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Special Incidence Structures of Type (p, n) *

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Abstract

In [4] there are incidence structures of type (p, n) investigated. These are such incidence structures $\mathcal{J}$ that the corresponding incidence structure $\mathcal{J}^p$ of independent sets of $\mathcal{J}$ has its incidence graph in a simple-join-form. In this paper some special incidence structures of type (p, n) are examined. The conditions $R^i = R^{i+1}$ and $a'_i \not I\ m'_i$ (the donotation is introduced in [4]) are valid in them. The paper has two parts. At the end of part II the main theorem describing incidence graphs of such special incidence structures of type (p, n) is formulated.

Key words: Incidence structures, independent sets.

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Definition 1 Let G and M be sets and $I \subseteq G \times M$. Then the triple $\mathcal{J} = (G, M, I)$ is called an *incidence structure.*[1] If $A \subseteq G$, $B \subseteq M$, then we denote

$$A^{\uparrow} = \{m \in M \mid g\ I\ m \quad \forall g \in A\}, \quad B^{\downarrow} = \{g \in G \mid g\ I\ m \quad \forall m \in B\}.$$

And moreover, we denote $A^{\uparrow\downarrow} := (A^{\uparrow})^{\downarrow}$, $B^{\downarrow\uparrow} := (B^{\downarrow})^{\uparrow}$ for $A \subseteq G$, $B \subseteq M$ and $\{g\}^{\uparrow} := g^{\uparrow}$, $\{m\}^{\downarrow} := m^{\downarrow}$ for $g \in G$, $m \in M$.

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[1] The triple (G, M, I) is called an incidence structure with regard to consecutive applications. The name "kontext" is used more frequently in literature—see [1] where the denotations are taken from.

Definition 2 An incidence structure $\mathcal{J}_1 = (G_1, M_1, I_1)$ is *embedded* into an incidence structure $\mathcal{J} = (G, M, I)$ if $G_1 \subseteq G$, $M_1 \subseteq M$ and $I_1 \subseteq I \cap (G_1 \times M_1)$. If $I_1 = I \cap (G_1 \times M_1)$, then $\mathcal{J}_1$ is a *substructure* of $\mathcal{J}$.

A subset $A \subseteq G$ is *independent* in G if $a \notin (A - \{a\})^{\uparrow\downarrow}$ for all $a \in A$. In what follows we denote $A_a := A - \{a\}$.

If $A \subseteq G$, then we put $X^A(a) := A_a^{\uparrow} - \{a\}^{\uparrow}$ for $a \in A$. Then $X^A(a) = \emptyset$ iff $a \in A_a^{\uparrow\downarrow}$. Hence the set A is independent in G if and only if $X^A(a) \neq \emptyset$ for all $a \in A$. Moreover, $m \in X^A(a)$ iff $\{m\}^{\downarrow} \cap A = A_a$. (See [3].)

Let a non-empty set $A \subseteq G$ be independent in G. Then we put $\mathcal{X} = \{X^A(a) \mid a \in A\}$. For every choice $Q^A = \{m_a \in X^A(a) \mid X^A(a) \in \mathcal{X}\} \subseteq M$ from the set $\mathcal{X}$ (which exists according to the axiom of choice) we define an *A-norming map* $\alpha : A \to Q^A$ by the formula $\alpha(a) = m_a$ for all $a \in A$.

A set $B \subseteq M$ is independent in M if $m \notin (B - \{m\})^{\downarrow\uparrow} = B_m^{\downarrow\uparrow}$ for all $m \in M$. If $m \in B$, then we put $Y^B(m) = B_m^{\downarrow} - \{m\}^{\downarrow}$. B is independent in M if and only if $Y^B(m) \neq \emptyset$ for all $m \in B$. Moreover, $a \in Y^B(m)$ iff $\{a\}^{\uparrow} \cap B = B_m$.

Let a non-empty set $B \subseteq M$ be independent in M. Then we put $\mathcal{Y} = \{Y^B(m) \mid m \in B\}$. For every choice $Q^B = \{a_m \in Y^B(m) \mid Y^B(m) \in \mathcal{Y}\} \subseteq G$ we consider a map $\beta : B \to Q^B$ given by the formula $\beta(m) = a_m$. It will be called a *B-norming map*.

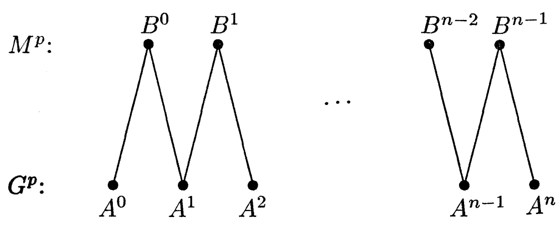
Let $A \subseteq G$, $B \subseteq M$ be independent sets in G, M, respectively. Then each A-norming map $A \to Q^A$ and each B-norming map $B \to Q^B$ are injective and the sets Q^A, Q^B are independent in M, G, respectively. (For the proof see [3].)

Definition 3 Let us consider an incidence structure $\mathcal{J} = (G, M, I)$ and a positive integer $p \geq 2$. Let G^p and M^p be the sets of all independent sets of G and M of cardinality p, respectively. Then $\mathcal{J}^p = (G^p, M^p, I^p)$ is an *incidence structure of independent sets* of $\mathcal{J}$, where AI^pB if and only if there exists an A-norming map $\alpha : A \to B$ for $A \in G^p$, $B \in M^p$.

Remark 1 If $G^p = \emptyset$, then $M^p = \emptyset$ and $\mathcal{J}^p = (\emptyset, \emptyset, \emptyset)$.

Definition 4 $\mathcal{J} = (G, M, I)$ is said to be an *incidence structure of type* (p, n), where $p > 1$, $n \geq 1$ are positive integers, if there is $G^p = \{A^0, \ldots, A^n\}$, $M^p = \{B^0, \ldots, B^{n-1}\}$ in $\mathcal{J}^p = (G^p, M^p, I^p)$ and $A^i\ I^p\ B^j$ iff $i = j$ or $i = j + 1$ for all $j \in \{0, \ldots, n-1\}$.

Remark 2 If $\mathcal{J}$ is a structure of type (p, n), then the incidence graph of the structure $\mathcal{J}^p$ can be drawn in the form



and $\mathcal{J}^p$ is called a *simple join*.

Theorem 1 *If $\mathcal{J}=(G,M,I)$ is an incidence structure of type (p,n), then*

(a) $|A^i \cap A^{i+1}| = p-1$ *for all* $i \in \{0,\dots,n-1\}$,

(b) $|B^i \cap B^{i+1}| = p-1$ *for all* $i \in \{0,\dots,n-2\}$.

Proof For the proof see [4].

Denotation In what follows we suppose that $R^i = A^i \cap A^{i+1}$, $A^i = \{a'_i\} \cup R^i$, $A^{i+1} = \{a_{i+1}\} \cup R^i$ for $i \in \{0,\dots,n-1\}$ and $Q^i = B^i \cap B^{i+1}$, $B^i = \{m'_i\} \cup Q^i$, $B^{i+1} = \{m_{i+1}\} \cup Q^i$ for $i \in \{0,\dots,n-2\}$.

In the following theorems there is always an incidence structure $\mathcal{J}=(G,M,I)$ of type (p,n) given and the previous denotations are respected.

Theorem 2

a) $a_i'^{\uparrow} \cap B^i = a_{i+1}^{\uparrow} \cap B^i$ *for all* $i \in \{0,\dots,n-1\}$,

b) $m_i'^{\downarrow} \cap A^{i+1} = m_{i+1}^{\downarrow} \cap A^{i+1}$ *for all* $i \in \{0,\dots,n-2\}$.

Theorem 3

a) $a'_i \in m_i'^{\downarrow} \iff a'_i \notin m_{i+1}^{\downarrow}$,

b) $m'_i \in a_{i+1}'^{\uparrow} \iff m'_i \notin a_{i+2}^{\uparrow}$,

for all $i \in \{0,\dots,n-2\}$.

Theorem 4

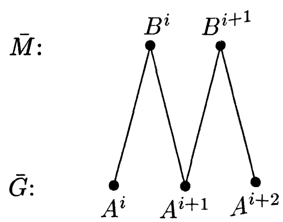
a) If $0 \le i \le n-2$, *then* $a'_i \neq a_{i+1}, a'_{i+1}, a_{i+2}, a'_{i+2}$,

b) If $0 \le i \le n-3$, *then* $m'_i \neq m_{i+1}, m'_{i+1}, m_{i+2}, m'_{i+2}$.

For the proofs of Theorems 2–4 see [4].

We will investigate special incidence structures $\mathcal{J}=(G,M,I)$ of type (p,n) in which $n \ge 2$ and $A^i \cap A^{i+1} = A^{i+1} \cap A^{i+2}$, that is $R^i = R^{i+1}$ for certain $i \in \{0,\dots,n-2\}$. We assume that $R^i = \{g_1,\dots,g_{p-1}\}$.

