Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 39 (2000), No. 1, 107--121

Persistent URL: `http://dml.cz/dmlcz/120420`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **39** (2000) 107–121

Two point boundary value problems for linear third order differential equations in the Colombeau algebra

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(Received September 9, 1999)

Abstract

The existence and uniqueness of solutions to the two point boundary value problem for ordinary linear third order differential equations in the Colombeau algebra are considered.

Key words: Generalized ordinary differential equations, boundary value problem, generalized functions, distributions, Colombeau algebra.

1991 Mathematics Subject Classification: 34A10, 46F99

1 Introduction

For a fixed $j = 1, 2, 3$ we will consider the boundary value problem

$$\ell(x)(t) := x'''(t) + p_1(t)x''(t) + p_2(t)x'(t) + p_3(t)x(t) = q(t), \tag{1.1}$$

$$L_{ij}(x) = d_i, \quad i = 1, 2, 3, \tag{1.2_j}$$

where

$$\begin{array}{lll} L_{11}(x) = x(0), & L_{12}(x) = x(T), & L_{13}(x) = x'(T), \\ L_{21}(x) = x(0), & L_{22}(x) = x(T), & L_{23}(x) = x'(0), \\ L_{31}(x) = x(0), & L_{12}(x) = x'(0), & L_{13}(x) = x''(T) \end{array}$$

and $0 < T < \infty$.

We assume that p_i, $i = 1,2,3$, and q are elements of the Colombeau algebra $\mathcal{G}(\mathbb{R})$ of generalized functions, d_i, $i = 1,2,3$, are elements of the Colombeau algebra $\overline{\mathbb{R}}$ of generalized real numbers, $x(0)$, $x(T)$, $x'(0)$, $x'(T)$, $x''(0)$, $x''(T)$ are understood as the values of the generalized functions x, x' and x'' at the points 0 and T, respectively. The elements p_i, d_i and q are given. The multiplication, the differentiation, the sum and the equality are meant in the Colombeau algebra sense. We prove theorems on the existence and uniqueness of solutions of the problems (1.1), ((1.2$_j$). In certain cases they generalize some of the results given in [1], [7] and [8].

2 Notation

Here we recall some basic definitions which are needed later on. For more details concerning generalized functions and generalized real numbers as well as for the proofs of the assertions mentioned in this section, see [2].

Let $\mathcal{D}(\mathbb{R})$ be the set of all C^∞ functions $\mathbb{R} \to \mathbb{R}$ with a compact support. For $q \in \mathbb{N}$, we denote by $\mathcal{A}_q$ the set of all functions $\varphi \in \mathcal{D}(\mathbb{R})$ with the following properties:

$$\int_{-\infty}^{\infty} \varphi(t)\,\mathrm{d}t = 1, \qquad \int_{-\infty}^{\infty} t^k \varphi(t)\,\mathrm{d}t = 0, \quad k = 1,2,\ldots,q.$$

Furthermore, $\mathcal{E}[\mathbb{R}]$ is the set of all functions $R : \mathcal{A}_1 \times \mathbb{R} \to \mathbb{R}$ such that $R(\varphi,.) \in C^\infty$ for every $\varphi \in \mathcal{A}_1$.

For $R \in \mathcal{E}[\mathbb{R}]$, $\varphi \in \mathcal{A}_1$, $t \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$, we define

$$D_k R(\varphi, t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k} R(\varphi, t).$$

(In particular, $D_0 R(\varphi,t) = R(\varphi,t)$.) Furthermore, if $\varphi \in \mathcal{D}(\mathbb{R})$ and $\varepsilon > 0$, then $\varphi_\varepsilon \in \mathcal{D}(\mathbb{R})$ is defined by

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right), \quad t \in \mathbb{R}.$$

$R \in \mathcal{E}(\mathbb{R})$ is said to be *moderate*, if for every compact subset K of $\mathbb{R}$ and every $k \in \mathbb{N} \cup \{0\}$ there is $N \in \mathbb{N}$ with the following property: for every $\varphi \in \mathcal{A}_N$ there are $C > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t \in K} |D_k R(\varphi_\varepsilon, t)| \leq C\,\varepsilon^{-N} \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

The set of all moderate elements of $\mathcal{E}[\mathbb{R}]$ is denoted by $\mathcal{E}_M[\mathbb{R}]$.

By Γ we denote the set of functions $\alpha : \mathbb{N} \to \mathbb{R}^+$ which are increasing and such that $\lim_{q\to\infty} \alpha(q) = \infty$. Furthermore, we define an ideal $\mathcal{N}[\mathbb{R}]$ in $\mathcal{E}_M[\mathbb{R}]$ as follows: $R \in \mathcal{N}[\mathbb{R}]$ if for every compact subset K of $\mathbb{R}$ and every $k \in \mathbb{N} \cup \{0\}$ there are $N \in \mathbb{N}$ and $\alpha \in \Gamma$ with the property: for every $q \geq N$ and $\varphi \in \mathcal{A}_q$ there are $C > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t \in K} |D_k R(\varphi_\varepsilon, t)| \leq C\,\varepsilon^{\alpha(q)-N} \quad \text{if } \varepsilon \in (0, \varepsilon_0).$$

The algebra $\mathcal{G}(\mathbb{R})$ (the *Colombeau algebra*) is defined as the quotient algebra of $\mathcal{E}_M[\mathbb{R}]$ with respect to $\mathcal{N}[\mathbb{R}]$, i.e.

$$\mathcal{G}(\mathbb{R}) = \frac{\mathcal{E}_M[\mathbb{R}]}{\mathcal{N}[\mathbb{R}]}.$$

Its elements are called *generalized functions.* For $R \in \mathcal{E}_M[\mathbb{R}]$, the corresponding class $G \in \mathcal{G}(\mathbb{R})$ (i.e. $G = R + \mathcal{N}[\mathbb{R}]$) is denoted by $[R]$. Vice versa, if $G \in \mathcal{G}(\mathbb{R})$, then its representative in $\mathcal{E}_M[\mathbb{R}]$ is usually denoted by R_G. If $G_1 = [R_{G_1}] \in \mathcal{G}(\mathbb{R})$ and $G_2 = [R_{G_2}] \in \mathcal{G}(\mathbb{R})$, then we define $G_1G_2 := [R_{G_1}R_{G_2}]$ (This definition does not depend on the choice of R_1 and R_2.)

