sketching the graph of $S_1(v)$ against v, we note that there are asymptotes at v_1 and v_2. Furthermore,

$$S_1(v_3) = (ab - c) - \frac{2(ad - e)}{b - (b^2 - 4d)^{\frac{1}{2}}} \tag{3.10}$$

and

$$S_1(v_4) = (ab - c) - \frac{2(ad - e)}{b - (b^2 + 4d)^{\frac{1}{2}}} \tag{3.11}$$

On substituting $v = \frac{c}{2a}$ into $S_1(v)$, we note that $S_1(\frac{c}{2a}) < 0$. Obviously, $S_1(v_3)$ and $S_1(v_4)$ are positive with $S_1(v_3) > S_1(v_4)$ Hence, $M_1(c) = S(v_0) \le S_1(v_3)$ with $A_1(v_0) = 0$. □

Lemma 2 *Let*

$$S_2(v) = (-v^2 + vb - d) + \frac{(v^2a - vc + e)^2}{v(-v^2 + vb - d)}$$

where $\omega^2 = v$. *Then,* $\pi_{22}(\omega)$ *is positive for all* $v > 0$, *provided that*

$$\mu_2 < M_2(d) = S_2(v_0) = \min_{v_3<v<v_4} S_2(v)$$

and

$$S_2(v_0) = S_2(v_2) = \frac{1}{2a}\left(b - \frac{c}{a}\right)\left(c + (c^2 - 4ae)^{\frac{1}{2}}\right) - \left(d - \frac{e}{a}\right)$$

where v_0 *is the unique real root of* $A_2(v) = 0$ *with* $v_3 < v_0 < v_4$ *and* $M_2(d)$ *is the minimum of* $S_2(v)$ *and attainable at say* $v = v_0$. *Furthermore, if* $v_2 \neq \frac{b}{2}$, *then,* $S_2(v_2) > S_2(\frac{b}{2})$ *and if* $v = \frac{b}{2}$ *with* $e = \frac{2bc - b^2a}{4}$ *and* $\varepsilon < 2b(b^2 - 4d)(b - \frac{c}{a})$, $\varepsilon > 0$, *then,* $S_2(v_2) > S_2(\frac{b}{2})$.

Proof For $\pi_{22}(\omega)$ to be positive for all $\omega \in \mathbb{R}$, the following inequality must be valid; $(\omega^2 = v)$

$$\mu_2 < (-v^2 + vb - d) + \frac{(v^2a - vc + e)^2}{v(-v^2 + vc - d)} \tag{3.12}$$

Let

$$\mu_2 < S_2(v) = (-v^2 + vb - d) + \frac{(v^2a - vc + e)^2}{v(-v^2 + vc - d)} \tag{3.13}$$

On differentiating $S_2(v)$, we have;

$$\begin{aligned} A_2(v) &= S_2'(v).v^2(-v^2 + vb - d)^2 \\ &= (b - 2v)\{v^2(-v^2 + vb - d)^2 - v(v^2a - vc + e)^2\} \\ &\quad + (v^2a - vc + e)(v^2a - vc - e)(-v^2 + vb - d) \end{aligned}$$

Obviously, $S_2'(v)$ can be zero in the interval (v_3, v_4) if

$$\begin{aligned} A_2(v) &= 2v^7 + 7(a^2 - 5b)v^6 + (4b^2 - 6a^2b - 12ac + 4d)v^5 \\ &\quad + (5a^2d - b^3 + 5c^2 + 10abc - 2ae - 4bd)v^4 \\ &\quad + (4b^2d - 4acd + 4bc^2 + 4ce + 5d^2)v^3 \\ &\quad + (4d^2 - 3c^2d - 2ade - 2bce - 3e)v^2 - (2be + d^3 - cde)v + de \\ &= 0 \end{aligned} \tag{3.14}$$

We also note that, on sketching the graph of $S_2(v)$ against v, there are asymptotes at v_3 and v_4. On substituting $v = \frac{b}{2}$ into $S_2(v)$, we shall have;

$$S_2\left(\frac{b}{2}\right) = \frac{b^2 - 4d}{4} + \frac{b(ab^2 - 2bc + 4e)^2}{2(b^2 - 4d)} \tag{3.15}$$

Similarly, we obtain

$$S_2(v_1) = \frac{1}{2a}\left(b - \frac{c}{a}\right)\left(c - (c^2 - 4ae)^{\frac{1}{2}}\right) - \left(d - \frac{e}{a}\right) \tag{3.16}$$

and

$$S_2(v_2) = \frac{1}{2a}\left(b - \frac{c}{a}\right)\left(c + (c^2 - 4ae)^{\frac{1}{2}}\right) - \left(d - \frac{e}{a}\right) \tag{3.17}$$

Let us consider the following cases with the relation:

$$e = \frac{2bc - b^2 a}{4}$$

Case I
If $v_2 = \frac{b}{2}$, then,

$$S_2(v_2) = \frac{b^2 - 4d}{4} = S_2(\frac{b}{2}) \tag{3.18}$$

Therefore, $S_2(v_2) = S_2\left(\frac{b}{2}\right)$.
Case II
If $v_2 > \frac{b}{2}$, then for some $\varepsilon > 0$, $v_2 = \frac{b}{2} + \varepsilon$, Thus,

$$S_2(v_2) = \frac{b^2 - 4d}{4} + \left(b - \frac{c}{a}\right)\varepsilon \tag{3.19}$$

and

$$S_2\left(\frac{b}{2}\right) = \frac{b^2 - 4d}{4} + \frac{2\varepsilon^2}{b(b^2 - 4d)} \tag{3.20}$$