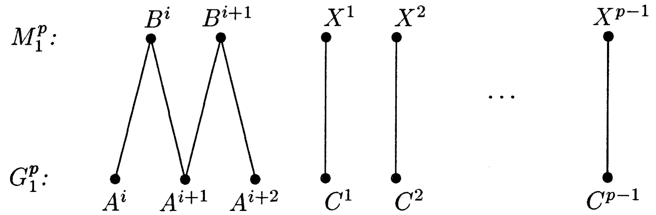
We have $A^i, A^{i+1}, A^{i+2} \subseteq G$; $B^i, B^{i+1} \subseteq M$ and $A^i, A^{i+1}, A^{i+2} \in G^p$; $B^i, B^{i+1} \in M^p$. Let us consider the substructure $\bar{\mathcal{J}} = (\bar{G},\bar{M},\bar{I})$ in $\mathcal{J}^p$ in which $\bar{G} = \{A^i, A^{i+1}, A^{i+2}\}$ and $\bar{M} = \{B^i, B^{i+1}\}$. Its incidence graph is



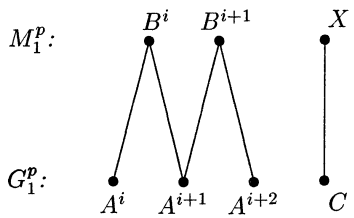
In what follows we consider a substructure $\mathcal{J}_1 = (G_1, M_1, I_1)$ in $\mathcal{J}$ where $G_1 = A^i \cup A^{i+1} \cup A^{i+2}$, $M_1 = B^i \cup B^{i+1}$.

By Theorem 3 in [3] $\mathcal{J}_1^p = (G_1^p, M_1^p, I_1^p)$ is a substructure of $\mathcal{J}^p$ and from Theorem 2 in [3] we get $A^i, A^{i+1}, A^{i+2} \in G_1^p$, $B^i, B^{i+1} \in M_1^p$. In the following theorem we show that $\mathcal{J}_1^p \neq \bar{\mathcal{J}}$.

Theorem 5 *If $a'_i \not{I}\, m'_i$, then the incidence structure $\mathcal{J}_1^p$ has a graph*



and if $a'_i\ I\ m'_i$, then it has a graph



Proof We have denoted $A^i = \{a'_i\} \cup R^i$, $A^{i+1} = \{a_{i+1}\} \cup R^i = \{a'_{i+1}\} \cup R^{i+1}$, $A^{i+2} = \{a_{i+2}\} \cup R^{i+1}$ and $B^i = \{m'_i\} \cup Q^i$, $B^{i+1} = \{m_{i+1}\} \cup Q^i$. Thus $a_{i+1} = a'_{i+1}$ because of $R^i = R^{i+1}$. Since $A^i\ I^p\ B^i$, there exists an A^i-norming mapping $\alpha : A^i \to B^i$.

1. Let us assume that $a'_i \not{I}\, m'_i$. Then $\alpha(a'_i) = m'_i$, $a_i'^{\uparrow} \cap B^i = B^i_{m'_i} = Q^i$ and $m_i'^{\downarrow} \cap A^i = A^i_{a'_i} = R^i$. If we put $\alpha(g_j) = n_j$ for $j \in \{1, \ldots, p-1\}$, then $\alpha(R^i) = Q^i$ and $g_j^{\uparrow} \cap B^i = B^i_{n_j}$ (for the incidence table of $\mathcal{J}_1$ see Figure 1).

I_1	n_1	. . . .	n_{p-1}	m'_i	m_{i+1}
g_1		- - - -	-	-	-
.	-	.	-	-	-
.	-	.	-	-	-
.	-	.	-	-	-
.	-	.	-	-	-
g_{p-1}	-	- - - -		-	-
a'_i	-	- - - -	-		-
a_{i+1}	-	- - - -	-		
a_{i+2}	-	- - - -	-	-	

Figure 1

From Theorem 4 it follows that $a'_i \neq a_{i+1}, a_{i+2}$ and $m'_i \neq m_{i+1}$. Since $a'_i \not I\, m'_i$ we obtain $a'_i \; I \; m_{i+1}$ by Theorem 3a). By Theorem 2a) we get $a'^{\uparrow}_i \cap B^i = a^{\uparrow}_{i+1} \cap B^i = Q^i$ which yields $a_{i+1} \not I\, m'_i$. From $a_{i+1} = a'_{i+1}$ and from Theorem 3b) we get $a_{i+2} \; I \; m'_i$. By Theorem 2b) we have $m'^{\downarrow}_i \cap A^{i+1} = m^{\downarrow}_{i+1} \cap A^{i+1}$. It follows that $R^i \subseteq m^{\downarrow}_{i+1}$ and $a_{i+1} \not I\, m_{i+1}$. Finally, $a^{\uparrow}_{i+1} \cap B^{i+1} = a^{\uparrow}_{i+2} \cap B^{i+1}$. This implies $Q^i \subseteq a^{\uparrow}_{i+2}$ and $a_{i+2} \not I\, m_{i+1}$. Now, the table of the incidence structure $\mathcal{J}_1$ is uniquely determined (up to isomorphism).

Let us consider a set $X^j = \{m'_i, m_{i+1}\} \cup (Q^i - \{n_j\}) = M_1 - \{n_j\}$ for each $j \in \{1, \ldots, p-1\}$. Then $\{a'_i\} = Y^{X_j}(m'_i)$, $\{a_{i+2}\} = Y^{X_j}(m_{i+1})$, $\{g_l\} = Y^{X_j}(n_l)$ for $l \in \{1, \ldots, j-1, j+1, \ldots, p-1\}$ and $X^j \in M^p_1$. If we put $C^j = \{a'_i, a_{i+2}\} \cup (R^i - \{g_j\})$, then $C^j \in G^p_1$ and $C^j \; I^p \; X^j$. We have obtained $G^p_1 = \{A^i, A^{i+1}, A^{i+2}, C^1, \ldots, C^{p-1}\}$, $M^p_1 = \{B^i, B^{i+1}, X^1, \ldots, X^{p-1}\}$ and $\mathcal{J}^p_1$ has a graph stated in Theorem.

2. Let us assume that $a'_i \; I \; m'_i$. Then $\alpha(a'_i) \neq m'_i$. By putting $\alpha(a'_i) = n$ we get $n \in Q^i$ and $a'^{\uparrow}_i \cap B^i = B^i_n$. Obviously, $\alpha(g_j) = m'_i$ for some $j \in \{1, \ldots, p-1\}$. For certainty we assume $j = p-1$ which yields $\alpha(g_{p-1}) = m'_i$ and $g^{\uparrow}_{p-1} \cap B^i = B^i_{m'_i} = Q^i$, $g_{p-1} \not I\, m'_i$. Let us denote $\alpha(g_l) = n_l$ for $l \in \{1, \ldots, p-2\}$. Then $g^{\uparrow}_l \cap B^i = B^i_{n_l}$ (Figure 2).

I_1	n	n_1	. . .	m'_i	m_{i+1}
g_1	-		- - -	-	-
.	-	-	.	-	-
.	-	-	.	-	-
.	-	-	.	-	-
g_{p-1}	-	-	- - -		
a'_i		-	- - -	-	
a_{i+1}		-	- - -	-	-
a_{i+2}		-	- - -		-

Figure 2

From Theorem 2a) we know that $a^{\uparrow}_{i+1} \cap B^i = B^i_n$. Since $a'_i \; I \; m'_i$, we obtain $a'_i \not I\, m_{i+1}$ according to Theorem 3a). Moreover, $m^{\downarrow}_{i+1} \cap A^{i+1} = A^{i+1}_{g_{p-1}}$ by Theorem 2b) and $a_{i+2} \not I\, m'_i$ by Theorem 3b). Finally, we get $a^{\uparrow}_{i+2} \cap B^{i+1} = B^{i+1}_n$. The table of the incidence structure $\mathcal{J}_1$ is completely determined again.

Let us put $X = \{m'_i, m_{i+1}\} \cup (Q^i - \{n\}) = M_1 - \{n\}$. Then $\{a'_i\} = Y^X(m_{i+1})$, $\{a_{i+2}\} = Y^X(m'_i)$ and $\{g_l\} = Y^X(n_l)$ for $l \in \{1, \ldots, p-2\}$. Thus $X \in M^p_1$. If we put $C = \{a'_i, a_{i+2}\} \cup (R^i - \{g_{p-1}\})$, then $C \in G^p_1$ and $C \; I^p \; X$. We have obtained $G^p_1 = \{A^i, A^{i+1}, A^{i+2}, C\}$, $M^p_1 = \{B^i, B^{i+1}, X\}$ and $\mathcal{J}^p_1$ has a graph stated in Theorem. □

Remark 3 My colleague V. Tichý has devised a computer program assigning to every incidence structure $\mathcal{J} = (G, M, I)$ for $|G|, |M| < 12$ all incidence structures $\mathcal{J}^p$ of independent sets of $\mathcal{J}$. In Example 1 there is for $p = 5$ and $a'_i \not I\, m'_i$ in a) an incidence table of the structure $\mathcal{J}_1$ presented (please compare to Figure

1), in b) and c) there are introduced all independent sets of cardinality p of G_1, M_1, respectively, and in d) there is a graph of the incidence structure $\mathcal{J}_1^5$ drawn (similarly in the other Examples). For technical reasons the denotation in Example 1 is rather different from the text. Example 2 shows the situation in case of $p = 5$ and $a'_i\ I\ m'_i$. Also in the following text some computer picture will be enclosed occasionally.