$\mathcal{E}_0$ is the set of the functions mapping $\mathcal{A}_1$ into $\mathbb{R}$ and $\mathcal{E}_M$ is the set of all moderate elements of $\mathcal{E}_0$, i.e.

$$\begin{aligned}\mathcal{E}_M = \{R \in \mathcal{E}_0 :\ &\text{there is } N \in \mathbb{N} \text{ such that for every } \varphi \in \mathcal{A}_N \text{ there are}\\ &C > 0,\ \eta_0 > 0 \text{ such that } |R(\varphi_\varepsilon)| \le C\,\varepsilon^{-N} \text{ for } \varepsilon \in (0,\eta_0)\}.\end{aligned}$$

The ideal $\mathcal{N}$ of $\mathcal{E}_M$ is defined by

$$\begin{aligned}\mathcal{N} = \{R \in \mathcal{E}_0 :\ &\text{there are } N \in \mathbb{N},\ \alpha \in \Gamma \text{ such that for any } q \ge N,\ \varphi \in \mathcal{A}_q\\ &\text{there are } C > 0,\ \eta_0 > 0 \text{ such that } |R(\varphi_\varepsilon)| \le C\,\varepsilon^{\alpha(q)-N} \text{ for } \varepsilon \in (0,\eta_0)\}.\end{aligned}$$

and

$$\overline{\mathbb{R}} = \frac{\mathcal{E}_M}{\mathcal{N}}$$

It is known that $\overline{\mathbb{R}}$ is an algebra, while it is not a field. The elements of $\overline{\mathbb{R}}$ are called *generalized real numbers.*

If $R \in \mathcal{E}_M[\mathbb{R}]$ and $G = [R] \in \mathcal{G}(\mathbb{R})$, then for any $t \in \mathbb{R}$ the map $Y : \varphi \to R(\varphi, t) \in \mathbb{R}$ is defined on $\mathcal{A}_1$ and belongs to $\mathcal{E}_M$. Furthermore, Y depends only on G and t and we denote it by $G(t)$. We say that $G(t)$ is *the value of the generalized function G at the point t.* $G \in \mathcal{G}(\mathbb{R})$ is said to be a *constant generalized function* on $\mathbb{R}$ if it admits a representative $R(\varphi,t)$ which does not depend on t. With any $Z \in \overline{\mathbb{R}}$ we associate a constant generalized function $Z \in \mathcal{G}(\mathbb{R})$ which admits $R_Z(\varphi,t) \equiv Z(\varphi)$ as its representative.

Throughout the paper $[0,T]$ stands for the compact interval $0 \le t \le T$. For a given matrix M with elements from $\overline{\mathbb{R}}$, its transpose is denoted by $M^{\mathcal{T}}$. Finally, for $x \in C^\infty(\mathbb{R})$ and $n \in \{0,1,2,3\}$ we put

$$\|D_n x\|_K = \max_{t\in K} |D_n x(t)| \quad \text{and} \quad \|x\|_{K,n} = \sum_{i=0}^{n} \|D_i x\|_K.$$

We say that $x \in \mathcal{G}(\mathbb{R})$ is a *solution of the equation* (1.1) if there is $\eta \in \mathcal{N}[\mathbb{R}]$ such that for any representative R_x of x the relations

$$\ell_\varphi(R_x)(t) - R_q(\varphi,t) = \eta(\varphi,t), \tag{2.1}$$

where

$$\begin{aligned}\ell_\varphi(R_x)(t) := &\ D_3R_x(\varphi,t) + R_{p_1}(\varphi,t)D_2R_x(\varphi,t) + R_{p_2}(\varphi,t)D_1R_x(\varphi,t)\\ &+ R_{p_3}(\varphi,t)R_x(\varphi,t)\end{aligned} \tag{2.2}$$

are satisfied for all $\varphi \in \mathcal{A}_1$ and $t \in \mathbb{R}$.

3 Main results

In this section we will formulate several theorems on the existence and uniqueness of the solutions to the problems (1.1), (1.2_j) To this aim we will need the following hypotheses:

Hypothesis ($\mathbf{H_0}$) For every compact subset K of $\mathbb{R}$ containing 0 there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are $C > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t \in K} \left| \int_0^t |R_{p_i}(\varphi_\varepsilon, s)| \mathrm{d}s \right| \leq C \quad \text{for } \varepsilon \in (0, \varepsilon_0) \text{ and } i = 1, 2, 3.$$

Hypothesis ($\mathbf{H_j}$) There is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there exist ε_0 and $\gamma > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the condition

$$a_j \, I_{0\varepsilon}(p_1, p_2, p_3) \leq 1 - \gamma$$

is satisfied with

$$I_{0\varepsilon}(p_1, p_2, p_3) = \int_0^T (|R_{p_1}(\varphi_\varepsilon, t)| + |R_{p_2}(\varphi_\varepsilon, t)| + |R_{p_3}(\varphi_\varepsilon, t)|) \, \mathrm{d}t \tag{3.1}$$

and

$$a_j = \begin{cases} \frac{T^2}{4}(5\sqrt{5} - 11) + \frac{T}{4} + 1 & \text{if } j = 1 \text{ or } j = 2, \\ \frac{T^2}{2} + T + 1 & \text{if } j = 3. \end{cases} \tag{3.2}$$

Hypothesis ($\mathbf{G_j}$) There is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there exist ε_0 and $\gamma > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the condition

$$b_j \, J_{j\varepsilon} \leq 1 - \gamma$$

is satisfied with

$$J_{j\varepsilon} = \int_0^T |R_{p_j}(\varphi_\varepsilon, t)| \, \mathrm{d}t \quad \text{and} \quad b_j = \begin{cases} 1 & \text{if } j = 1, \\ \frac{T}{4} & \text{if } j = 2, \\ \frac{T^2}{4}(5\sqrt{5} - 11) & \text{if } j = 3. \end{cases} \tag{3.3_j}$$

We will also deal with the associated homogeneous problems

$$\ell(x)(t) = 0, \quad L_{ji}(x) = 0, \quad i = 1, 2, 3 \tag{3.4_j}$$

Theorem 3.1 *Let $j \in \{1, 2, 3\}$. Assume that* ($\mathrm{H_0}$) *is true and that $x = 0$ is the only solution of* (3.4_j). *Then the problem* (1.1), (1.2_j) *has a unique solution in* $\mathcal{G}(\mathbb{R})$ *for any $q \in \mathcal{G}(\mathbb{R})$ and any $(d_1, d_2, d_3)^T \in \overline{\mathbb{R}}^3$.*

Remark 3.2 The generalized function $R_\delta(\varphi, t) = \varphi(t)$, $\varphi \in \mathcal{A}_1$, satisfies ($\mathrm{H_0}$).