Case III
If $v_2 < \frac{b}{2}$, then for some $\varepsilon > 0$, $v_2 = \frac{b}{2} - \varepsilon$. Thus,

$$S_2(v_2) = \frac{b^2 - 4d}{4} - \left(b - \frac{c}{a}\right)\varepsilon \tag{3.21}$$

and

$$S_2\left(\frac{b}{2}\right) = \frac{b^2 - 4d}{4} + \frac{2\varepsilon^2}{b(b^2 - 4d)} \tag{3.22}$$

On choosing $\varepsilon < 2b(b^2 - 4d)\left(b - \frac{c}{a}\right)$, we obtain the inequality $S_2(v_2) > S_2(\frac{b}{2})$. Hence, $M_2(d) = S(v_0) \leq S_2(v_2)$ with $A_2(v_0) = 0$. This completes the proof of Lemma 2. □

Lemma 3 *For all* $v > 0$, $\det \pi(\omega) > 0$ $(\omega^2 = v)$

Proof

$$\begin{aligned} \det \pi(\omega) &= \pi_{11}\pi_{22} - |\pi_{12}|^2 \\ &= \tau_1\tau_2\left(\frac{1}{\mu_1\mu_2} + \frac{v}{|\Delta|^2}\left(\frac{v^2 - vb + d}{\mu_1} - \frac{v^2 a - vc + e}{\mu_2} - \frac{{\tau_2}^2 + v{\tau_1}^2}{4\tau_1\tau_2}\right)\right) \end{aligned} \tag{3.23}$$

This will be positive for all $v > 0$ in $\mathbb{R}$, if

$$v^5 + (a^2 - 2b)v^4 + (b^2 - 2ac - 2d + \mu_2 - a\mu_1)v^3 + \left(2ae - c^2 - 2bd - b\mu_2 + c\mu_1 - \frac{\tau_1\mu_1\mu_2}{4\tau_2}\right)v^2 + \left(d^2 - 2ac + d\mu_2 - e\mu_1 - \frac{\tau_2\mu_1\mu_2}{4\tau_1}\right)v + e^2 > 0 \tag{3.24}$$

If $v = 0$, then $\det \pi(\omega) > 0$. But if $v \neq 0$, then by our choice of μ_1, μ_2 and the following inequalities;

$$a^2 > 2b, \quad b^2 > 2(ac + d), \quad c^2 < 2(ae - bd), \quad d^2 > 2ac \tag{3.25}$$

we have

$$\frac{\mu_1\mu_2}{4(c\mu_1 - b\mu_2)} < \frac{\tau_2}{\tau_1} < \frac{4(d\mu_2 - e\mu_1)}{\mu_1\mu_2} \tag{3.26}$$

Thus, $\pi_{22}\pi_{33} - |\pi_{23}|^2 > 0$ for all v in $\mathbb{R}$, if

$$(\mu_1\mu_2)^2 < 16(d\mu_2 - e\mu_1)(c\mu_1 - b\mu_2) \qquad \square$$

4 Proofs of the Main Results

We shall now give the proofs of the theorems stated in Section 2.

Proof of Theorem 2.1

Let $f(z) = cz + \hat{f}(z)$ and $g(z) = dz + \hat{g}(z)$ where c and d are positive parameters. By setting $x_1 = x$, the equation;

$$x^{(v)} + ax^{(iv)} + bx''' + f(x'') + g(x') + ex = p(t)$$

reduces to the equivalent form:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= x_5 \\ x_5' &= -ex_1 - dx_2 - cx_3 - bx_4 - ax_4 - \hat{f}(x_3) - \hat{g}(x_2) + p(t) \end{aligned}$$

and in vector form

$$X' = AX - B\varphi(\sigma) + P(t), \qquad \sigma = C^*X$$

with X, A, B, C, P and $\varphi(\sigma)$ as given in system (3.1). The frequency-domain condition reduces to the matrix inequality (3.3) which is satisfied for all ω in $\mathbb{R}$. This is true by using Lemmas 1, 2 and 3. The conclusions of Theorem 2.1 thus follow from the generalized theorem of Yacubovich.

Proof of Theorem 2.2

Let $f(z) = cz + \hat{f}(z)$ and $\int_0^x g_1(s)ds = dx + \hat{g}_1(x)$ with c and d as positive parameters. Then equation (1.2):

$$x^{(v)} + ax^{(iv)} + bx''' + f(x'') + g_1(x)x' + ex = p(t)$$

or its equivalent form:

$$\begin{aligned} x_1' &= -ex_5 \\ x_2' &= x_1 - dx_5 - \hat{g}(x_5) + p(t) \\ x_3' &= x_2 - cx_5 - \hat{f}(x_5) \\ x_4' &= x_3 - bx_5 \\ x_5' &= x_4 - ax_5 \end{aligned}$$

is a dual to system (3.1) with a non singular matrix transformation given by

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The conclusions to the proof thus follow from Theorem B and the proof of Theorem 2.1.

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