Corollary 1 *Let $\mathcal{J} = (G, M, I)$ be an incidence structure of type (p, n), $n \geq 2$. Then there exist distinct $i, j \in \{0, \ldots, n-1\}$ such that $R^i \neq R^j$.*

Proof Assume $R^i = R$ for each possible i. If we consider sets $A^0, A^1, A^2 \in G^p$ and $B^0, B^1 \in M^p$, then $A^0 \cap A^1 = A^1 \cap A^2 = R$ and $A^2 = \{a_2\} \cup R$. According to the proof of Theorem 5 there exists a set $C \in G^p$ such that $a_2 \in C$ and $C = A^l$ for certain $l \neq 0, 1, 2$. Then $A^l = \{a_l\} \cup R$ and since $a_2 \in A^l$, $a_2 \notin R$, we get $a_l = a_2$. Thus $A^2 = A^l$ and this is a contradiction. □

In what follows we will investigate only special incidence structures of type (p, n) in which $a'_i\ \not I\ m'_i$, that is $G_1^p = \{A^i, A^{i+1}, A^{i+2}, C^1, \ldots, C^{p-1}\}$, $M_1^p = \{B^i, B^{i+1}, X^1, \ldots, X^{p-1}\}$, $R^i = \{g_1, \ldots, g_{p-1}\}$, $Q^i = \{n_1, \ldots, n_{p-1}\}$ and $X^j = \{m'_i, m_{i+1}\} \cup (Q^i - \{n_j\})$, $C^j = \{a'_i, a_{i+2}\} \cup (R^i - \{g_j\})$.

Theorem 6 *If $A\ I^p\ X^j$ for $A \in G^p - G_1^p$, then $A\ I^p\ X^r$ for some $r \neq j$ and $C^j \cap A = C^r \cap A$.*

Proof Since $A \in G^p - G_1^p$ there exists $a \in A$, $a \in G - G_1$. From $A, C^j\ I^p\ X^j$ and by Theorem 1 we get $|C^j \cap A| = p-1$. It means that $A = \{a, a'_i\} \cup (R^i - \{g_j\})$ or $A = \{a, a_{i+2}\} \cup (R^i - \{g_j\})$ or $A = \{a, a'_i, a_{i+2}\} \cup (R^i - \{g_j, g_l\})$ where $l \neq j$. There exists an A-norming mapping $\alpha : A \to X^j$ because $A\ I^p\ X^j$.

a) Assume that $A = \{a, a'_i\} \cup (R^i - \{g_j\})$. It is clear that $A_a = \{a'_i\} \cup (R^i - \{g_j\}) \subseteq m_{i+1}^{\downarrow}$. Thus $m_{i+1}^{\downarrow} \cap A = A_a$, $a^{\uparrow} \cap X^j = X^j_{m+1}$ which implies $\alpha(a) = m_{i+1}$ (see Figure 1). Moreover, $\alpha(g_l) = n_l$ for all $l \neq j$ and $\alpha(a'_i) = m'_i$. Let $a\ \not I\ n_j$. Then $A\ I^p\ B^i$ but this is a contradiction to $A \neq A^i, A^{i+1}$ (see Example 3 where $p = 5$, $j = 2$, $a = g_8$, $B^i = B^1$, $A = A^8$). If $a\ I\ n_j$, then $A'\ I^p\ B^{i+1}$ where $A' = \{a\} \cup R^i$. That is a contradiction again (see Example 4, $p = 5$).

For brevity we will not bring all the incidence tables or computer pictures in the following text.

b) Assume that $A = \{a, a_{i+2}\} \cup (R^i - \{g_j\})$. Hence $\alpha(a) = m'_i, \alpha(a_{i+2}) = m_{i+1}$ and $\alpha(g_l) = n_l$ for all $l \neq j$. If $a\ \not I\ n_j$, then $A\ I^p\ B^{i+1}$ and this is a contradiction. In case of $a\ I\ n_j$ we have $A'\ I^p\ B^i$ where $A' = \{a\} \cup R^i$ and that is a contradiction again.

c) We have obtained $A = \{a, a'_i, a_{i+2}\} \cup (R^i - \{g_j, g_r\})$ for certain $r \neq j$. Thus $\alpha(a) = n_r$, $\alpha(a'_i) = m'_i$, $\alpha(a_{i+2}) = m_{i+1}$ and $\alpha(g_l) = n_l$ for $l \neq j, r$. If $a\ I\ n_j$, then $A'\ I^p\ B^i$ where $A' = \{a, a'_i\} \cup (R^i - \{g_r\})$ and that is a contradiction. Thus $a\ \not I\ n_j$ and $A\ I^p\ X^r$. In Example 5 there is an incidence table of a substructure $\mathcal{J}_2 = (G_2, M_2, I_2)$ of $\mathcal{J}$ and a graph of the incidence structure $\mathcal{J}_2^p$

where $G_2 = G_1 \cup \{a\}$, $M_2 = M_1$, $p = 5$, $j = 2$, $r = 1$, $A = A^8$, $X^j = B^5$, $X^r = B^6$. Obviously $A \cap C^j = \{a'_i, a_{i+2}\} \cup (R^i - \{g_j, g_r\}) = A \cap C^r$. □

Theorem 7 *Let $\mathcal{J}$ be a special incidence structure of type (p, n) where $a'_i \not I\, m'_i$. Then p is odd.*

Proof Let us denote $L = \{1, \ldots, p-1\}$. For $j \in L$ we have $C^j \in G_1^p$, $X^j \in M_1^p$ and $C^j\ I^p\ X^j$. Since $\mathcal{J}$ is of type (p, n) there exists a (unique) set $A \in G^p - G_1^p$ such that $A\ I^p\ X^j$. By Theorem 6, $A\ I^p\ X^r$ for certain $r \in L$, $r \neq j$. Then $\varphi : j \mapsto r$ is an involutory mapping of the set L. Hence $p - 1$ is even and p is odd. □

Since $\mathcal{J}$ is of type (p, n) it follows from the graph of the substructure $\mathcal{J}_1^p$ that there exists either a set $B^{i+2} \in M^p$ where $A^{i+2}\ I^p\ B^{i+2}$ or a set $B^{i-1} \in M^p$ where $A^i\ I^p\ B^{i-1}$.

Proposition 1 *If $i < n-2$, then there exists a set B^{i+2} and $B^{i+2} = \{b, m_{i+2}\} \cup (Q^i - \{n_k\})$ for certain $k \in \{1, \ldots, p-1\}$ where $b \in G - G_1$. If $i > 0$, then there exists a set B^{i-1} and $B^{i-1} = \{c, m'_i\} \cup (Q^i - \{n_l\})$ for certain $l \in \{1, \ldots, p-1\}$ where $c \in G - G_1$.*

Proof 1. Let $i < n - 2$. Then $i + 2 < n$ and there exists a set $B^{i+2} \in M^p$ such that $A^{i+2}\ I^p\ B^{i+2}$. Obviously $B^{i+2} \notin M_1^p$ and there is $b \in B^{i+2}, b \notin M_1$. Since $|B^{i+1} \cap B^{i+2}| = p - 1$ and $B^{i+1} = \{m_{i+1}\} \cup Q^i$ we get either $B^{i+2} = \{b\} \cup Q^i$ or $B^{i+2} = \{b, m_{i+1}\} \cup (Q^i - \{n_k\})$ for certain $k \in \{1, \ldots, p-1\}$.

Assume that $B^{i+2} = \{b\} \cup Q^i$. There exists a B^{i+2}-norming mapping $\beta : B^{i+2} \to A^{i+2}$ because $A^{i+2}\ I^p\ B^{i+2}$. It is easy to see that $\beta(b) = a_{i+2}$, $\beta(n_j) = g_j$ for $j \in \{1, \ldots, n-1\}$. If $a_{i+1} \not I\, b$, then $A^{i+1}\ I^p\ B^{i+2}$—a contradiction. Similarly, $a'_i \not I\, b$ implies $A^i\ I^p\ B^{i+2}$ and we get a contradiction again. Thus $a'_i, a_{i+1}\ I\ b$ and $b^{\downarrow} \cap G_1 = G_1 - \{a_{i+2}\}$. Let $\mathcal{J}_2 = (G_2, M_2, I_2)$ be a substructure of $\mathcal{J}$ where $G_2 = G_1, M_2 = M_1 \cup \{b\}$. Then $Y_j = \{b, m'_i\} \cup (Q^i - \{n_j\}) \in M_2^p$, $D_j = \{a_{i+1}, a_{i+2}\} \cup (R^i - \{g_j\}) \in G_2^p$ and $C^j, D_j\ I^p\ Y_j$ for $j \in \{1, \ldots, n-1\}$. In Example 6 there is an incidence table of the substructure $\mathcal{J}_2$ for $p = 5$ and an incidence graph of $\mathcal{J}_2^5$.

Since $\mathcal{J}$ is of type (p, n) there exists $A^{i+3} \in G^p$ such that $A^{i+3}\ I^p\ B^{i+2}$ where $A^{i+3} \notin G_1^p$. Hence there exists $a \in A^{i+3}, a \notin G_1$. From $A^{i+2} = \{a_{i+2}\} \cup R^i$ and $|A^{i+2} \cap A^{i+3}| = p - 1$ we get $A^{i+3} = \{a\} \cup R^i$ or $A^{i+3} = \{a, a_{i+2}\} \cup (R^i - \{g_p\})$ for certain $p \in \{1, \ldots, p-1\}$. Let us consider a substructure $\mathcal{J}_3 = (G_3, M_3, I_3)$ of $\mathcal{J}$ where $G_3 = G_1 \cup \{a\}$, $M_3 = M_2 = M_1 \cup \{b\}$.