Remark 3.3 Under the assumption (H_0), the initial value problem (1.1), $x(t_0) = r_0$, $x'(t_0) = r_1$, $x''(t_0) = r_2$, has for every $t_0 \in \mathbb{R}$, $q \in \mathcal{G}(\mathbb{R})$ and $r = (r_0, r_1, r_2)^{\mathcal{T}} \in \overline{\mathbb{R}}^3$ a unique solution $y \in \mathcal{G}(\mathbb{R})$. Moreover, it can be expressed in the form

$$x = c_0\psi_0 + c_1\psi_1 + c_2\psi_2 + Q, \tag{3.5}$$

where for $i \in \{0,1,2\}$, ψ_i is a solution of the initial value problem

$$\ell(\psi_i)(t) = 0, \quad \psi_i^{(i)}(t_0) = 1, \quad \psi_i^{(k)}(t_0) = 0, \quad k \in \{0,1,2\} \setminus \{i\}, \tag{3.6}$$

Q is a particular solution of the equation (1.1) and c_0, c_1 and $c_2 \in \overline{\mathbb{R}}$. Moreover, x is the class of solutions R_x of the problems

$$\begin{aligned} \ell_\varphi(R_x)(t) &= R_q(\varphi, t), \quad \varphi \in \mathcal{A}_1, \\ R_x(\varphi, t_0) &= R_{r_0}(\varphi),\ D_1R_x(\varphi, t_0) = R_{r_1}(\varphi),\ D_2R_x(\varphi, t_0) = R_{r_2}(\varphi),\ \varphi \in \mathcal{A}_1, \end{aligned}$$

where $\ell_\varphi(x)(t)$ is defined by (2.2). For the proofs, see [11, Theorem 3.3].

Theorem 3.4 *Let the assumptions of Theorem* 3.1 *be satisfied and let $q \in \mathcal{G}(\mathbb{R})$ and $d_i \in \overline{\mathbb{R}}$, $i = 1,2,3$. Then the problem*

$$\ell_{\varphi_\varepsilon}(z)(t) = R_q(\varphi_\varepsilon, t), \quad L_{ji}(z) = R_{d_i}(\varphi_\varepsilon), \quad i = 1,2,3,\ \varphi \in \mathcal{A}_N, \tag{3.7}$$

has for any $N \in \mathbb{N}$ sufficiently large and any $\varepsilon > 0$ sufficiently small exactly one solution $z = z_{\varphi_\varepsilon}$. If, in addition, we define $z_\varphi(t) \equiv 0$ on $\mathbb{R}$ for the remaining $\varphi \in \mathcal{A}_1$ and

$$R(\varphi, t) = z_\varphi(t) \quad \textit{for } (\varphi, t) \in \mathcal{A}_1 \times \mathbb{R},$$

then $R \in \mathcal{E}_M[\mathbb{R}]$ and $x = [R]$ is a solution of the problem (1.1), (1.2$_j$).

Theorem 3.5 *Let $j \in \{1,2,3\}$ and let* (H_0) *and* (H_j) *be satisfied. Then the problem* (3.4$_j$) *has in $\mathcal{G}(\mathbb{R})$ only the trivial solution $x = 0$.*

Theorem 3.6 *Let $j \in \{1,2,3\}$ and let* (H_0) *and* (G_j) *be satisfied. Then the problem*

$$x'''(t) + p_j(t)x^{(3-j)}(t) = 0, \quad x(0) = x(T) = x'(T) = 0 \qquad 3.8_j$$

has in $\mathcal{G}(\mathbb{R})$ only the trivial solution $x = 0$.

Remark 3.7 If

$$b_3 = \frac{T^2}{4}(5\sqrt{5} - 11) < 1, \quad R_{p_3}(\varphi, t) = \frac{b_3\varphi(t)}{\int_{-\infty}^{\infty} |\varphi(t)|\,dt}, \quad \varphi \in \mathcal{A}_1,\ t \in \mathbb{R},$$

and $p_3 = [R_{p_3}]$, then p_3 verifies the assumptions of Theorem 3.6 for $j = 3$.

Theorem 3.8 *Let* (H_0) *be true. Furthermore, assume that p_3 admits a representative R_{p_3} with the following property: there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there exist $\varepsilon_0 > 0$ such that*

$$R_{p_3}(\varphi_\varepsilon, t) \geq 0 \quad \textit{for all } t \in [0,T] \textit{ and } \varepsilon \in (0, \varepsilon_0).$$

Then the problem (3.8$_3$) *has only the trivial solution in $\mathcal{G}(\mathbb{R})$.*

4 Proofs

Proof of Thorem 3.1

Let $t_0 \in \mathbb{R}$, let Q be an arbitrary particular solution of (1.1) and let ψ_i, $i = 0, 1, 2$, be solutions of (3.6). Using [11, Theorem 3.3] (see Remark 3.3) and inserting (3.5) into (1.2_j), we get that $x \in \mathcal{G}(\mathbb{R})$ is a solution to (1.1), (1.2_j) if and only if x is given by (3.5), where $c = (c_1, c_2, c_3)^{\mathcal{T}} \in \overline{\mathbb{R}}^3$ satisfies the following system of linear algebraic equations in $\overline{\mathbb{R}}$

$$H_{(j)}\, c = b, \tag{4.1}$$

where

$$H_{(j)} = \left(L_{ji}(\psi_{r-1}^{(i-1)})\right)_{\substack{i=1,2,3\\ r=1,2,3}}, \quad b = (b_1, b_2, b_3)^{\mathcal{T}}, \quad b_i = d_i - L_{ji}(Q), \quad i = 1, 2, 3.$$

Similarly, $y \in \mathcal{G}(\mathbb{R})$ is a solution to (3.4_j) if and only if $y = \psi_1 \overline{c}_1 + \psi_2 \overline{c}_2 + \psi_3 \overline{c}_3$, where $\overline{c} = (\overline{c}_1, \overline{c}_2, \overline{c}_3)^{\mathcal{T}} \in \overline{\mathbb{R}}^3$ satisfies the corresponding homogeneous system

$$H_{(j)}\, \overline{c} = 0 \tag{4.2}$$

Under our assumptions (4.2) has only the trivial solution. Furthermore, by [12, Theorem 3.1], (4.1) has a unique solution for any $b \in \overline{\mathbb{R}}^3$ and this completes the proof of Theorem 3.1. □

Remark 4.1 Notice that under the assumptions of Theorem 3.1, $\det(H_{(j)})$ is invertible element of $\overline{\mathbb{R}}$ (see [13, Corollary of Theorem 51]).