First suppose that $A^{i+3} = \{a\} \cup R^i$. There exists an A^{i+3}-norming mapping $\alpha : A^{i+3} \to B^{i+2}$ such that $\alpha(a) = b$ and $\alpha(g_j) = n_j$ for $j \in \{1, \ldots, p-1\}$. If $a \not I\, m'_i$, then $A^{i+3}\ I^p\ B^i$—a contradiction. Similarly, $a \not I\, m_{i+1}$ yields $A^{i+3}\ I^p\ B^{i+1}$. Thus $a\ I\ m'_i, m_{i+1}$ and $A_j\ I^p\ Y_j$ where $A_j = \{a, a'_i\} \cup (R^i - \{g_j\})$ for $j \in \{1, \ldots, p-1\}$. However, we have obtained a contradiction because $C^j, D_j\ I^p\ Y_j$.

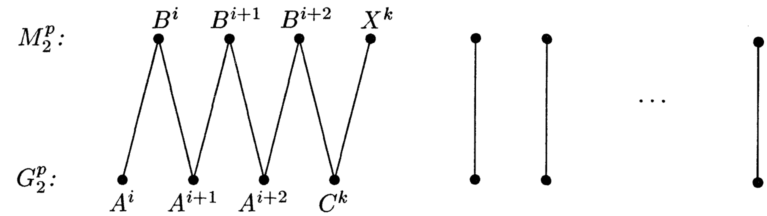
Now it is clear that $A^{i+3} = \{a, a_{i+2}\} \cup (R^i - \{g_p\})$. Then $\alpha(a) = n_p, \alpha(a_{i+2}) = b, \alpha(g_j) = n_j$ for $j \neq p$ where $\alpha : A^{i+3} \to B^{i+2}$ is an A^{i+3}-norming mapping. If

$a\ I\ m_i'$, then $A'\ I^p\ B^i$ where $A' = \{a, a_i'\} \cup (R^i - \{g_p\})$. It is a contradiction. Similarly, $a \not I\ m_i'$ implies $A'\ I^p\ Y^p$ where $A' = \{a, a_{i+2}\} \cup (Q^i - \{g_p\})$ and this is a contradiction again since $C^p, D_p\ I^p\ Y^p$. We get $B^{i+2} \neq \{b\} \cup Q^i$ and hence $B^{i+2} = \{b, m_{i+1}\} \cup (Q^i - \{n_k\})$.

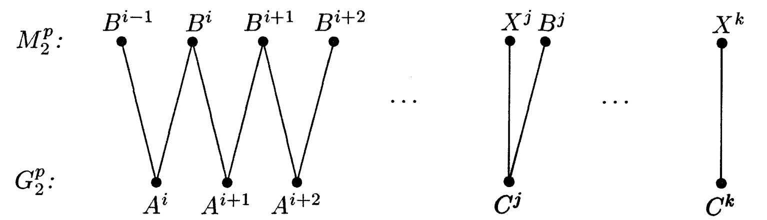
2. In case of $i > 0$ there exists $B^{i-1} \in G^p$ such that $A^i\ I^p\ B^{i-1}$. As in 1. we can prove that $B^{i-1} = \{c, m_i'\} \cup (Q^i - \{n_l\})$, $c \in G - G_1$. □

Proposition 2 *Let $i < n-2$ and $B^{i+2} = \{b, m_{i+1}\} \cup (Q^i - \{n_k\})$. If $\mathcal{J}_2 = (G_2, M_2, I_2)$ is a substructure of $\mathcal{J}$ such that $G_2 = G_1$ and $M_2 = M_1 \cup \{b\}$, then:*

a) In case of $a_i' \not I\ b$ the graph of $\mathcal{J}_2^p$ is



b) If $a_i'\ I\ b$, then $i > 0$, $B^{i-1} = \{b, m_i'\} \cup (Q^i - \{n_k\})$ and the graph of $\mathcal{J}_2^p$ (where $B_j = \{b, m_i', m_{i+1}\} \cup (Q^i - \{n_k, n_j\})$, $j \neq k$) is



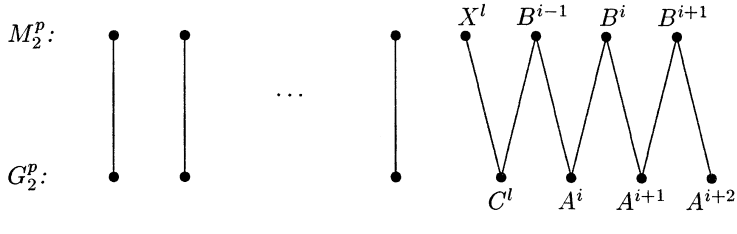
Proof There exists an B^{i+2}-norming mapping $\beta : B^{i+2} \to A^{i+2}$ such that $\beta(b) = g_k$, $\beta(m_{i+1}) = a_{i+2}$ and $\beta(n_j) = g_j$ for $j \neq k$. From $a_{i+1}\ I\ b$ we get $A^{i+1}\ I^p\ B^{i+2}$ and it is a contradiction. Thus $a_{i+1} \not I\ b$.

a) Let $a_i' \not I\ b$. Then $C^k\ I^p\ B^{i+2}$. Since $a_i' \not I\ b, m_i'$ and $a_{i+1} \not I\ b, m_i'$ there does not exist $B \in M_2^p$ containing elements b, m_i' and $\mathcal{J}_2^p$ has a graph stated above. In Example 7 there is an incidence table of $\mathcal{J}_2$ and a graph of the incidence structure $\mathcal{J}_2^p$ for $p = 5$, $k = 2$.

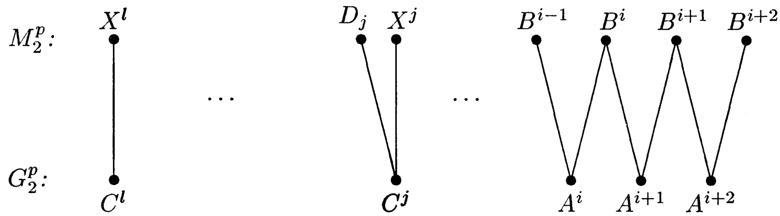
b) Let $a_i'\ I\ b$. Then $A^i\ I^p\ B'$ where $B' = \{b, m_i'\} \cup (Q^i - \{n_k\})$, that is $B' \neq B^i$. Thus $B' = B^{i-1}$ and $i > 0$. For $j \neq k$ we get $B_j = \{b, m_i', m_{i+1}\} \cup (Q^i - \{n_k, n_j\}) \in M_2^p$ and $C^j\ I^p\ B_j$. Example 8 shows a table of $\mathcal{J}_2$ and a graph of $\mathcal{J}_2^5$. □

Proposition 3 *Let* $i > 0$ *and* $B^{i-1} = \{c, m_i'\} \cup (Q^i - \{n_l\})$. *If* $\mathcal{J}_2 = (G_2, M_2, I_2)$ *is a substructure of* $\mathcal{J}$ *where* $G_2 = G_1, M_2 = M_1 \cup \{c\}$, *then:*

a) In case of $a_{i+2} \not I\, c$ *the graph of* $\mathcal{J}_2^p$ *is*



b) If $a_{i+2} I c$, *then* $i < n-2$, $B^{i+2} = \{c, m_{i+1}\} \cup (Q^i - \{n_k\})$ *and the graph of* $\mathcal{J}_2^p$ *(where* $D_j = \{c, m_i', m_{i+1}\} \cup (Q^i - \{n_l, n_j\}), j \neq l$*) is*



The proof is similar to the previous proposition.

Proposition 4 *Let* $0 < i < n-2$ *and* $B^{i+2} = \{b, m_{i+1}\} \cup (Q^i - \{n_k\})$, $B^{i-1} = \{c, m_i'\} \cup (Q^i - \{n_l\})$. *Then the following statements are equivalent:*

(1) $k \neq l$,

(2) $a_i' \not I\, b$,

(3) $a_{i+2} \not I\, c$,

(4) $b^{\downarrow} \cap G_1 \neq c^{\downarrow} \cap G_1$.

Proof (1) $\Rightarrow$ (2) Let $k \neq l$ and $a_i' I b$. By Proposition 2b we have $B^{i-1} = \{b, m_i'\} \cup (Q^i - \{n_k\}) = \{c, m_i'\} \cup (Q^i - \{n_l\})$. Since $b, c \notin M_1$ we get $n_l = n_k$, $l = k$ and it is a contradiction. Thus $a_i' \not I\, b$.

(2) $\Rightarrow$ (3) Let $a_i' \not I\, b$. This implies $C^k\ I^p\ B^{i+2}$. If $a_{i+2}\ I\ c$, then $B^{i+2} = \{c, m_{i+1}\} \cup (Q^i - \{n_l\}) = \{b, m_{i+1}\} \cup (Q^i - \{n_k\})$ by Proposition 3b and hence $k = l$. Now we get $C^k I^p B^{i+2}$ by Proposition 2a and $C^k \not I^p\ B^{i+2}$ by Proposition 3b. This is a contradiction.

(3) $\Rightarrow$ (4) If $a_{i+2} \not I\, c$, then $a_{i+2} \notin c^{\downarrow}$. It follows from the proof of Proposition 2 that $a_{i+2} \in b^{\downarrow}$ and hence $b^{\downarrow} \cap G_1 \neq c^{\downarrow} \cap G_1$.

(4) $\Rightarrow$ (1) Let $b^{\downarrow} \cap G_1 \neq c^{\downarrow} \cap G_1$, that is $b \neq c$. If $a_i'\ I\ b$, then $B^{i-1} = \{b, m_i'\} \cup (Q^i - \{n_k\}) = \{c, m_i'\} \cup (Q^i - \{n_l\})$. Thus $b = c$ and it is a contradiction.

Therefore $a'_i \not I\ b$ and $C^k\ I^p\ B^{i+2}$. We know that $a_{i+2} \not I\ c$ which yields $C^l\ I^p\ B^{i-1}$. If $k = l$, then $C^k\ I^p\ X^k, B^{i-1}, B^{i+2}$ and this is a contradiction again. Thus $k \neq l$. □

The folllowing proposition is also valid:

Proposition 5 *Let* $0 < i < n-2$ *and* $B^{i+2} = \{b, m_{i+1}\} \cup (Q^i - \{n_k\})$, $B^{i-1} = \{c, m'_i\} \cup (Q^i - \{n_l\})$. *Then the following statements are equivalent:*

(1) $k = l$,

(2) $a'_i I b'_i$,

(3) $a_{i+2} I c$,

(4) $b = c$.

Now let us assume that $B^{i+2} = \{b, m_{i+1}\} \cup (Q^i - \{n_k\})$, $a'_i \not I\ b$ for $i < n-2$ and $B^{i-1} = \{c, m'_i\} \cup (Q^i - \{n_l\})$, $a_{i+2} \not I\ c$ for $i > 0$. It means that for $0 < i < n-2$ the conditions (1)–(4) from Proposition 4 hold.

Theorem 8 *If there exists* $B \in M^p$, $B \neq X^j$ *for* $j \neq k, l$ *such that* $C^j\ I^p\ B$, *then* $C^r\ I^p\ B$ *for certain* $r \neq j, k, l$ *and* $X^j \cap B = X^r \cap B$.

Proof Let $B \in M^p$ has the properties described above. Then there exists a B-norming mapping $\beta : B \to C^j$. If $\mathcal{J}_2 = (G_2, M_2, I_2)$ is a substructure of $\mathcal{J}$ where $G_2 = G_1, M_2 = M_1 \cup \{b, c\}$, then $B \notin M_2^p$ since $B \neq B^{i-1}, B^{i+2}$. Moreover, there exists $d \in B$ such that $d \in M - M_2$. From $C^j\ I^p\ X^j, B$ we get $|X^j \cap B| = p-1$ and hence $B = \{d, m'_i\} \cup (Q^i - \{n_j\})$ or $B = \{d, m_{i+1}\} \cup (Q^i - \{n_j\})$ or $B = \{d, m'_i, m_{i+1}\} \cup (Q^i - \{n_j, n_r\})$.

a) Let $B = \{d, m'_i\} \cup (Q^i - \{n_j\})$. Then $\beta(d) = a_{i+2}, \beta(m'_i) = a'_i$ and $\beta(n_q) = g_q$ for $q \neq j$. If $g_j \not I\ d$, then $A^i I^p B$. This yields $B = B^{i-1}$ since $B \neq B^i$. In case of $i = 0$ we get a contradiction. If $i > 0$, then $C^l\ I^p\ B$ by Proposition 3a and it is a contradiction again. Let $g_j\ I\ d$. Then $A^{i+2}\ I^p\ B'$ where $B' = \{d\} \cup Q^i$ and that is a contradiction.

b) Assume that $B = \{d, m_{i+1}\} \cup (Q^i - \{n_j\})$. Then $\beta(d) = a'_i$, $\beta(m_{i+1}) = a_{i+2}$ and $\beta(n_q) = a_q$ for $q \neq j$. If $g_j \not I\ d$, then $A^{i+2}\ I^p\ B$ and $B = B^{i+2}$. In case of $i = n-2$ we get a contradiction. If $i < n-2$, then $C^k\ I^p\ B$ by Proposition 2a and it is a contradiction again. Let $g_j\ I\ d$. Then $A^i\ I^p\ B'$ where $B' = \{d\} \cup Q^i$ and that is a contradiction.

c) Now it is obvious that $B = \{d, m'_i, m_{i+1}\} \cup (Q^i - \{n_j, n_r\})$. Then $\beta(d) = g_r$, $\beta(m_{i+1}) = a_{i+2}$, $\beta(m'_i) = a'_i$ and $\beta(n_q) = g_q$ for $q \neq j, r$. If $g_j\ I\ d$, then $A^{i+2}\ I^p\ B'$ where $B' = \{d, m_{i+1}\} \cup (Q^i - \{n_r\})$. Since $B' \neq B^{i+1}$ we obtain $B' = B^{i+2}$. In case of $i = n-2$ we get a contradiction. If $i < n-2$, then $B^{i+2} = \{b, m_{i+1}\} \cup (Q^i - \{n_k\})$ and this is a contradiction again because $b \neq d$. Thus $g_j \not I\ d$ and $C^r\ I^p\ B$.

Let $i < n-2$. If $r = k$, then $C^k\ I^p\ X^k, B^{i+2}, B$ and that is a contradiction. Hence $r \neq k$. For $i > 0$ we get $r \neq l$. Obviously $X^j \cap B = \{m'_i, m_{i+1}\} \cup (Q^i - \{n_j, n_r\}) = B \cap X^r$. □

In Examples 9, 10 there are tables of substructures $\mathcal{J}_2 = (G_2, M_2, I_2)$ of $\mathcal{J}$ for $G_2 = G_1$, $M_2 = M_1 \cup \{b, d\}$ where $p = 5, j = 1, r = 3$. The case of $a_{i+1} I d$ is in 9 and the case of $a_{i+1} \not{I}\, d$ is in 10. It is showed that both these incidence structures have the same graph. Furthermore, $B = B_8$, $C^j = A^7$, $C^r = A_5$.

For certainty let us suppose $0 < i < n-2$ and denote $L = \{1, \ldots, p-1\}$. Let $\varphi : L \to L$ be the mapping mentioned in Theorem 7. If $C^j \neq A^0, A^n$, then there exists $B \in M^p$, $B \neq X^j$ such that $C^j I^p B$. It follows from Theorem 8 that there exists a unique $j' \in L$ such that $C^{j'}\ I^p\ B$. We put $\xi(j) = j'$.

Let us consider a substructure $\mathcal{J}_2 = (G_2, M_2, I_2)$ of $\mathcal{J}$ where $G_2 = G_1, M_2 = M_1 \cup \{b, c\}$. According to Proposition 2 we get $C^k\ I^p\ B^{i+2}, X^k$. There is $A \in G^p$, $A \neq C^k$ such that $A\ I^p\ X^k$. If we put $\varphi(k) = k_2 \in L$, then $A\ I^p\ X^{k_2}$ and $C^k\ I^p\ X^{k_2}$. Moreover, let us put $A_{k_2} := A$ (see Figure 3).

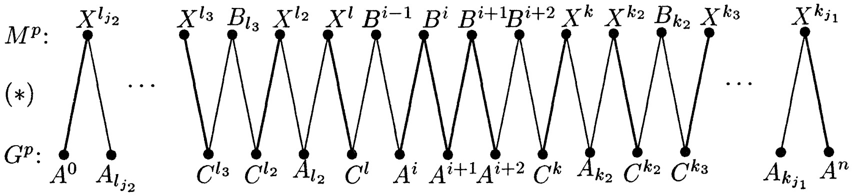


Figure 3

If $C^{k_2} \neq A^n$, then there exists $B \in M^p$, $B \neq X^{k_2}, C^{k_2}\ I^p\ B$. From Theorem 8 we get $C^{k_3}\ I^p\ B$ where $k_3 = \xi(k_2) = \xi\varphi(k)$. We know that $C^{k_3}\ I^p\ X^{k_3}$ and let us put $B_{k_3} := B$. There exists $A \in G^p$, $A \neq C^{k_3}$ such that $A\ I^p\ X^{k_3}$ and so on. For certain $j_1 \in L$ we obtain $C^{k_{j_1}} = A^n$ where $\varphi\xi \ldots \xi\varphi(k) = k_{j_1}$.

Similarly, it follows from Proposition 3 that $C^l\ I^p\ B^{i-1}, X^l$ and for certain $j_2 \in L$ we get $C^{l_{j_2}} = A^0$ where $\varphi\xi \ldots \xi\varphi(l) = l_{j_2}$. The numbers j_1, j_2 are even and $j_1 + j_2 = p - 1$. See Figure 3 for a graph of the incidence structure $\mathcal{J}^p$ where the graph of $\mathcal{J}_1^p$ is emphasized. By assumption we have $A^i \cap A^{i+1} = A^{i+1} \cap A^{i+2}$. It follows from Theorem 6 that $C^{k_{j-1}} \cap A_{k_j} = A_{k_j} \cap C^{k_j}$ where $k_1 := k$, $j \in \{1, \ldots, j-1\}$ and $C^{l_{j-1}} \cap A_{l_j} = A_{l_j} \cap C^{l_j}$ where $l_1 := l$, $j \in \{1, \ldots, j-2\}$. Now we have $B^{i+1} \cap B^{i+2} = B^{i+2} \cap X^k, B^i \cap B^{i-1} = B^{i-1} \cap X^l$ and $X^{k_{j-1}} \cap B_{k_j} = B_{k_j} \cap X^{k_j}$ for $j \in \{2, \ldots, j_1\}$, $X^{l_{j-1}} \cap B_{l_j} = B_{l_j} \cap X^{l_j}$ for $j \in \{2, \ldots, j_2\}$. By Theorem 7 the number p is odd, thus $p = 2q + 1$. For $q \geq 1$ we get $n = 3q + 2$.

See Figure 4 for a graph of such an incidence structure $\mathcal{J}^p$ that $i = 0$.