Proof of Theorem 3.4

Let $t_0 \in \mathbb{R}$, and let $R_{\psi_i}(\varphi_\varepsilon, t)$, $i = 0, 1, 2$, be solutions of the family of initial value problems for ordinary differential equations

$$\ell_{\varphi_\varepsilon}(R_{\psi_i}) = 0, \quad R_{\psi_{i-1}^{(r-1)}}(\varphi_\varepsilon, t_0) = \begin{cases} 1 \text{ if } r = i, \\ 0 \text{ if } r \neq i, \end{cases} \quad \varphi \in \mathcal{A}_1,\ \varepsilon > 0. \tag{4.3}$$

Then, every solution z_{φ_ε} of (3.7) can be expressed in the form

$$z_{\varphi_\varepsilon}(t) = c_1(\varphi_\varepsilon)\, R_{\psi_0}(\varphi_\varepsilon, t) + c_2(\varphi_\varepsilon)\, R_{\psi_1}(\varphi_\varepsilon, t) + c_3(\varphi_\varepsilon)\, R_{\psi_2}(\varphi_\varepsilon, t) + Q(\varphi_\varepsilon, t) \tag{4.4}$$

with $c_i(\varphi_\varepsilon) \in \mathbb{R}$ for $i = 1, 2, 3$,

$$Q(\varphi_\varepsilon, t) = \int_{t_0}^{t} W^{-1}(\varphi_\varepsilon, s) U(t, s) R_q(\varphi_\varepsilon, s) \mathrm{d}s, \tag{4.5}$$

$$U(t, s) = R_{\psi_0}(\varphi_\varepsilon, t)\, D_{31}(s) + R_{\psi_1}(\varphi_\varepsilon, t)\, D_{32}(s) + R_{\psi_2}(\varphi_\varepsilon, t)\, D_{33}(s),$$

$$W(\varphi_\varepsilon, t) = \exp\Big(-\int_{t_0}^{t} R_{p_1}(\varphi_\varepsilon, s) \mathrm{d}s\Big), \tag{4.6}$$

and

$$\begin{aligned} D_{31}(s) &= R_{\psi_1}(\varphi_\varepsilon, s) R_{\psi_2'}(\varphi_\varepsilon, s) - R_{\psi_1'}(\varphi_\varepsilon, s) R_{\psi_2}(\varphi_\varepsilon, s), \\ D_{32}(s) &= R_{\psi_0'}(\varphi_\varepsilon, s) R_{\psi_2}(\varphi_\varepsilon, s) - R_{\psi_0}(\varphi_\varepsilon, s) R_{\psi_2'}(\varphi_\varepsilon, s), \qquad (4.7) \\ D_{33}(s) &= R_{\psi_0}(\varphi_\varepsilon, s) R_{\psi_1'}(\varphi_\varepsilon, s) - R_{\psi_0'}(\varphi_\varepsilon, s) R_{\psi_1}(\varphi_\varepsilon, s), \end{aligned}$$

Inserting (4.4) into the boundary conditions in (3.7) we obtain that z_{φ_ε} will verify (3.7) for a given $\varphi \in \mathcal{A}_1$ and $\varepsilon > 0$ if and only if

$$A_{(j)}(\varphi_\varepsilon)\, c(\varphi_\varepsilon) = b_{(j)}(\varphi_\varepsilon) \qquad (4.8)$$

is true, where

$$A_{(j)}(\varphi_\varepsilon) = \big(L_{ji}(R_{\psi_{r-1}}(\varphi_\varepsilon, t)\big)_{\substack{i=1,2,3 \\ r=1,2,3}},$$

and

$$b_{(j)}(\varphi_\varepsilon) = \big(R_{d_{ji}}(\varphi_\varepsilon) - L_{ji}(Q(\varphi_\varepsilon, t))\big)_{i=1}^{i=3}.$$

Now, since we assume that (3.4_j) has only the trivial solution, it follows that $\det H_{(j)}$ is an invertible element of $\overline{\mathbb{R}}$. From this we can deduce that there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are $C > 0$ and $\varepsilon_0 > 0$ such that

$$|\det A_{(j)}(\varphi_\varepsilon)| \geq C\varepsilon^N, \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \qquad (4.9)$$

This means that for $\varphi \in \mathcal{A}_q$, $q \geq N$ and $\varepsilon \in (0, \varepsilon_0)$ the problem (3.7) has exactly one solution z_{φ_ε} which is given by (4.4), where

$$c_i(\varphi_\varepsilon) = (\det A_{(j)}(\varphi_\varepsilon))^{-1} \det A_{(j),i}(\varphi_\varepsilon), \quad i = 1, 2, 3, \qquad (4.10)$$

and $A_{(j),i}(\varphi_\varepsilon)$ denotes the matrix obtained from $A_{(j)}(\varphi_\varepsilon)$ by replacing its i-th column by $b_{(j)}(\varphi_\varepsilon)$. Now, if we extend the definition of c_i on the whole $\mathcal{A}_1$ by setting $c_i(\varphi) = 0$ in the cases that $\det A_{(j)}(\varphi) = 0$, we get $c_i \in \mathcal{E}_M[\mathbb{R}]$, $i = 1, 2, 3$. Similarly, for $\varphi \in \mathcal{A}_1$ such that $\det A_{(j)}(\varphi) = 0$ put $z_\varphi(t) \equiv 0$ on $\mathbb{R}$. It is known (cf. [11]), that

$$R_{\psi_r}(\varphi, t) \in \mathcal{E}_M[\mathbb{R}] \quad \text{for } r = 0, 1, 2.$$

Now, define

$$R(\varphi, t) := z_\varphi(t) \quad \text{for } (\varphi, t) \in \mathcal{A}_1 \times \mathbb{R}.$$

Then $R \in \mathcal{E}_M[\mathbb{R}]$ whefrom the proof of Theorem 3.4 already follows. □

In the proof of Theorem 3.5 we will make use of the Green functions $G_j(t, s)$, $j = 1, 2, 3$, of the boundary value problems

$$x''' = 0, \quad L_{j1}(x) = L_{j2}(x) = L_{j3}(x) = 0,$$

which are respectively defined by the following presciptions:

$$G_1(t, s) = \begin{cases} \frac{s^2(T-t)^2}{2T^2} - \frac{1}{2}(s-t)^2 & \text{if } 0 \leq t \leq s \leq T \\ \frac{s^2(T-t)^2}{2T^2} & \text{if } 0 \leq s \leq t \leq T, \end{cases} \qquad (4.11)$$

$$G_2(t, s) = \begin{cases} -\frac{1}{2}\left(1 - \frac{s}{T}\right)^2 t^2 & \text{if } 0 \leq t \leq s \leq T, \\ \frac{s}{T}\left(1 - \frac{s}{2T}\right)t^2 - st + \frac{1}{2}s^2 & \text{if } 0 \leq s \leq t \leq T. \end{cases} \qquad (4.12)$$

and

$$G_3(t,s) = \begin{cases} -\frac{1}{2}t^2 & \text{if } 0 \le t \le s \le T, \\ -st + \frac{1}{2}s^2 & \text{if } 0 \le s \le t \le T. \end{cases} \tag{4.13}$$

The properties of the functions G_j needed later on are described by the following lemma.

Lemma 4.2 *Let $j \in \{1,2,3\}$. Then*

$$\sup_{t,s\in[0,T]} |G_j(t,s)| = a_{j0}, \qquad \sup_{t,s\in[0,T]} \left|\frac{\partial G_j}{\partial t}(t,s)\right| = a_{j1}, \tag{4.14}$$

$$\sup_{t,s\in[0,T]} \left|\frac{\partial^2 G_j(t,s)}{\partial t^2}\right| = 1,$$

where

$$a_{j0} = \begin{cases} \frac{T^2}{4}(5\sqrt{5}-11) & \text{for } j=1,2, \\ \frac{T^2}{2} & \text{for } j=3 \end{cases}$$

and

$$a_{j1} = \begin{cases} \frac{T}{4} & \text{for } j=1,2, \\ T & \text{for } j=3. \end{cases}$$

Remark 4.3 Notice that

$$a_{10} = a_{20} = |G_1(T-s_0, s_0)| = |G_2(s_0, T-s_0)|,$$

where

$$s_0 = \frac{1}{2}(-T + \sqrt{5}T).$$

Similarly,

$$a_{11} = a_{21} = \left|\frac{\partial}{\partial t}G_1\left(0, \frac{T}{2}\right)\right| = \left|\frac{\partial}{\partial t}G_2\left(T, \frac{T}{2}\right)\right|,$$

$$a_{30} = |G_3(T,T)| \quad \text{and} \quad a_{31} = \left|\frac{\partial G_3}{\partial t}(T,0)\right|.$$

Proof of Theorem 3.5

Let $j \in \{1,2,3\}$ be given and let $x = [R_x]$ be a solution of (3.4_j). Then

$$\ell_\varphi(R_x) = \eta_j(\varphi,t) \quad \text{and} \quad L_{ji}(R_x) = \overline{\eta}_{ji}(\varphi), \quad i=1,2,3, \ \varphi \in \mathcal{A}_1,$$

where $\eta_j \in \mathcal{N}[\mathbb{R}]$ and $\overline{\eta}_{ji} \in \mathcal{N}$, $i=1,2,3$. Hence

$$R_x(\varphi,t) = -\int_0^T G_j(t,s)M_x(\varphi,s)\,\mathrm{d}s + A_j(\varphi)t^2 + B_j(\varphi)t + C_j(\varphi), \tag{4.15}$$

where

$$\begin{aligned} M_x(\varphi,s) &= -R_{p_1}(\varphi,s)\,R_{x''}(\varphi,s) - R_{p_2}(\varphi,s)\,R_{x'}(\varphi,s) \\ &\quad - R_{p_3}(\varphi,s)\,R_x(\varphi,s) - \eta_j(\varphi,s) \end{aligned} \tag{4.16}$$

and A_j, B_j, $C_j \in \mathcal{N}$. In virtue of (4.15), (4.16), (H_1) and Lemma 4.2 for sufficiently small $\varepsilon > 0$, sufficiently large $n \in \mathbb{N}$ and $\varphi \in \mathcal{A}_N$ we have

$$\begin{aligned} \|R_x(\varphi_\varepsilon, t)\|_{[0,T]} &\le a_{j0}\, I_{0\varepsilon}(p_1,p_2,p_3)\, \|R_x(\varphi_\varepsilon, t)\|_{[0,T],2} \\ &+ a_{j0} \int_0^T |\eta_j(\varphi_\varepsilon, s)|\mathrm{d}s + |A_j(\varphi_\varepsilon)|T^2 + |B_j(\varphi_\varepsilon)|T + |C_j(\varphi_\varepsilon)|, \end{aligned} \tag{4.17}$$

$$\begin{aligned} \|R_{x'}(\varphi_\varepsilon, t)\|_{[0,T]} &\le a_{j_1}\, I_{0\varepsilon}(p_1,p_2,p_3)\|R_x(\varphi_\varepsilon, t)\|_{[0,T],2} \\ &+ a_{j1} \int_0^T |\eta_j(\varphi_\varepsilon, s)|\mathrm{d}s + 2\,|A_j(\varphi_\varepsilon)|T + |B_j(\varphi_\varepsilon)| \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} \|R_{x''}(\varphi_\varepsilon, t)\|_{[0,T]} &\le I_{0\varepsilon}(p_1,p_2,p_3)\, \|R_x(\varphi_\varepsilon, t)\|_{[0,T],2} \\ &+ \int_0^T |\eta_j(\varphi_\varepsilon, s)|\mathrm{d}s + 2\,|A_j(\varphi_\varepsilon)|, \end{aligned} \tag{4.19}$$

where $I_{0\varepsilon}(p_1,p_2,p_3)$ are defined by (3.1).

Now, taking into account the relations (4.17)–(4.19), it follows that

$$\|R_x(\varphi_\varepsilon, t)\|_{[0,T],2} \le I_{0\varepsilon}(p_1,p_2,p_3)\, \|R_x(\varphi_\varepsilon, t)\|_{[0,T],2} + \eta_j^*(\varphi_\varepsilon),$$

where $\eta_j^*(\varphi) \in \mathcal{N}$. This means that there are $\overline{C} > 0$, $N_1 \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that

$$\|R_x(\varphi_\varepsilon, t)\|_{[0,T],2} \le \overline{C}_2 \varepsilon^{\alpha(q)-N_1} \tag{4.20}$$

is true for $q \ge N_1$, $\varphi \in \mathcal{A}_q$ and $\varepsilon \in (0, \varepsilon_0)$. If $t_0 \in (0,T)$, then (4.20) implies that $R_{x^{(n)}}(\varphi, t_0) \in \mathcal{N}$ for $n = 0,1,2$. On the other hand, $z = R_x(\varphi, t)$ is a solution of

$$\begin{aligned} \ell_{\varphi_\varepsilon}(z) = \eta_j(\varphi_\varepsilon, t), \quad & z(t_0) = R_x(\varphi_\varepsilon, t_0), \\ z'(t_0) = R_{x'}(\varphi_\varepsilon, t_0), \quad & z''(t_0) = R_{x''}(\varphi_\varepsilon, t_0) \end{aligned}$$

on $\mathbb{R}$. So, by virtue of Remark 3.3 and [11, Theorem 3.3] we obtain

$$R_x(\varphi, t) \in \mathcal{N}[\mathbb{R}] \tag{4.21}$$

and this proves Theorem 3.5. □

Proof of Theorem 3.6.

a) Let $j = 1$ and let $x = [R_x]$ be a solution of (3.8_1), i.e.