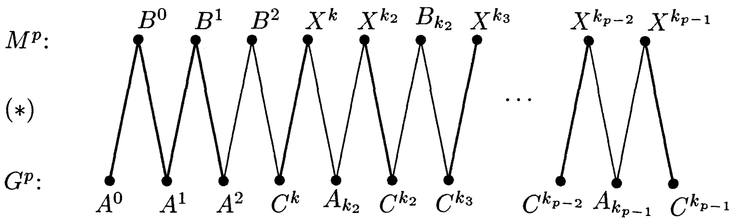


Figure 4

Similarly for $i = n - 2$.

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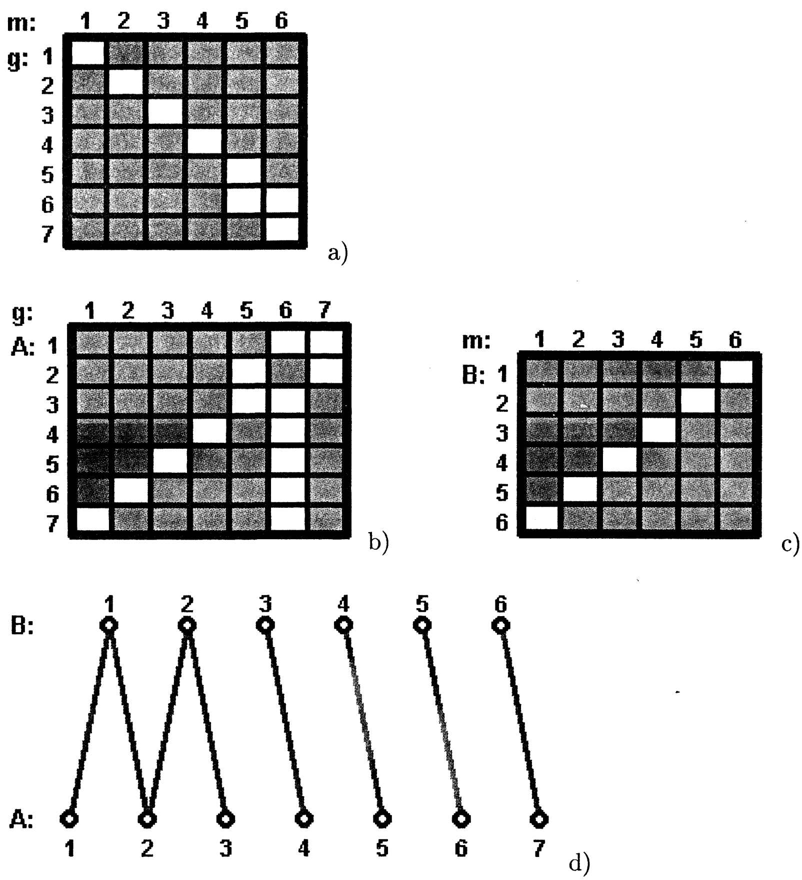
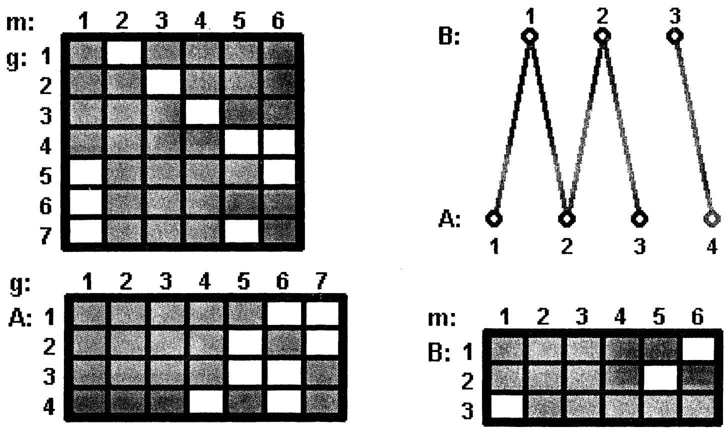
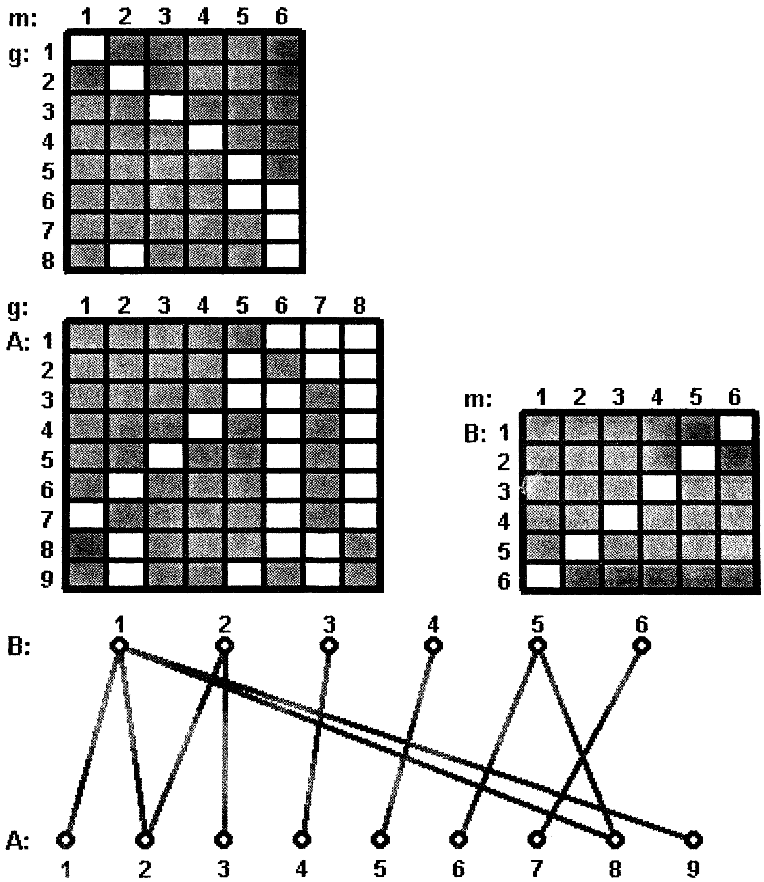
Examples