$$\begin{aligned} R_x(\varphi_\varepsilon, t) = &- \int_0^T G_1(t,s)(R_{p_1}(\varphi_\varepsilon, s)R_{x''}(\varphi_\varepsilon, s) - \eta(\varphi_\varepsilon, s))\mathrm{d}s \\ &+ \widetilde{A}_1(\varphi_\varepsilon)t^2 + \widetilde{B}_1(\varphi_\varepsilon) + \widetilde{C}_1(\varphi_\varepsilon), \end{aligned}$$

where G_1 is defined by (4.11), $\eta \in \mathcal{N}[\mathbb{R}]$, $\widetilde{A}_1$, $\widetilde{B}_1$ and $\widetilde{C}_1 \in \mathcal{N}$. For some $\overline{C} > 0$, $N \in \mathbb{N}$ and $\varepsilon_0 > 0$ we have

$$\|R_{x''}(\varphi_\varepsilon, t)\|_{[0,T]} \le \overline{C}\varepsilon^{\alpha(q)-N}, \quad q \ge N,\ \varphi \in \mathcal{A}_q,\ \varepsilon \in (0, \varepsilon_0).$$

Since

$$R_{x'}(\varphi_\varepsilon,t) = \int_T^t R_{x''}(\varphi_\varepsilon,s)\mathrm{d}s + R_{x'}(\varphi_\varepsilon,T),$$
$$R_x(\varphi_\varepsilon,t) = \int_0^t R_{x'}(\varphi_\varepsilon,s)\mathrm{d}s + R_x(\varphi_\varepsilon,0),$$

$R_{x'}(\varphi,T) \in \mathcal{N}$ and $R_x(\varphi,0) \in \mathcal{N}$, $R_x(\varphi,t)$ has properties (4.20) and (4.21), which completes the proof of Theorem 3.6 for $j=1$.

b) Let $j=2$ and let $x=[R_x]$ be a solution of (3.8_2), i.e.

$$R_{x'''}(\varphi_\varepsilon,t) = -R_{p_2}(\varphi_\varepsilon,t)R_{x'}(\varphi_\varepsilon,t) + \eta(\varphi_\varepsilon,t),$$

and

$$R_x(\varphi_\varepsilon,0) = \eta_1(\varphi_\varepsilon), \quad R_x(\varphi_\varepsilon,T) = \eta_2(\varphi_\varepsilon), \quad R_{x'}(\varphi_\varepsilon,T) = \eta_3(\varphi_\varepsilon),$$

where $\eta \in \mathcal{N}[\mathbb{R}]$, $\eta_i \in \mathcal{N}$, $i=1,2,3$. Hence

$$\begin{aligned} R_x(\varphi_\varepsilon,t) = &-\int_0^T G_2(t,s)(R_{p_2}(\varphi_\varepsilon,s)\,R_{x'}(\varphi_\varepsilon,s) - \eta(\varphi_\varepsilon,s)\mathrm{d}s \\ &+ \overline{A}_2(\varphi_\varepsilon)t^2 + \overline{B}_2(\varphi_\varepsilon)t + \overline{C}_2(\varphi_\varepsilon), \end{aligned} \tag{4.22}$$

where G_2 is given by (4.12) and $\overline{A}_2(\varphi), \overline{B}_2(\varphi) \in \mathcal{N}$, $\overline{C}_2(\varphi) \in \mathcal{N}$. According to (4.14) we have

$$||R_{x'}(\varphi_\varepsilon,t)||_{[0,T]} \leq J_{2\varepsilon}(p_2)_\varepsilon ||R_{x'}(\varphi_\varepsilon,t)||_{[0,T]} + \eta_4(\varphi_\varepsilon), \tag{4.23}$$

where $\eta_4 \in \mathcal{N}$. By (3.6) and (4.23) there are $N \in \mathbb{N}$, $\varepsilon_0 > 0$ and $\widetilde{C} > 0$ such that for $q \geq N$, $\varphi \in \mathcal{A}_q$ and $\varepsilon \in (0,\varepsilon_0)$

$$||R_{x'}(\varphi_\varepsilon,t)||_{[0,T]} \leq \widetilde{C}\varepsilon^{\alpha(q)-N} \tag{4.24}$$

Applying the Schwarz inequality to the equality

$$R_x(\varphi_\varepsilon,t) = \int_0^t R_{x'}(\varphi_\varepsilon,s)\mathrm{d}s + R_x(\varphi_\varepsilon,0)$$

we obtain that there are $\widetilde{C}_1 > 0$, $N_1 \in \mathbb{N}$ and $\varepsilon_1 > 0$ such that

$$||R_x(\varphi_\varepsilon,t)||_{[0,T]} \leq \widetilde{C}_1\varepsilon^{\alpha(q)-N_1} \quad \text{for all } q \geq N_1 \text{ and } \varepsilon \in (0,\varepsilon_1). \tag{4.25}$$

Similarly, the relations (4.22), (4.24) and (4.25) yield the existence of $\widetilde{C}_2 > 0$, $N_2 \in \mathbb{N}$ and $\varepsilon_2 > 0$ such that

$$||R_{x''}(\varphi_\varepsilon,t)||_{[0,T]} \leq \widetilde{C}_2\,\varepsilon^{\alpha(q)-N_2}$$

is true for $q \geq N_2$, $\varphi \in \mathcal{A}_q$ and $\varepsilon \in (0,\varepsilon_2)$ and we see that R_x has the property (4.20), i.e. $R_x \in \mathcal{N}[\mathbb{R}]$, which completes the proof of Theorem 3.6 for $j=2$.

c) Assume $j = 3$, let $x \in \mathcal{G}(\mathbb{R})$ be a solution of (3.8_3) and let there are $\eta \in \mathcal{N}[\mathbb{R}]$, $\eta_i \in \mathcal{N}$, $i = 1, 2, 3$, such that the relations

$$\begin{aligned} &R_{x'''}(\varphi, t) = -R_{p_3}(\varphi, t) R_x(\varphi, t) + \eta(\varphi, t), \\ &R_x(\varphi, 0) = \eta_1(\varphi), \quad R_x(\varphi, T) = \eta_2(\varphi), \quad R_{x'}(\varphi, T) = \eta_3(\varphi) \end{aligned} \tag{4.26}$$

are satisfied for all $\varphi \in \mathcal{A}_1$. Applying (4.15) with $j = 3$ and G_3 given by (4.13) we can see that

$$||R_x(\varphi_\varepsilon, t)||_{[0,T]} \le b_3 \, J_{3\varepsilon}(p_3) \, ||R_x(\varphi_\varepsilon, t)||_{[0,T]} + \eta(\varphi_\varepsilon),$$

where $\eta \in \mathcal{N}$, $\varepsilon > 0$ is sufficiently small and b_3 and $J_{3\varepsilon}$ are defined by (3.3_3). As in the proof of Theorem 3.4 we conclude that R_x has property (4.21), which completes the proof of Theorem 3.6. □