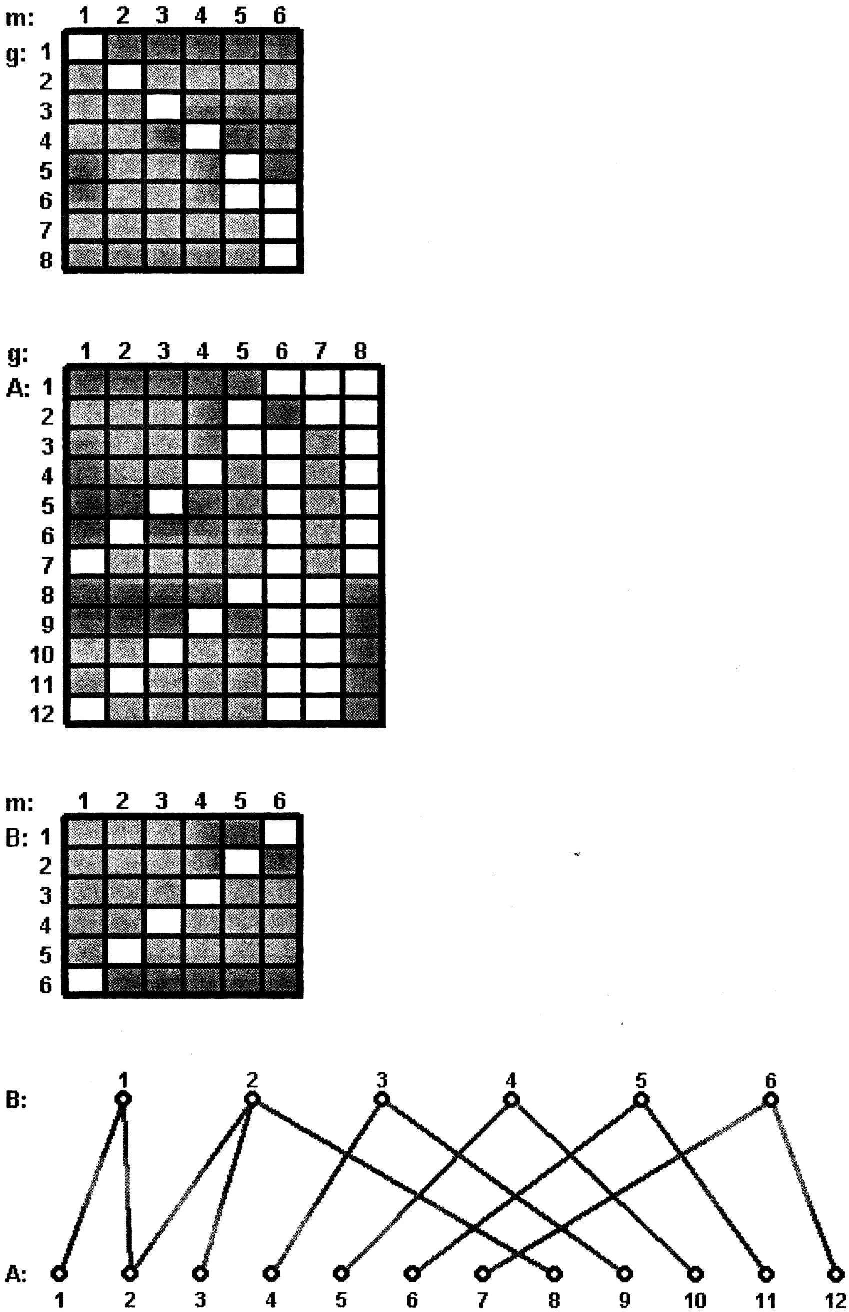
Example 1



Example 2



Example 3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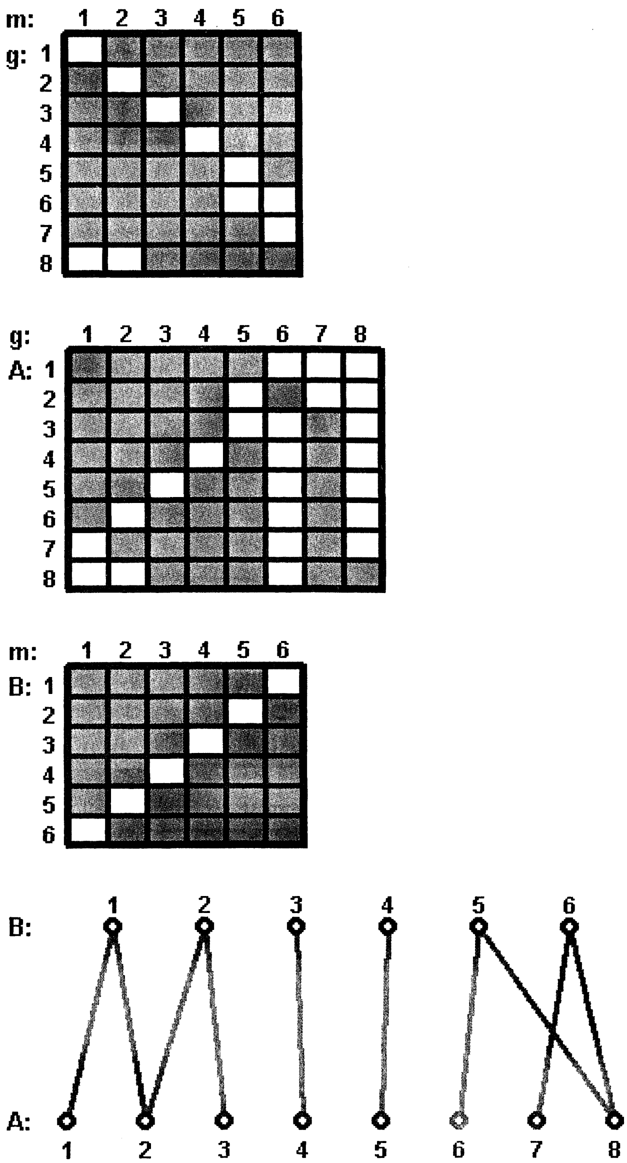
g: 1 2 3 4 5 6 7 8
g: 1 2 3 4 5 6 7 8
A: 1 2 3 4 5 6 7 8 9 10 11 12
m: 1 2 3 4 5 6
B: 1 2 3 4 5 6
B: 1 2 3 4 5 6
A: 1 2 3 4 5 6 7 8 9 10 11 12

Example 4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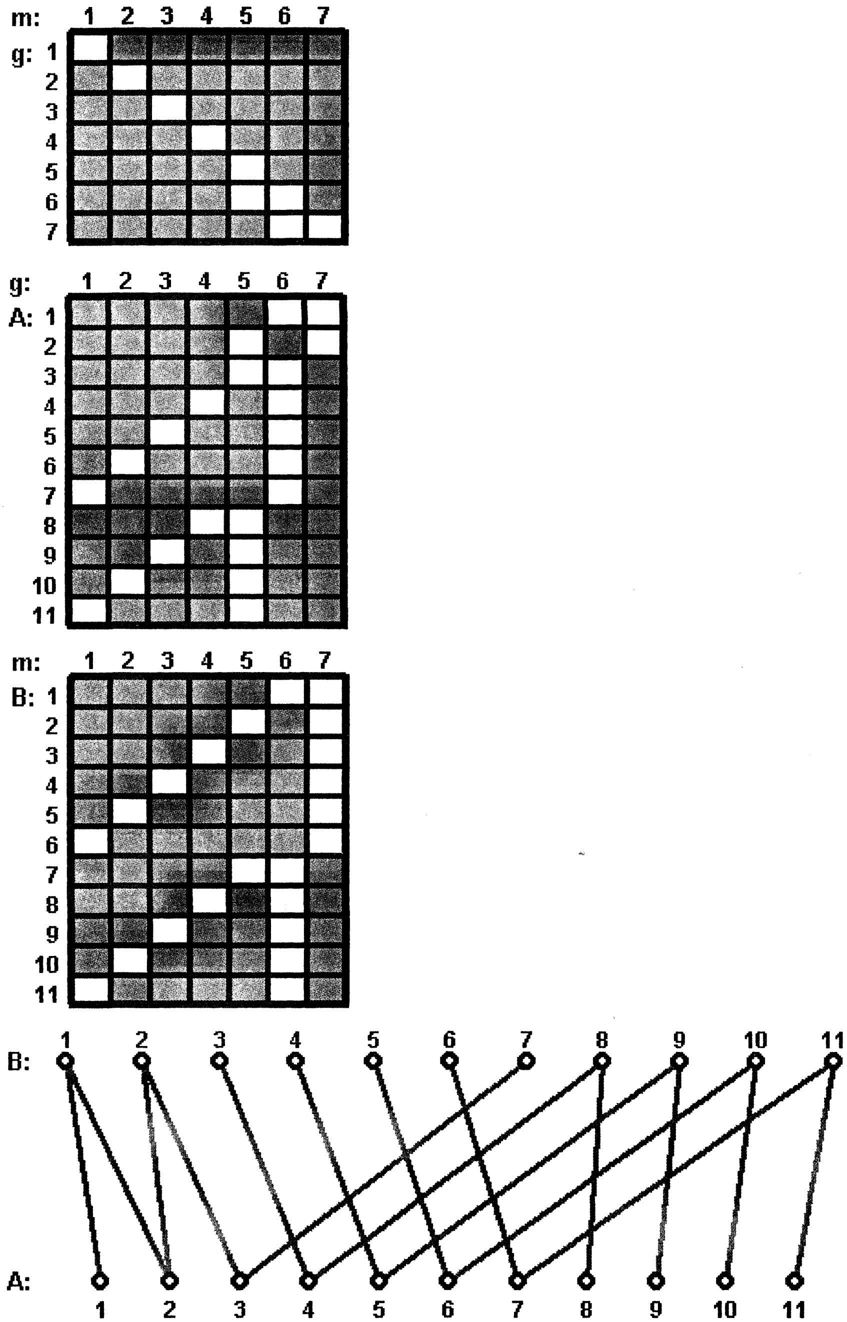
m: 1 2 3 4 5 6
B: 1 2 3 4 5 6
B: 1 2 3 4 5 6
A: 1 2 3 4 5 6 7 8

Example 5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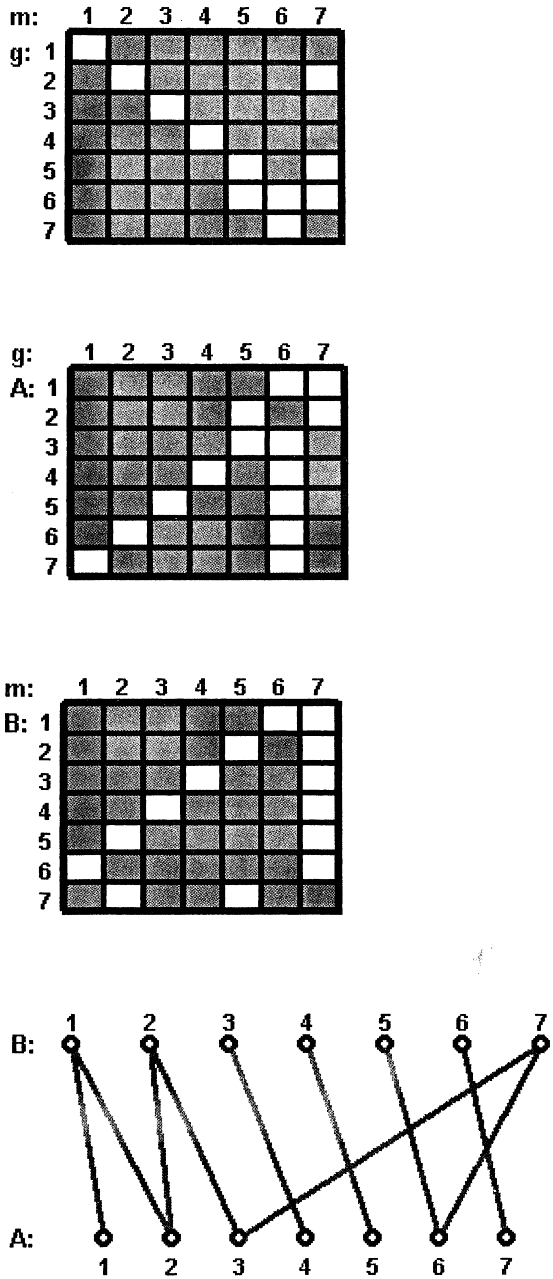
m: 1 2 3 4 5 6 7
B: 1 2 3 4 5 6 7 8 9 10 11
B: 1 2 3 4 5 6 7 8 9 10 11
A: 1 2 3 4 5 6 7 8 9 10 11