Proof of Theorem 3.8

As in the proof of Theorem 3.6, assume that there are $\eta \in \mathcal{N}[\mathbb{R}]$, $\eta_i \in \mathcal{N}$, $i = 1, 2, 3$, such that the relations (4.26) and

$$R_x(\varphi, 0) = \eta_1(\varphi), \quad R_x(\varphi, T) = \eta_2(\varphi), \quad R_{x'}(\varphi, T) = \eta_3(\varphi) \tag{4.27}$$

are true for all $\varphi \in \mathcal{A}_1$. Any solution R_x of (4.26) can be expressed in the form (4.4), where R_{ψ_i}, $i = 1, 2, 3$, and Q are respectively defined by (4.3) and (4.5), where R_q is replaced by η, $c_i \in \mathcal{E}_M[\mathbb{R}]$ for $i = 1, 2, 3$ and $t_0 = T$. Consequently, for $\varphi \in \mathcal{A}_1$, we have

$$R_x(\varphi_\varepsilon, t) = c_3(\varphi_\varepsilon) \, R_{\psi_2}(\varphi_\varepsilon, t) + \eta_4(\varphi_\varepsilon), \tag{4.28}$$

where $\eta_4 \in \mathcal{N}[\mathbb{R}]$.

Now, we will prove that $c_3 \in \mathcal{N}$. Indeed, $y = R_{\psi_2}$ is a solution to

$$y''' = -R_{p_3}(\varphi_\varepsilon, t) \, y, \quad y(T) = y'(T) = 0, \quad y''(T) = 1. \tag{4.29}$$

Multliplying the differential equation in (4.29) by y and integrating-by-parts from T to t, we get

$$\int_T^t y'''(s) \, y(s) \mathrm{d}s = -\int_T^t R_{p_3}(\varphi_\varepsilon, s) \, y^2(s) \mathrm{d}s = y''(t) \, y(t) - \frac{{y'}^2(t)}{2}.$$

Thus,

$$y''(t) \, y(t) = \frac{{y'}^2(t)}{2} - \int_T^t R_{p_3}(s) \, y^2(s) \mathrm{d}s \ge 0 \quad \text{for } t \in [0, T]. \tag{4.30}$$

Hence, for $\varphi \in \mathcal{A}_N$, $t \in [0, T]$ and sufficiently small $\varepsilon > 0$, we have

$$y(t) = \frac{(T - t)^2}{2} + \int_t^T \frac{(t - s)^2}{2} \, R_{p_3}(\varphi_\varepsilon, s) \, y(s) \mathrm{d}s \ge \frac{(T - t)^2}{2}. \tag{4.31}$$

By virtue of (4.27), (4.29), (4.30) and (4.31) we conclude that

$$y(0) \geq \frac{T^2}{2} \tag{4.32}$$

and

$$\eta_1(\varphi_\varepsilon) = R_x(\varphi_\varepsilon, 0) = c_3(\varphi_\varepsilon)\, y(0) + \eta_4(\varphi_\varepsilon), \tag{4.33}$$

where $\eta_4 \in \mathcal{N}$. Relations (4.32) and (4.33) imply that there are $C_1 \geq 0$, $N_1 \in \mathbb{N}$ and $\varepsilon_1 > 0$ such that for $q \geq N_1$, $\varphi \in \mathcal{A}_{N_1}$ and $\varepsilon \in (0, \varepsilon_1)$ the following inequality is true

$$|c_3(\varphi_\varepsilon)|\, \frac{T^2}{2} \leq |c_3(\varphi_\varepsilon)|\, |y(0)| \leq C_1\, \varepsilon^{\alpha(q)-N_1},$$

i.e. $c_3 \in \mathcal{N}$, which completes the proof of Theorem 3.8. □

5 Relations between Carathéodory's and Colombeau's concepts of solutions of differential equations

In this section we denote the product of g_1 and $g_2 \in \mathcal{G}(\mathbb{R})$ in $\mathcal{G}(\mathbb{R})$ by $g_1 \odot g_2$. If $g_1, g_2 \in C^\infty$, then their classical product $g_1 g_2$ and the product $g_1 \odot g_2$ in $\mathcal{G}(\mathbb{R})$ give rise to the same element of $\mathcal{G}(\mathbb{R})$. Hence we have the following assertion.

Theorem 5.1 *Let* $j \in \{1, 2, 3\}$,

$$p_r, q \in C^\infty, \quad r = 1, 2, 3, \qquad d_{ji} \in \mathbb{R}, \quad i = 1, 2, 3. \tag{5.1}$$

Furthermore, assume that the problem (1.1), (1.2_j) *has a generalized solution* $x_1 \in \mathcal{G}(\mathbb{R})$ *in the Colombeau sense and let the zero function be the only classical solution of the problem* (3.4_j) *Then the problem* (1.1), (1.2_j) *has also a classical solution* x_2 *and* x_1 *and* x_2 *give rise to the same element of* $\mathcal{G}(\mathbb{R})$.

Proof The existence of the classical solution x_2 of (1.1), (1.2_j) is evident. In particular, we have

$$\ell_{\varphi_\varepsilon}(R_{x_1})(t) = q(t) + \eta(\varphi_\varepsilon, t), \quad L_{ji}(R_{x_1}) = d_{ji} + \eta_{ji}(\varphi_\varepsilon), \tag{5.2}$$

and

$$\ell(x_2) = q(t), \quad L_{ji}(x_2) = d_{ji}, \tag{5.3}$$

where $\eta \in \mathcal{N}[\mathbb{R}]$, $\eta_{ji} \in \mathcal{N}$, $\varphi \in \mathcal{A}_1$ and $i = 1, 2, 3$. Furthermore, for

$$R_x(\varphi_\varepsilon, t) = x_2(t) - R_{x_1}(\varphi_\varepsilon, t), \quad \varphi \in \mathcal{A}_1, \quad t \in \mathbb{R} \text{ and } i = 1, 2, 3 \tag{5.4}$$

we get

$$\ell(R_x(\varphi_\varepsilon, t)) = -\eta(\varphi_\varepsilon, t), \quad L_{ji}(R_x(\varphi_\varepsilon, t)) = -\eta_{ji}(\varphi_\varepsilon), \tag{5.5}$$

On the other hand, $R_x(\varphi_\varepsilon, t)$ is given by (4.4) and (4.5), where R_q is replaced with η. Taking into account (4.4), (4.6)–(4.9) and (5.5) we deduce that $c_r \in \mathcal{N}$ for $r = 1, 2, 3$ and consequently

$$x_2 - R_{x_1} \in \mathcal{N}[\mathbb{R}],$$

as well. This completes the proof of Theorem 5.1. $\square$

Remark 5.2 It is known that every distribution is moderate (see e.g. [3, Proposition 2.2]). In general, the multiplication in $\mathcal{G}(\mathbb{R})$ does not coincide with the usual multiplication of continuous functions (see [2]). As a consequence, solutions of ordinary differential equations in the Carathéodory sense and in the Colombeau sense are different (in general). To "repair" the consistency problem for multiplication we give the definition introduced by J. F. Colombeau in [2].

A generalized function $u \in \mathcal{G}(\mathbb{R})$ is said to admit a member $w \in \mathcal{D}'(\mathbb{R})$ as the associated distribution, if it has a representative R_u with the following property: for every $\psi \in \mathcal{D}(\mathbb{R})$ there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ we have

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} R_u(\varphi_\varepsilon, t)\psi(t)\,\mathrm{d}t = w(\psi).$$

If $f \in \mathrm{L}_{loc}(\mathbb{R})$, we define

$$R_f(\varphi, t) = \int_{-\infty}^{\infty} f(t+u)\varphi(u)\,\mathrm{d}u = (f * \varphi)(t), \quad \varphi \in \mathcal{A}_1.$$

Obviously $R_f \in \mathcal{E}_M[\mathbb{R}]$.

Theorem 5.3 *Let $j \in \{1, 2, 3\}$, $p_r \in \mathrm{L}_{loc}(\mathbb{R})$ for $r = 1, 2, 3, 4$ and let the zero function be the unique solution of the problem (3.4_j) in the Carathéodory sense. Let x denote the solution of the problem (1.1), (1.2_j) in the Carathéodory sense. Furthermore, assume that there exists a solution $\widetilde{x}$ of the problem (1.1), (1.2_j) generalized in the Colombeoau sense. Then $\widetilde{x}$ admits an associated distribution which equals x.*

Proof follows from the fact that $p_k * \varphi_\varepsilon \to p_k$ and $q * \varphi_\varepsilon \to q$ in $\mathrm{L}_{loc}(\mathbb{R})$ as $\varepsilon \to 0+$ for $k = 1, 2, 3$ and from the continuous dependence of x on the coefficients p_1, p_2, p_3 and q. Indeed, let $R_{\psi_0}(\varphi_\varepsilon, t)$, $R_{\psi_1}(\varphi_\varepsilon, t)$, $R_{\psi_2}(\varphi_\varepsilon, t)$ be solutions of the problems (4.3). Then for every $\varphi \in \mathcal{A}_1$ and $r, i = 1, 2, 3$, we have

$$\lim_{\varepsilon \to 0} R_{\psi_{i-1}}^{(r-1)}(\varphi_\varepsilon, t) = \psi_{i-1}^{(r-1)}(t) \tag{5.6}$$

almost uniformly on $\mathbb{R}$. This yields

$$\lim_{\varepsilon \to 0} |\det A_{(j)\varepsilon}| = g \neq 0, \quad g \in \mathbb{R}, \tag{5.7}$$

for every $\varphi \in \mathcal{A}_1$, $\det A_{(j)\varepsilon}$ is defined by (4.9). Let $R_x(\varphi_\varepsilon, t)$ be a solution of the equation (3.6) satisfying the conditions

$$L_{ji}(R_x(\varphi_\varepsilon, t)) = d_{ji}, \quad i = 1, 2, 3. \tag{5.8}$$

Thus, by virtue of the relations (4.5)–(4.7), (4.10) and (5.6)–(5.8), for $r = 1, 2, 3$ and for every fixed $\varphi \in \mathcal{A}_1$, we get

$$\lim_{\varepsilon \to 0} R_{x^{(r-1)}}(\varphi_\varepsilon, t) = x^{(r-1)}(t)$$

almost uniformly on $\mathbb{R}$ and x is a solution of (1.1), (1.2_j) in the Carathéodory sense. On the other hand, $\widetilde{x} = [R_x]$ is a generalized solution of (1.1), (1.2_j) (we put $R_x(\varphi_\varepsilon, t) = 0$ if $\det A_{(j)\varepsilon} = 0$). This proves Theorem 5.3. □

Remark 5.4 If $p_k \in \mathrm{L}_{loc}(\mathbb{R})$, then

$$|R_{p_k}(\varphi_\varepsilon, t)| - \left|\int_{-\infty}^{\infty} p_k(t)\varphi(u)\,\mathrm{d}u\right| \leq \left|R_p(\varphi_\varepsilon, t) - \int_{-\infty}^{\infty} p_k(t)\varphi(u)\,\mathrm{d}u\right|$$
$$\leq \int_{-\infty}^{\infty} |p_k(t+\varepsilon_u) - p(t)|\,|\varphi(u)|\mathrm{d}u,$$

i.e. p_k satisfy (H_0).

Corollary 5.5 *Let* $j \in \{1, 2, 3, \}$, $p_r \in \mathrm{L}_{loc}(\mathbb{R})$, $r = 1, 2, 3$, *and*

$$a_j\Big(\sum_{r=1}^{3} \int_0^T |p_r(t)|\mathrm{d}t\Big) < 1,$$

with a_j *given by* (3.2).

Then the problem (3.4_j) *has only the trivial solution in the Carathéodory sense.*

Corollary 5.6 *Let* $j \in \{1, 2, 3, \}$, $p_j \in \mathrm{L}_{loc}(\mathbb{R})$ *and*

$$b_j \int_0^T |p_j(t)|\mathrm{d}t < 1,$$

with b_j *given by* (3.4_j). *Then the problem* (3.8_j) *has only the trivial solution in the Carathéodory sense.*

Corollary 5.7 (cf. [8]) *Let* $j \in \{1, 2, 3, \}$, $p_r \in \mathrm{L}_{loc}(\mathbb{R})$, $r = 1, 2, 3$, *and* $p_3(t) \geq 0$ *a.e. on* $[0, T]$. *Then the problem* (3.8_3) *has only the trivial solution in the Carathéodory sense.*

Remark 5.8 The concept of generalized solutions of ordinary differential equations can be considered also in other ways. See e.g. [4], [5], [6], [9], [10], [14], [15], [16] or [17].

Remark 5.9 The definition of Colombeau generalized functions on a given open subinterval of $\mathbb{R}$ is analogous to the definition used in this paper (see [2]). It is not defficult to observe that if reformulated our assumptions (H_0), (H_j) and (G_j) in a proper way, the results of this paper would remain true also in the case when the generalized functions are considered on some open subinterval (a, b) of $\mathbb{R}$.

Acknowledgement I should like to express may gratitude to prof. M. Tvrdý for efficient help in preparing this paper to print.

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