Example 6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B: 1 2 3 4 5 6 7
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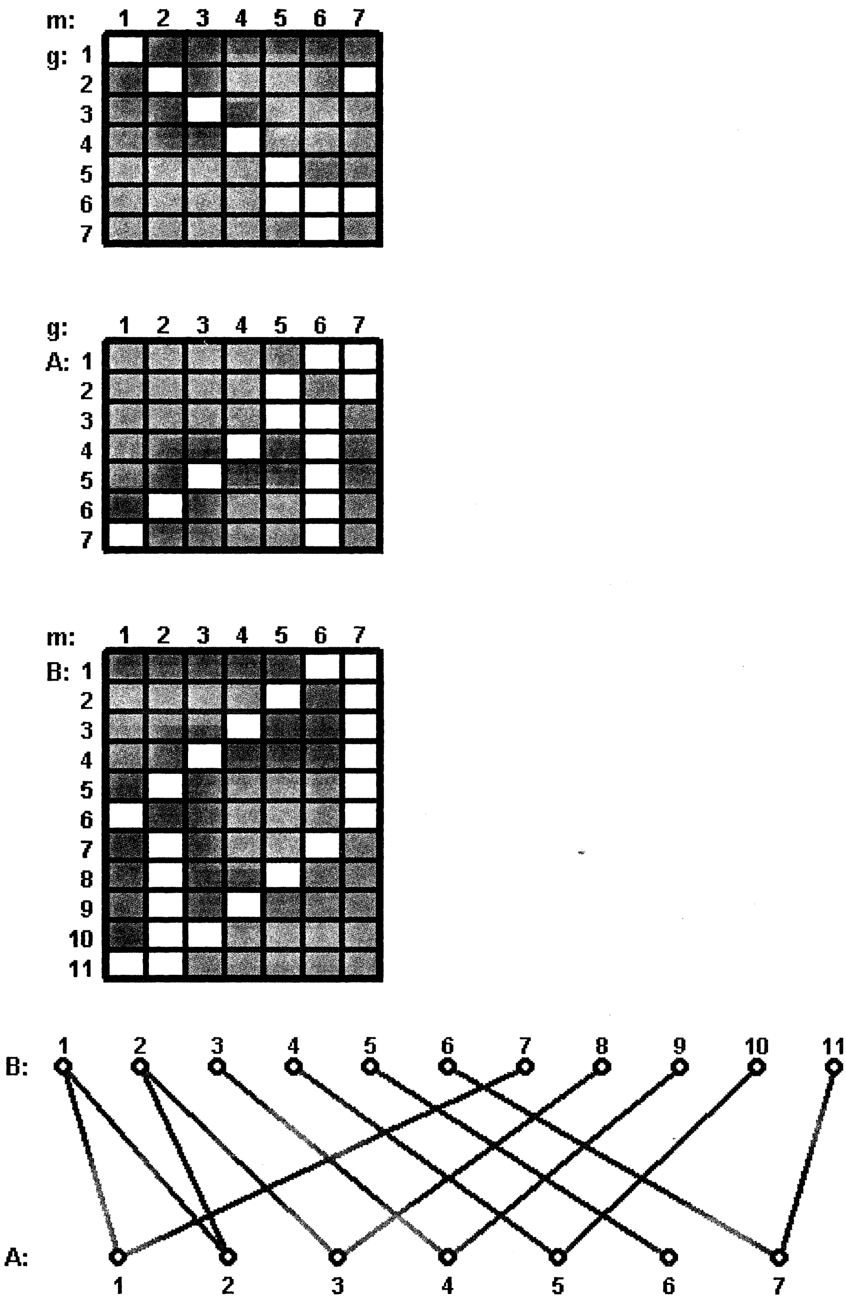
Example 7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g: 1 2 3 4 5 6 7
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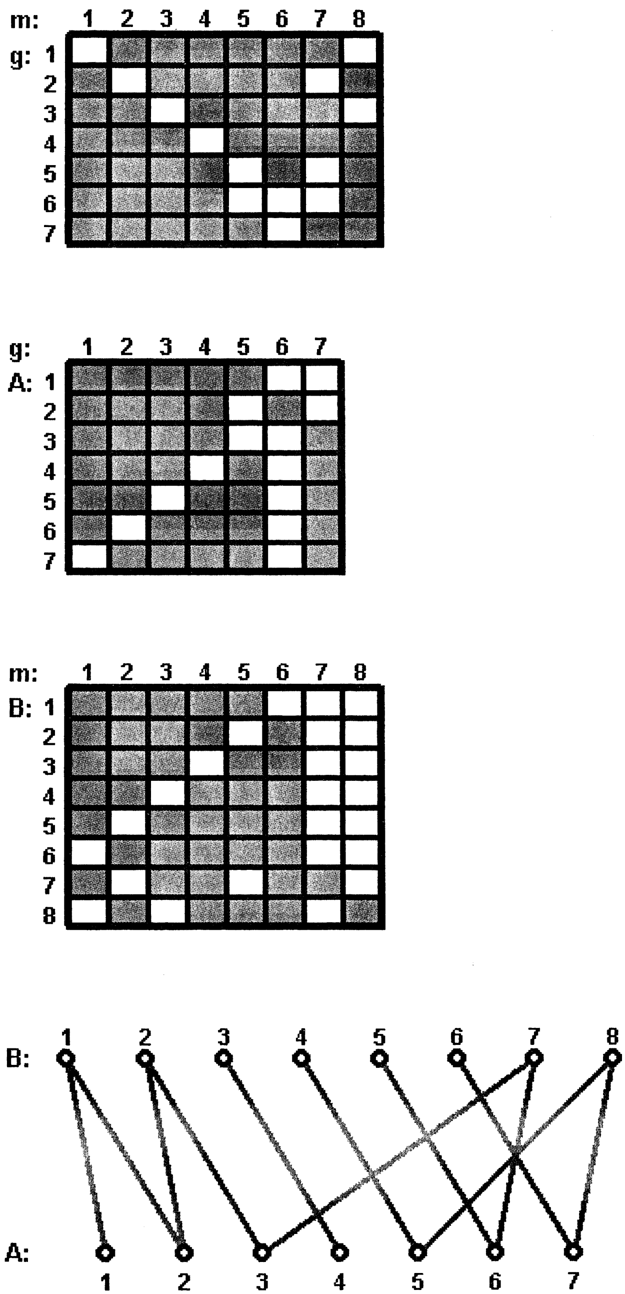
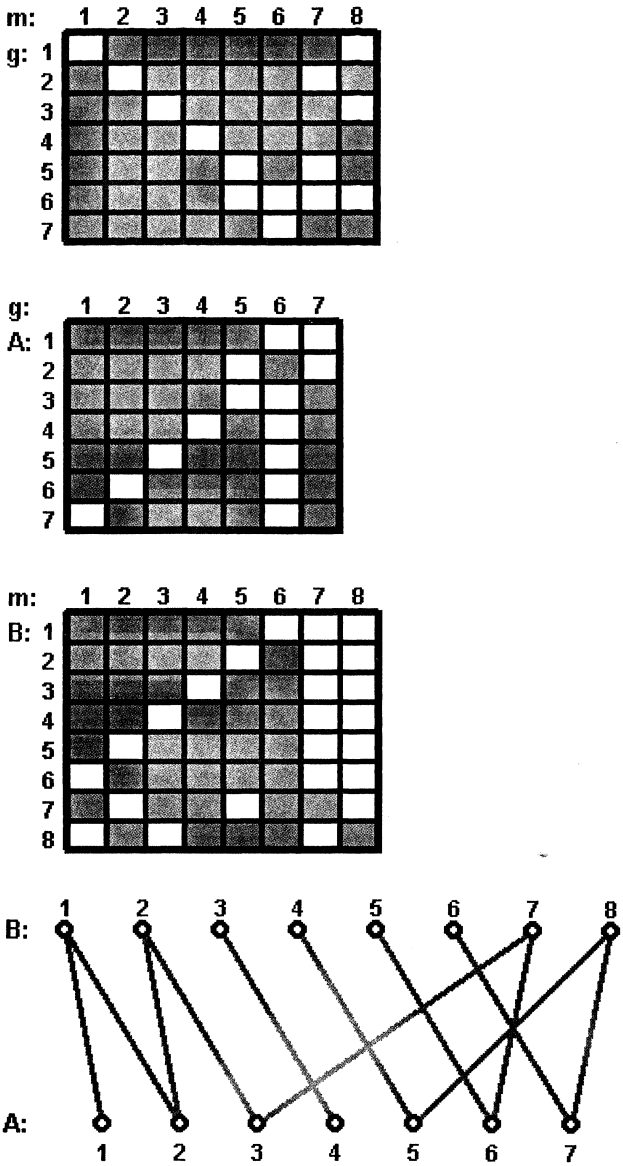
A: 1 2 3 4 5 6 7
m: 1 2 3 4 5 6 7
B: 1 2 3 4 5 6 7 8 9 10 11
B: 1 2 3 4 5 6 7 8 9 10 11
A: 1 2 3 4 5 6 7

Example 8


m: 1 2 3 4 5 6 7 8
g: 1 2 3 4 5 6 7
g: 1 2 3 4 5 6 7
A: 1 2 3 4 5 6 7
m: 1 2 3 4 5 6 7 8
B: 1 2 3 4 5 6 7 8
B: 1 2 3 4 5 6 7 8
A: 1 2 3 4 5 6 7

Example 9


m: 1 2 3 4 5 6 7 8
g: 1 2 3 4 5 6 7
g: 1 2 3 4 5 6 7
A: 1 2 3 4 5 6 7
m: 1 2 3 4 5 6 7 8
B: 1 2 3 4 5 6 7 8
B: 1 2 3 4 5 6 7 8
A: 1 2 3 4 5 6 7

Example 10